FVNDAMENTA NOVA

INVESTIGATIONIS ORBITAE VERAE

QVAM

LVNA PERLVSTRAT,

QVIBVS ANNEXA EST

SOLVTIO PROBLEMATIS QVATVOR CORPORVM

BREVITER EXPOSITA

AVCTORE

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PRAEFATIO.

Casus duos problematis trium corporum, quod vocatur, quos systema nostrum solare nobis offert, alterum puta, ubi massa perturbans revera parvula est, alterum, ubi massa perturbans respectu massae corporis primarii permagna est, sed propter distantiam permagnam cum a corpore primario, tum a corpore perturbato parvulam efficit vim perturbantem; casus, inquam, hos duos unius eiusdemque problematis geometrae ad hoc usque tempus duobus modis inter se plane diversis solverunt, et neque solutionem, quam pro casu priori dedere, ad casum alterum, neque solutionem huius casus ad illum applicari licet. Solutionem generaliorem utrumque casum complectentem adhuc nemo dedit, quin etiam sunt, qui talem problematis huius solutionem rei naturam non admittere, et semper problema hoc in his duobus casibus diverso modo tractari debere dixerint. Nihilominus solutio talis generalior inveniri potest, et pagellis sequentibus eam, quam assequutus sum, exponam. Quae solutio immediate ad casum posteriorem, qui theoriam Lunae constituit, pertinet, ita autem comparata est, ut factis tribus quantitatibus quibusdam cifrae aequalibus, solutio eliciatur, quam pro casu priori, qui in theoria planetarum locum habet, iam publici iuris feci. Theoria igitur pla-

netarum mea casum specialem theoriae Lunae hoc volumine expositae sese praestat.

Antequam methodi huius expositionem aggredior, certam huius problematis conditionem praemisisse haud ab re est. Integratis formulis, quas ad motum corporis coelestis cuiuscunque determinandum dedi, eo modo, quem in theoria Iovis atque Saturni explicavi, termini oriuntur, qui per sinus aut cosinus multiplicium anomaliarum mediarum multiplicati, tempori aut temporis potestatibus proportionales sunt. Qui termini ex integrandi methodo oriuntur, et revera originem trahunt e perturbationibus pure periodicis et plerumque longissimae periodi, in quarum argumentis tempus ipsum per functiones vis perturbantis multiplicatum existit. Termini igitur illi, qui in planetarum theoria et per tempus temporisque potestates, et per sinus cosinusve multiplicium anomaliarum mediarum multiplicati sunt, non nisi pro terminis primis evolutionis in seriem infinitam illarum functionum vis perturbantis habendi sunt, et semper quoties series hae admodum convergunt sub forma hac tutissime in usum vocari possunt. Quoties vero series istae non convergunt vel etiam paullulum convergunt, iidem termini efficiunt, ut brevi temporis spatio praeterlapso locus corporis coelestis computatus a vero eius loco aberrare incipiat, et magis magisque ab eo discrepaturus sit.

In planetarum systematis nostri solaris motu termini, de quibus loquimur, series maxime convergentes constituunt, itaque primis tantum harum serierum terminis in calculum vocatis, locus planetae absque errore sensibili per longam annorum seriem optime repraesentari potest, imo hoc in casu functiones periodicae, e quibus termini illi oriuntur, propter incertitudinem, quae massis planetarum semper inhaerebit, magna cum praecisione determinari nequeunt. Nam longitudinis, latitudinis et radii vectoris perturbationes generaliter formam induunt hanc

$$a\sin(ig+i'g'+i''g''+\text{etc.}+\alpha nt+A)+b\sin(ig+i'g'+i''g''+\text{etc.}+\beta nt+B)+\text{etc.}$$

ubi t tempus, n motum medium planetae perturbati, g eius anomaliam mediam, g', g'', etc. anomalias medias planetarum perturbantium, a, α, A, b, β, B, etc. constantes et i, i', i'', etc. numeros integros positivos seu negativos, inclusa cifra, denotant. In theoria quidem planetarum α, β, etc. semper minutissimi e massis perturbantibus pendentes numeri sunt, sed saltem in terminis ubi $i' = i'' = \text{etc.} = o$, ipsae a, b, etc. valorem permagnum habent, et tum α, β, etc., quae ex aequatione algebraica gradus altioris pendent, propter incertitudinem, cui massae planetarum semper subiectae erunt; nunquam accurate computare nobis licebit. Evolutis vero terminis allatis in seriem infinitam hanc

$$\begin{aligned}
&(a\cos A + b\cos B + \text{etc.})\sin(ig + i'g' + i''g'' + \text{etc.})\\
&+ (a\sin A + b\sin B + \text{etc.})\cos(ig + i'g' + i''g'' + \text{etc.})\\
&- nt(\alpha a\sin A + \beta b\sin B + \text{etc.})\sin(ig + i'g' + i''g'' + \text{etc.})\\
&+ nt(\alpha a\cos A + \beta b\cos B + \text{etc.})\cos(ig + i'g' + i''g'' + \text{etc.})\\
&- \tfrac{1}{2}n^2t^2(\alpha^2 a\cos A + \beta^2 b\cos B + \text{etc.})\sin(ig + i'g' + i''g'' + \text{etc.})\\
&- \tfrac{1}{2}n^2t^2(\alpha^2 a\sin A + \beta^2 b\sin B + \text{etc.})\cos(ig + i'g' + i''g'' + \text{etc.})
\end{aligned}$$

terminos omnes accurate computari licet. In casu enim ubi $i' = i'' = \text{etc.} = o$ eliciuntur ex hac serie primi termini hi

$$(a\cos A + b\cos B + \text{etc.})\sin ig + (a\sin A + b\sin B + \text{etc.})\cos ig$$

quorum quidem coefficientes permagni sunt, sed cum terminis, quos ellipsis pura in longitudine etc. profert, sese coniungunt, itaque simul cum elementis ellipticis per observationes astronomicas determinantur, neque in expressionibus perturbationum seorsum apponuntur. Termini vero reliqui seriei praecedentis omnes ita comparati sunt, ut accurate computari possint, quia producta αa, βb, etc. semper minutissimi numeri sunt. Propter exiguitatem massarum planetarum, et quoniam n in hoc casu est numerus haud magnus, series praecedens admodum convergit, et intra temporis intervallum plus quam

millia annorum ante et post epocham complectens termini per altiores temporis potestates multiplicati vim non habent.

In Lunae motu, quatenus a Sole perturbatur, res secus se habet; hoc enim in casu α, β, etc. maiores numeri sunt, et coefficientes a, b, etc. semper minores, quam in illo problemate, in terminis etiam ubi $i' = o$. Quum vero illi maiores sint, et n respectu motus Solis sive Terrae magnus numerus sit, factum est, ut termini seriei praecedentis per superiores temporis potestates multiplicati, parvulo temporis spatio praeterlapso, tam magni evadant, ut nullo modo negligi possint. Quin etiam inde a certo quodam temporis momento ab epocha non longe remoto termini per altiores temporis potestates multiplicati reliquis terminis maiores evaderent, quo fit, ut loca Lunae hoc modo computata cum veris eius locis nequaquam congruere possint. Quamobrem in theoria Lunae ad huius seriei functiones originales, hoc est ad terminos huius formae

$$a \sin(ig + i'g' + \alpha nt + A) + b \sin(ig + i'g' + \beta nt + B) + \text{etc.}$$

nobis refugiendum est, et aequationum differentialium perturbationes suppeditantium integrationes ita instituendae sunt, ut perturbationes statim hac forma expressae obtineantur.

Geometrae saeculi praeteriti, qui problema hoc tractabant, theoriam Lunae ad hunc usque praecisionis gradum non expolierunt, ut cum observato Lunae motu congrueret. Summus Laplace quoque, qui et Lunae theoriam perscrutans multa et gravissima huius theoriae momenta detegebat, computationes non eo usque extendit, ut tabulis motum Lunae exhibentibus omnes superstrui possent. Observationibus ipsis coefficientes perturbationum Lunae inveniendi fuere, et usque ad tempus recentissimum tabulis sola theoria gravitatis nitentibus caruimus. Ill. Damoiseau methodum perturbationum computandarum sequutus, cuius fundamenta ill. Laplace quoque adhibuerat, primus tabulas Lunae sola theoria gravitatis fundatas dedit. Quae computandarum perturbatio-

num methodus praecipue eo consistit, quod tempus, unitas per radium vectorem ad eclipticam proiectum divisa et tangens latitudinis versus eclipticam tanquam functiones longitudinis verae ad eclipticam proiectae exhibentur. Hinc vero factum est, ut coordinatae Solis, quae quidem, quatenus in motu Lunae vim habent, expressiones quam simplicissimae sunt, per longitudinem illam Lunae exprimendae sint, unde formam admodum implicitam accipiunt; praeterea, computatione perturbationum peracta, series, quae tempus per longitudinem illam expressum suppeditat, revertenda, et in seriebus perturbationes reliquarum coordinatarum exhibentibus loco illius longitudinis expressio eius per tempus substituenda est. Quae conditiones perturbationum Lunae computationes secundum hanc methodum satis longas reddunt. Ill. Damoiseau evolutiones analyticas ad ultimum finem non produxit, sed methodo coefficientium indeterminatorum eadem, qua ill. Clairaut et Laplace usi erant, adhibita, aequationes mixtas post valores numericos quantitatum, quas continent, substitutos resolvit. Quantum laboris haec problematis huius solvendi methodus poposcerit ex commentatione ill. Damoiseau praemio ornata et in libri: Mémoires présentés par divers savans etc. intitulati volumine I° typis excusa vides, et auctor sane laboris compendium sibi non conciliavisset, si evolutiones analyticas ad ultimum finem perduxisset. Ill. Plana in opere suo de theoria Lunae nuperrime edito eandem fere methodum adhibuit, evolutiones vero analyticas usque ad metam produxit; sed non dubium est, quin hoc computandi modo adhuc plus laboris sibi conciliaverit. Praeter haec de theoria Lunae confecta opera, ill. Poisson nec non ill. Lubock formulas notas, quibus perturbationes immediate in functione temporis exprimuntur, ad perturbationes Lunae computandas, et quidem ille formulas variabilia elementa elliptica suppeditantes, hic vero formulas coordinatas polares subministrantes proposuerunt, et uterque evolutionum analyticarum tantum, quantum fieri possit, extendendarum consilium cepisse videtur.

Ut opera et labor, quem hae methodi requirant, recte iudicari possit, nihil est quod malim, quam ut hi geometrae integram perturbationum Lunae computationem secundum methodos propositas confici curent. Egomet secundum methodum in hac commentatione explicatam huius computationis maiorem partem iam confeci, et reliquam, simul atque occupationes quaedam aliae absolutae fuerint, conficere in animo est.

Formulis modo memoratis adhibitis, computationes breviores quam computationes ill. Damoiseau et Plana evadere debere videntur, quoniam reversiones serierum, quas methodus ab his adhibita poscit, opus non sunt; evolutiones vero analyticae aliam ob causam aliquid incerti involvunt. Qui calculus postulat, ut quantitates quaedam pro parvulis quantitatibus primi ordinis habeantur, secundum quas evolutiones analyticae procedant, sed quum de coefficientium numericorum ordine analytico sermo esse nequeat, fieri potest, ut propter hos coefficientes singuli perturbationum termini in fine calculi valorem habeant multo maiorem, quam eum, cui pro ordine suo analytico adnumerandi sint. Revera fit, ut propter coefficientes numericos divergentes in perturbationum complurium expressionibus analyticis duo termini consequutivi, qui pro indole sua ordinibus analyticis diversis adnumerandi sunt, eundem fere valorem numericum habeant, uti in opere ill. Plana inspici potest. E. g. coefficientem perturbationis longitudinis Lunae verae, cuius argumentum duplex distantia perigaei Lunae a nodo eius est, invenit ill. Plana =

$$\left(\frac{1}{8} + \frac{135}{64}m\right)e^2\gamma^2$$

Iam quum m, quae in hac expressione rationem motus medii Lunae ad motum medium Solis repraesentat, in hac computatione pro quantitate primi ordinis habeatur, prior expressionis praecedentis terminus, quod attinet ad compositionem suam analyticam, ordinis finiti, et posterior ordinis primi est. Quum vero valor numericus ipsius m sit proxime $= \frac{1}{13}$, habetur

$$\frac{135}{64}m = \frac{135}{832} = \frac{1}{6,16..}$$

Hic igitur terminus, qui in evolutione analytica pro parvula quantitate primi ordinis habetur, revera maior est quam terminus prior, qui pro quantitate finita habetur. Quae quum ita sint, in evolutione tali, etiamsi summa cura industriaque maxima instituta sit, terminos omnes, qui vim habeant, receptos esse, quis pro certo affirmare potest? nisi evolutiones longe ultra terminos in fine calculi retinendos produxerit. Conditio enim eadem, quam modo ante oculos posuimus, in terminis quoque ordinum altiorum locum habere potest, quo factum erit, ut termini quidam ordinis analytici altissimo in calculis recepto ordine altioris et hac ratione neglecti propter coefficientes numericos suos valorem nanciscantur numericum, qui revera negligendus non est.

Quibus considerationibus inductus, iam pridem in computandis planetarum perturbationibus saltem evolutiones analyticas usque ad quosdam terminos generales tantum produxi, quos immediate ad calculum numericum absolvendum adhibui. Hoc modo effeci, ut terminus quisque secundum valorem absolutum aestimari posset, et omnes limitem numericum ex lubito praedeterminatum superantes termini, paucis adhibitis regulis, in calculum vocari possent. Eandem methodum ad computandas Lunae perturbationes adhibeo, et hoc modo, nullum terminum qui vim habeat omitti, pro certo habere possum. Computationes brevissimae evadunt, quia series, ad quas methodus haec perducit, rapidissime convergunt, et termini limite proposito minores statim omitti possunt. Integrationes ita institui, ut termini per tempus ipsum multiplicati, quatenus e coordinatis Lunae ipsius orirentur, non adsint, id quod introductis quantitatibus tribus y, α atque η nominatis effeci, quarum $y-2\eta$ motum directum perigaei Lunae in orbita, et $\alpha+\eta$ motum retrogradum nodorum orbitae Lunae cum orbita terrae quam proxime denotant, aggregatum vero $(y-2\eta)+(\alpha+\eta)$ sive $y+\alpha-\eta$ horum motuum aggregatum rigorose designat. Perturbationes longitu-

dinis ita computavi, ut ad longitudinem **mediam** addendae sint, quo factum est, ut formulae simpliciores evaderent, et tum series perturbationum ipsarum, tum series, quibus quisque coefficiens computandus est, magis convergentes fierent. Formulas ita comparavi, ut is excentricitatis Lunae valor numericus, qui ex observato maximo aequationis centri termino ope formulae pure ellipticae elicitur, in perturbationum computatione adhibendus sit, calculo igitur indirecto olim ad eum excentricitatis Lunae valorem, qui computandis perturbationibus superstruendus fuerit, indagandum adhibito opus non sit.

Praeterea problema ita solvi, ut longitudo ad planum respectu situs eius in spatio penitus arbitrarium relata et latitudo versus idem planum obtineatur. Solutio igitur hic data hac quoque ex parte solutionibus reliquis, quae longitudinem Lunae latitudinemque versus eclipticam solummodo suppeditant, generalior est. Quum vero planum proiectionis in solutione nostra quodlibet sit, aequatorem planum hoc statuere licet, quo factum erit, ut ex tabulis Lunae motum exhibentibus ascensio recta eius et declinatio immediate computari possint.

Quibus in formulis generalibus y, x atque η cifrae aequalibus factis, formulas nanciscimur, quae immediate perturbationibus planetarum computandis inserviunt, et cum iis identicae sunt, quas iam pro hac computatione dedi. Externa vero facie formulae hic datae ab illis paullulum discrepant, quia aequationis pro $\frac{d^2\zeta}{d\tau\, dt}$ integrationem alio modo institui; illic, ipsis ζ et w computatis, ipsam z ex ζ, mutata in fine calculi numerici τ in t, elicui; hic vero ζ non computatur, sed statim et ipsa z et ipsa w, qua formularum transmutatione magnum calculi compendium assequutus sum.

Quantitates quoque auxiliares in hac commentatione $p_{,}$, $q_{,}$, P, Q, K, $p_{,,}$ atque $q_{,,}$ nominatae resp. cum quantitatibus auxiliaribus, quibus in theoria planetarum usus sum, cum quantitatibus puta in Astr. Nachr. No. 244. seqq. p, q, P, Q atque $\varphi_{,} - \varphi$, et cum quantitatibus ibidem No. 295. seqq. l' atque

identitatis, quamquam externa facie ab his discrepant, factis y, a et η cifrae aequalibus, identicae sunt. Evolutiones perturbationum primi ordinis respectu vis perturbantis, quatenus his utor, ita institui potuissent, ut immediate ad planetas quoque applicari possent: talis enim functionis perturbatricis Ω evolutio, qualis in planetarum theoria adhibenda est, in Lunae quoque theoria in usum vocari posset. In hac vero commentatione evolutionem hanc ita institui, ut quantitatem Ω in seriem secundum potestates rationis distantiarum Terrae a Luna et a Sole progredientem evolverem, quae evolutio in Lunae theoria propter huius rationis exiguitatem omnibus aliis praeferenda est, in planetarum vero theoria in usum vocari nequit.

Causas non omnes, quae motum Lunae perturbant, in hac commentatione tractavi, perturbationum enim earum, quae e vi attractiva Solis oriuntur, computationem solummodo explicavi, futuro tempori edendas perturbationum a figura sphaeroidica terrae et planetarum perturbationibus pure periodicis orientium explicationes relinquens.

Problema illud

Determinare corporum coelestium systematis nostri Solaris motus secundum legem gravitatis Newtonianam

saepissime problema trium corporum appellatum est, quamquam revera inter corporum corpus primarium ambientium numerum distinguendum est. Ita proprie hoc problema duorum corporum, quoties motus unius, trium corporum, quoties motus duorum, quatuor corporum, quoties motus trium corporum respectu corporis primarii, quod illa ambiunt, investigandi sunt, dicendum est, et sic porro, quoties plura corpora adsunt. Problema duorum corporum expressionibus finitis solvi posse, et haecce corpora sectiones conicas circum commune centrum eorum gravitatis describere, aut, si motus relativi investigantur, utrumque sectionem conicam eiusdem excentricitatis periodique circum alterum perlustrare, nemo nescit. Problemata plurium quam duorum corporum expressionibus finitis solvi

nequeunt, sed quum in systemate nostro Solari corpora omnia in se invicem parvulam efficiant vim respectu vis corporis primarii, haec problemata ad series convergentes perducunt, quarum effectus in eo consistit, quod motum in sectione conica alioqui absolutum haud multum perturbant. Quas perturbationes duplici exhiberi posse modo, altero quo termini per tempus ipsum multiplicati evitentur, altero quo iidem admittantur, iam in praecedentibus explicavimus. Problema Lunae determinandi motus, si perturbationes a planetis productas non respexeris, proprie quidem problema trium corporum est, sed quum ei conditio insit, ut Terrae a Luna perturbatus motus a priori et ab ipso Lunae perturbato motu independenter fere exhiberi possit, itaque in hoc problemate propemodum solus Lunae motus investigandus sit: hoc casum tantum specialem problematis generalis trium corporum, in quo utriusque corporis motus relativus respectu tertii corporis investigandus est, ponit, licet perturbationes illo supra explicato modo exhibeantur, ut termini scilicet per tempus ipsum multiplicati non adsint.

Quamobrem ex Lunae in hoc volumine exposita theoria solutio problematis generalis trium corporum peti non potest, nec, etiamsi plura non obstiterint, problematis plurium corporum solutio in ea continetur. Quum vero huius problematis solutioni ita sumtae, ut termini per tempus ipsum multiplicati evitentur, in systemate nostro Solari applicatio pulcherrima pateat, et problematis quatuor corporum solutio ad problematis plurium corporum solutionem perfacile extendatur: eam, quam assequutus sum, hoc loco silentio praetermittere nolui. Breviter igitur expositam solutionem problematis quatuor corporum annexui, quam eo usque exposui, ut functiones perturbatrices in series evolvendae et in formulis datis substituendae restent.

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RERVM IN HOC VOLVMINE CONTENTARVM.

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SECTIO I.

DE AEQVATIONIBVS DIFFERENTIALIBVS MOTVM SYSTEMATIS CVIVSCVNQVE CORPORVM ATTRACTIONI RECIPROCAE SVBIECTORVM DEFINIENTIBVS.

1.

Consideremus numerum indefinitum corpusculorum infinite parvulorum, quorum massae resp. sunt m, m', m'', etc. $m^{(\mu)}$, $m^{(\mu')}$, $m^{(\mu'')}$, etc. $m^{(\nu)}$, $m^{(\nu')}$, $m^{(\nu'')}$, etc. etc. et quorum coordinatae orthogoniae sunt ξ, η, ζ, ξ', η', ζ', etc. $\xi^{(\mu)}$, $\eta^{(\mu)}$, $\zeta^{(\mu)}$, $\xi^{(\mu')}$, $\eta^{(\mu')}$, $\zeta^{(\mu')}$, etc. etc. Supponamus corpuscula haec viribus acceleratricibus X, Y, Z, X', Y', Z', etc. $X^{(\mu)}$, $Y^{(\mu)}$, $Z^{(\mu)}$, etc. etc. secundum directiones coordinatarum harum sollicitari, viresque has coordinatas augere tendere. Si conditio adiungitur, ut inter corpusculorum horum loca aequationes conditionales non adsint, principia Mechanicae docent, vires acceleratrices easdem per quotientes differentiales coordinatarum secundi ordinis respectu temporis quoque hos

$$\frac{d^2\xi}{dt^2},\ \frac{d^2\eta}{dt^2},\ \frac{d^2\zeta}{dt^2},\ \frac{d^2\xi'}{dt^2},\ \frac{d^2\eta'}{dt^2},\ \frac{d^2\zeta'}{dt^2},\ \text{etc. etc.}$$

exprimi, unde aequationes motum corpusculorum horum definientes nanciscimur has

$$\frac{d^2\xi}{dt^2} = X, \quad \frac{d^2\eta}{dt^2} = Y, \quad \frac{d^2\zeta}{dt^2} = Z$$

$$\frac{d^2\xi'}{dt^2} = X', \quad \frac{d^2\eta'}{dt^2} = Y, \quad \frac{d^2\zeta'}{dt^2} = Z'$$

etc. etc.

$$\frac{d^2\xi^{(\mu)}}{dt^2} = X^{(\mu)}, \quad \frac{d^2\eta^{(\mu)}}{dt^2} = Y^{(\mu)}, \quad \frac{d^2\zeta^{(\mu)}}{dt^2} = Z^{(\mu)}$$

etc. etc.

etc. etc.

2.

Si nunc ponimus corpuscula haec solummodo sollicitari attractionibus reciprocis, quae in ratione directa massarum et in ratione inversa quadratorum distantiarum reciprocarum urgeant, vis, quam corpusculum m' in m exercet, aequalis est

$$\varkappa \frac{m'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}^2}$$

designante $\varkappa$ intensitatem vis huius, quoties et distantia et massa unitati aequalis est; vis, quam m'' in m exercet, aequalis

$$\varkappa \frac{m''}{\sqrt{(\xi-\xi'')^2+(\eta-\eta'')^2+(\zeta-\zeta'')^2}^2}$$

et sic porro. Dissolutis his viribus in tres secundum directiones axium coordinatarum urgentes vires, habemus, quia attractiones distantias deminuere tendent,

$$X = \frac{\varkappa m'(\xi'-\xi)}{[(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2]^{\frac{3}{2}}} + \frac{\varkappa m''(\xi''-\xi)}{[(\xi-\xi'')^2+(\eta-\eta'')^2+(\zeta-\zeta'')^2]^{\frac{3}{2}}} + \text{etc.}$$

et similes expressiones nanciscimur pro Y, Z, X', etc. etc. Quae ipsius X expressio sub hac redigi potest forma

$$X = \frac{\varkappa}{m} \frac{d.\frac{mm'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}}{d\xi} + \frac{\varkappa}{m} \frac{d.\frac{mm''}{\sqrt{(\xi-\xi'')^2+(\eta-\eta'')^2+(\zeta-\zeta'')^2}}}{d\xi}$$

et eodem modo obtinetur

$$X' = \frac{\varkappa}{m'} \frac{d.\frac{mm'}{\sqrt{(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2}}}{d\xi'} + \frac{\varkappa}{m'} \frac{d.\frac{m'm''}{\sqrt{(\xi'-\xi'')^2+(\eta'-\eta'')^2+(\zeta'-\zeta'')^2}}}{d\xi'} + \text{etc.}$$

et sic porro. Hinc et quum expressiones analogae pro Y, Z, Y', etc. inveniri possint, sequitur, ponendo

$$\lambda = \varkappa \Sigma \frac{mm'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}$$

ubi summatio ad binas massas mm'', $m'm''$, etc. $mm^{(\mu)}$, $m'm^{(\mu)}$, etc. etc. extendenda est, aequationes art. 1. in has transire aequationes

$$\frac{d^2\xi}{dt^2} = \frac{1}{m}\left(\frac{d\lambda}{d\xi}\right), \quad \frac{d^2\eta}{dt^2} = \frac{1}{m}\left(\frac{d\lambda}{d\eta}\right), \quad \frac{d^2\zeta}{dt^2} = \frac{1}{m}\left(\frac{d\lambda}{d\zeta}\right) \quad \ldots\ldots(1)$$

$$\frac{d^2\xi'}{dt^2} = \frac{1}{m'}\left(\frac{d\lambda}{d\xi'}\right), \quad \frac{d^2\eta'}{dt^2} = \frac{1}{m'}\left(\frac{d\lambda}{d\eta'}\right), \quad \frac{d^2\zeta'}{dt^2} = \frac{1}{m'}\left(\frac{d\lambda}{d\zeta'}\right) \quad \ldots\ldots(2)$$

etc. etc.

$$\frac{d^2\xi^{(\mu)}}{dt^2} = \frac{1}{m^{(\mu)}}\left(\frac{d\lambda}{d\xi^{(\mu)}}\right), \quad \frac{d^2\eta^{(\mu)}}{dt^2} = \frac{1}{m^{(\mu)}}\left(\frac{d\lambda}{d\eta^{(\mu)}}\right), \quad \frac{d^2\zeta^{(\mu)}}{dt^2} = \frac{1}{m^{(\mu)}}\left(\frac{d\lambda}{d\zeta^{(\mu)}}\right) \quad \ldots\ldots(3)$$

$$\frac{d^2\xi^{(\mu')}}{dt^2} = \frac{1}{m^{(\mu')}}\left(\frac{d\lambda}{d\xi^{(\mu')}}\right), \quad \frac{d^2\eta^{(\mu')}}{dt^2} = \frac{1}{m^{(\mu')}}\left(\frac{d\lambda}{d\eta^{(\mu')}}\right), \quad \frac{d^2\zeta^{(\mu')}}{dt^2} = \frac{1}{m^{(\mu')}}\left(\frac{d\lambda}{d\zeta^{(\mu')}}\right) \quad \ldots\ldots(4)$$

etc. etc.

$$\frac{d^2\xi^{(\nu)}}{dt^2} = \frac{1}{m^{(\nu)}}\left(\frac{d\lambda}{d\xi^{(\nu)}}\right), \quad \frac{d^2\eta^{(\nu)}}{dt^2} = \frac{1}{m^{(\nu)}}\left(\frac{d\lambda}{d\eta^{(\nu)}}\right), \quad \frac{d^2\zeta^{(\nu)}}{dt^2} = \frac{1}{m^{(\nu)}}\left(\frac{d\lambda}{d\zeta^{(\nu)}}\right) \quad \ldots\ldots(5)$$

$$\frac{d^2\xi^{(\nu')}}{dt^2} = \frac{1}{m^{(\nu')}}\left(\frac{d\lambda}{d\xi^{(\nu')}}\right), \quad \frac{d^2\eta^{(\nu')}}{dt^2} = \frac{1}{m^{(\nu')}}\left(\frac{d\lambda}{d\eta^{(\nu')}}\right), \quad \frac{d^2\zeta^{(\nu')}}{dt^2} = \frac{1}{m^{(\nu')}}\left(\frac{d\lambda}{d\zeta^{(\nu')}}\right) \quad \ldots\ldots(6)$$

etc. etc.

Quae sunt aequationes differentiales motuum systematis corpusculorum, quae legi gravitationis universalis Newtonianae sunt subiecta.

3.

Consideremus seorsim et corpuscula m, $m^{(\mu)}$, $m^{(\nu)}$, etc. et m', $m^{(\mu')}$, $m^{(\nu')}$, etc. et m'', $m^{(\mu'')}$, $m^{(\nu'')}$, etc. etc. Formentur primum aggregata ex aequationibus praecedentibus (1.), (3.), (5.), etc. postquam per massas respectivas multiplicatae erunt, deinde aggregata ex aequationibus (2.), (4.), (6.), etc. postquam per massas quoque respectivas multiplicatae erunt, et sic porro. Quibus factis nanciscimur has

$$\left.\begin{array}{l} \Sigma \frac{d^2\xi}{dt^2} m = \Sigma \left(\frac{d\lambda}{d\xi}\right), \ \Sigma \frac{d^2\eta}{dt^2} m = \Sigma \left(\frac{d\lambda}{d\eta}\right), \ \Sigma \frac{d^2\zeta}{dt^2} m = \Sigma \left(\frac{d\lambda}{d\zeta}\right) \\ \Sigma \frac{d^2\xi'}{dt^2} m' = \Sigma \left(\frac{d\lambda}{d\xi'}\right), \ \Sigma \frac{d^2\eta'}{dt^2} m' = \Sigma \left(\frac{d\lambda}{d\eta'}\right), \ \Sigma \frac{d^2\zeta'}{dt^2} m' = \Sigma \left(\frac{d\lambda}{d\zeta'}\right) \end{array}\right\} \ldots\ldots(7)$$

etc. etc.

ubi igitur summationes in linea prima ad corpuscula m, $m^{(\mu)}$, $m^{(\nu)}$, etc., in linea secunda ad corpuscula m', $m^{(\mu')}$, $m^{(\nu')}$, etc. referendae sunt, et sic porro. Perfacile perspicitur esse respectu terminorum ipsius λ per massas $m\,m^{(\mu)}$, $m\,m^{(\nu)}$, $m^{(\mu)}\,m^{(\nu)}$, etc. multiplicatorum

$$o = \left(\frac{d\lambda}{d\xi}\right) + \left(\frac{d\lambda}{d\xi^{(\mu)}}\right) + \left(\frac{d\lambda}{d\xi^{(\nu)}}\right) + \text{etc.}$$

et respectu terminorum per massas $m'm^{(\mu')}$, $m'm^{(\nu')}$, $m^{(\mu')}m^{(\nu')}$, etc. multiplicatorum

$$o = \left(\frac{d\lambda}{d\xi'}\right) + \left(\frac{d\lambda}{d\xi^{(\mu')}}\right) + \left(\frac{d\lambda}{d\xi^{(\nu')}}\right) + \text{etc.}$$

et sic porro. Quae quum pro aggregatis reliquis $\Sigma\left(\frac{d\lambda}{d\eta}\right)$, $\Sigma\left(\frac{d\lambda}{d\zeta}\right)$, $\Sigma\left(\frac{d\lambda}{d\eta'}\right)$, etc. etiam locum habere debeant, in ipsa λ, quatenus in aequationibus (7) vim habet, terminos solummodo recipere opus est, qui per binas massas mm', $mm^{(\mu')}$, etc. $m^{(\mu)}m'$, etc. etc., quae ad diversa aggregata pertinent, multiplicati sunt.

Quibus positis, sumamus in aequationibus (7) numerum corpusculorum, quae in quoque aggregato continentur, infinite magnum evadere, et simul distantias reciprocas inter haec corpuscula infinite parvulas fieri, quo factum erit, ut aggregatum quodque in corpus dimensionum finitarum abeat, quae corpora per massas eorum m, m', m'', etc. designabo. Iam nunc, quum sit

$$\lambda = \varkappa\Sigma\frac{m\,m'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}$$

in qua expressione termini per massarum combinationes $mm^{(\mu)}$, $mm^{(\nu)}$, etc. $m'm^{(\mu')}$, etc. quae ad idem corpus pertinent, multiplicati secundum praecedentia excludi possunt: haec quantitas λ ex numero infinite magno terminorum constat, quorum quisque per massarum particulas infinite parvulas mm', $mm^{(\mu')}$, etc. $m^{(\mu)}m'$, etc., quarum utraque ad aliud corpus pertinet, multiplicatus est. Summatio igitur in integrale duplex abit, ita ut habeatur

$$\lambda = \varkappa\iint\frac{dm\,.\,dm'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}$$

ubi integrationes per massas integras omnes extendendae sunt. Quum vero supponatur corpus finitum quodque a reliquis corporibus spatio finito di-

stare, et figuras superficierum corporum horum a se invicem independentes esse, integratio integra, quam λ requirit, in tot integrationes duplices distribuitur, quot binae combinationes corporum adsunt, et ipsa λ ex totidem terminis constat, in quibus quoque integrationes a se invicem independentes sunt. Habemus igitur denique

$$\lambda = \varkappa \Sigma \iint \frac{dm \,.\, dm'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}} \quad \ldots\ldots(8)$$

ubi signum Σ ad binas corporum m, m', m'', etc. combinationes referendum est, ita ut ex. gr. signum Σ quantitatem λ ex $\frac{(n-1)}{2}$ terminis constare indicet, quoties numerus corporum est n. Quantitas $\Sigma\left(\frac{d\lambda}{d\xi}\right)$, quae primae aequationi (7) inest, designat ipsam λ respectu coordinatae ξ puncti cuiusque massae m differentiandam esse et post differentiationem hanc peractam quotientes differentiales omnes addendos esse. Quum vero differentiationes hae ab integrationibus, quas λ postulat, independentes sint, $\Sigma\left(\frac{d\lambda}{d\xi}\right)$ etiam obtinetur differentiando ipsam λ respectu coordinatae ξ puncti cuiusvis corporis m sive post integrationes peractas sive sub signo integrationis. Hinc concludere licet, loco $\Sigma\left(\frac{d\lambda}{d\xi}\right)$ in prima aequatione (7) simpliciter $\left(\frac{d\lambda}{d\xi}\right)$ poni posse, dummodo quotiens differentialis hic ex aequatione (8) computetur. Denique facile intelligitur terminum $\Sigma\,\frac{d^2\xi}{dt^2}\,m$ primae aequationis (7) in integrale $\int\frac{d^2\xi}{dt^2}\,dm$ per totam massam m extendendum abire. Quum aequationes reliquae (7) transmutationes analogas in casu quem consideramus patiantur, habemus loco earum

$$\left.\begin{array}{l}
\int\frac{d^2\xi}{dt^2}\,dm = \left(\frac{d\lambda}{d\xi}\right),\ \int\frac{d^2\eta}{dt^2}\,dm = \left(\frac{d\lambda}{d\eta}\right),\ \int\frac{d^2\zeta}{dt^2}\,dm = \left(\frac{d\lambda}{d\zeta}\right) \\
\int\frac{d^2\xi'}{dt^2}\,dm' = \left(\frac{d\lambda}{d\xi'}\right),\ \int\frac{d^2\eta'}{dt^2}\,dm' = \left(\frac{d\lambda}{d\eta'}\right),\ \int\frac{d^2\zeta'}{dt^2}\,dm' = \left(\frac{d\lambda}{d\zeta'}\right) \\
\qquad\qquad \text{etc.} \qquad\qquad\qquad\qquad \text{etc.}
\end{array}\right\}\ldots\ldots(9)$$

in quibus non minus quam in aequatione (8) ξ, η, ζ, coordinatas puncti cuiusvis corporis m, ξ', η', ζ', coordinatas puncti cuiusvis corporis m', et sic porro, denotant, quae coordinatae omnes originem eandem arbitrariam habent.

4.

Aequationes (9) una cum aequatione (8) eam corporum m, m', m'', etc. motuum partem, quae pro omnibus cuiusque corporis punctis eadem est, sive motum absolutum translationis corporum singulorum determinant, et ex iisdem aequationibus (1) . . . (6) etc. ea motuum corporum m, m', etc. pars, quae pro diversis corporum punctis diversa est, sive motus rotationis corporum singulorum determinari potest. Quem in finem aggregata $\Sigma \frac{\xi d^2\eta - \eta d^2\xi}{dt^2} m$, etc. $\Sigma \frac{\xi' d^2\eta' - \eta' d^2\xi'}{dt^2} m'$, etc. etc. formanda sunt, quae vero, quum ad scopum nostrum hic persequendum non spectent, hoc loco non evolvam.

Sint x, y, z coordinatae centri gravitatis corporis m; x', y', z' coordinatae centri gravitatis corporis m', etc. ab eadem origine uti ξ, η, ζ, ξ', etc. numeratae et his resp. parallelae: tum proprietas nota centri gravitatis suppeditat aequationes has

$$\begin{array}{ccc} \int\xi dm = xm , & \int\eta dm = ym , & \int\zeta dm = zm , \\ \int\xi' dm' = x'm' , & \int\eta' dm' = y'm' , & \int\zeta' dm' = z'm' \\ \text{etc.} & & \text{etc.} \end{array}$$

ubi integrationes per massas totas extendendae sunt. Quibus aequationibus bis differentiatis, habetur

$$\begin{array}{ccc} \int\frac{d^2\xi}{dt^2} dm = \frac{d^2x}{dt^2} m , & \int\frac{d^2\eta}{dt^2} dm = \frac{d^2y}{dt^2} m , & \int\frac{d^2\zeta}{dt^2} dm = \frac{d^2z}{dt^2} m \\ \text{etc.} & & \text{etc.} \end{array}$$

unde aequationes (9) abeunt in has

$$(10)\ldots\ldots\left\{\begin{array}{ccc} \frac{d^2x}{dt^2} = \frac{1}{m}\left(\frac{d\lambda}{d\xi}\right), & \frac{d^2y}{dt^2} = \frac{1}{m}\left(\frac{d\lambda}{d\eta}\right), & \frac{d^2z}{dt^2} = \frac{1}{m}\left(\frac{d\lambda}{d\zeta}\right) \\ \frac{d^2x'}{dt^2} = \frac{1}{m'}\left(\frac{d\lambda}{d\xi'}\right), & \frac{d^2y'}{dt^2} = \frac{1}{m'}\left(\frac{d\lambda}{d\eta'}\right), & \frac{d^2z'}{dt^2} = \frac{1}{m'}\left(\frac{d\lambda}{d\zeta'}\right) \\ \frac{d^2x''}{dt^2} = \frac{1}{m''}\left(\frac{d\lambda}{d\xi''}\right), & \frac{d^2y''}{dt^2} = \frac{1}{m''}\left(\frac{d\lambda}{d\eta''}\right), & \frac{d^2z''}{dt^2} = \frac{1}{m''}\left(\frac{d\lambda}{d\zeta''}\right) \\ \text{etc.} & & \text{etc.} \end{array}\right.$$

quae igitur aequationes differentiales motus absoluti systematis corporum m, m', m'', etc. sunt, et quidem motus absolutos centri gravitatis cuiusque corporis determinant. Quum vero in astronomia motus relativi observentur, aequationes differentiales motus relativi investigandae sunt. Quem in finem subtrahantur singulae aequationes (10) primae lineae primum a

singulis aequationibus secundae lineae, deinde a singulis aequationibus tertiae lineae et sic porro, quo facto emergunt

$$\frac{d^2(x'-x)}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\xi'}\right)-\frac{1}{m}\left(\frac{d\lambda}{d\xi}\right),\ \frac{d^2(y'-y)}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\eta'}\right)-\frac{1}{m}\left(\frac{d\lambda}{d\eta}\right),\ \frac{d^2(z'-z)}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\zeta'}\right)-\frac{1}{m}\left(\frac{d\lambda}{d\zeta}\right)$$
$$\frac{d^2(x''-x)}{dt^2}=\frac{1}{m''}\left(\frac{d\lambda}{d\xi''}\right)-\frac{1}{m}\left(\frac{d\lambda}{d\xi}\right),\ \frac{d^2(y''-y)}{dt^2}=\frac{1}{m''}\left(\frac{d\lambda}{d\eta''}\right)-\frac{1}{m}\left(\frac{d\lambda}{d\eta}\right),\ \frac{d^2(z''-z)}{dt^2}=\frac{1}{m''}\left(\frac{d\lambda}{d\zeta''}\right)-\frac{1}{m}\left(\frac{d\lambda}{d\zeta}\right)$$
etc. etc.

quae motum relativum corporum m', m'', etc. respectu corporis m determinant. Ut vero aequationes concinniores et ad evolutiones accommodatiores nanciscamur, modo sequenti progrediemur. Sumo inter corpora m, m', m'', etc. corpus M adesse, cuius respectu motus relativi investigandi sint. Sint coordinatae orthogoniae centri gravitatis corporis M ad originem quandam indeterminatam sed fixam relatae, ceterum vero coordinatis x, y, z, x', y', z', etc. resp. parallelae, X, Y, Z, et coordinatae puncti cuiusvis huius corporis ad originem eam, ad quam x, y, z, etc. referuntur, relatae et his quoque resp. parallelae Ξ, Υ, Z_{I}: tum ponendo

$$\Lambda=\varkappa\Sigma\iint\frac{dM.dm}{\sqrt{(\xi-\Xi)^2+(\eta-\Upsilon)^2+(\zeta-Z)^2}} \qquad ......(11)$$

ubi integrationes ad massas integras et summatio ad corpora omnia m, m', m'', etc. extendendae sunt, habemus secundum praecedentia

$$\frac{d^2X}{dt^2}=\frac{1}{M}\left(\frac{d\Lambda}{d\Xi}\right),\ \frac{d^2Y}{dt^2}=\frac{1}{M}\left(\frac{d\Lambda}{d\Upsilon}\right),\ \frac{d^2Z}{dt^2}=\frac{1}{M}\left(\frac{d\Lambda}{dZ_{\mathrm{I}}}\right) \qquad ......(12)$$

quae aequationes motum absolutum centri gravitatis corporis M determinant. Quum origo coordinatarum x, y, z, x', etc. ξ, η, ζ, ξ', etc. et Ξ, Υ, Z_{I} arbitraria sit, suppono originem hanc centrum gravitatis corporis M esse; hinc concluditur coordinatas absolutas centri gravitatis corporum m, m', etc. esse $X+x$, $Y+y$, $Z+z$, $X+x'$, $Y+y'$, $Z+z'$, etc., coordinatas absolutas puncti cuiusvis horum corporum esse $X+\xi$, $Y+\eta$, $Z+\zeta$, $X+\xi'$, $Y+\eta'$, $Z+\zeta'$, etc. et coordinatas absolutas puncti cuiusvis corporis M esse $X+\Xi$, $Y+\Upsilon$, $Z+Z_{\mathrm{I}}$. Aequationes igitur (8) et (11) λ et Λ suppeditantes non mutantur, aequationes vero (10) abeunt in has

$$\frac{d^2(X+x)}{dt^2}=\frac{1}{m}\left(\frac{d\lambda}{d\xi}\right)+\frac{1}{m}\left(\frac{d\Lambda}{d\xi}\right),\ \frac{d^2(Y+y)}{dt^2}=\frac{1}{m}\left(\frac{d\lambda}{d\eta}\right)+\frac{1}{m}\left(\frac{d\Lambda}{d\eta}\right),\ \frac{d^2(Z+z)}{dt^2}=\frac{1}{m}\left(\frac{d\lambda}{d\zeta}\right)+\frac{1}{m}\left(\frac{d\Lambda}{d\zeta}\right)$$
$$\frac{d^2(X+x')}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\xi'}\right)+\frac{1}{m'}\left(\frac{d\Lambda}{d\xi'}\right),\ \frac{d^2(Y+y')}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\eta'}\right)+\frac{1}{m'}\left(\frac{d\Lambda}{d\eta'}\right),\ \frac{d^2(Z+z')}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\zeta'}\right)+\frac{1}{m'}\left(\frac{d\Lambda}{d\zeta'}\right)$$
etc. etc.

Subductis aequationibus (12) a praecedentibus nanciscimur

$$(13)\begin{cases}\frac{d^2x}{dt^2}=\frac{1}{m}\left(\frac{d\lambda}{d\xi}\right)+\frac{1}{m}\left(\frac{d\Lambda}{d\xi}\right)-\frac{1}{M}\left(\frac{d\Lambda}{d\Xi}\right),\ \frac{d^2y}{dt^2}=\frac{1}{m}\left(\frac{d\lambda}{d\eta}\right)+\frac{1}{m}\left(\frac{d\Lambda}{d\eta}\right)-\frac{1}{M}\left(\frac{d\Lambda}{d\Upsilon}\right),\ \frac{d^2z}{dt^2}=\frac{1}{m}\left(\frac{d\lambda}{d\zeta}\right)+\frac{1}{m}\left(\frac{d\Lambda}{d\zeta}\right)-\frac{1}{M}\left(\frac{d\Lambda}{dZ_1}\right)\\ \frac{d^2x'}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\xi'}\right)+\frac{1}{m'}\left(\frac{d\Lambda}{d\xi'}\right)-\frac{1}{M}\left(\frac{d\Lambda}{d\Xi}\right),\ \frac{d^2y'}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\eta'}\right)+\frac{1}{m'}\left(\frac{d\Lambda}{d\eta'}\right)-\frac{1}{M}\left(\frac{d\Lambda}{d\Upsilon}\right),\ \frac{d^2z'}{dt^2}=\frac{1}{m'}\left(\frac{d\lambda}{d\zeta'}\right)+\frac{1}{m'}\left(\frac{d\Lambda}{d\zeta'}\right)-\frac{1}{M}\left(\frac{d\Lambda}{dZ_1}\right)\\ \qquad\qquad\text{etc.}\qquad\qquad\qquad\qquad\text{etc.}\end{cases}$$

quae aequationes differentiales motuum relativorum corporum m, m', etc. respectu corporis M sunt, et motum relativum centrorum gravitatis horum corporum determinant, quomodocunque dimensiones et figurae corporum omnium sunt, dummodo distantiae particularum unius corporis a particulis reliquorum corporum infinite parvae non evadant.

5.

Aequationes praecedentes ad formam simpliciorem redigere licet. Ex indole quantitatis Λ per aequationem (11) datae facile reperitur esse

$$\left(\frac{d\Lambda}{d\xi}\right)=-\varkappa\iint\frac{(\xi-\Xi)\,dM.dm}{[(\xi-\Xi)^2+(\eta-\Upsilon)^2-(\zeta-Z_1)^2]^{\frac{3}{2}}}$$

$$\left(\frac{d\Lambda}{d\Xi}\right)=\varkappa\Sigma\iint\frac{(\xi-\Xi)\,dM.dm}{[(\xi-\Xi)^2+(\eta-\Upsilon)^2-(\zeta-Z_1)^2]^{\frac{3}{2}}}$$

si igitur ponitur

$$\Gamma=-\varkappa\Sigma'\iint\frac{(\xi-\Xi)(\xi'-\Xi)+(\eta-\Upsilon)(\eta'-\Upsilon)+(\zeta-Z_1)(\zeta'-Z_1)}{[(\xi'-\Xi)^2+(\eta'-\Upsilon)^2+(\zeta'-Z_1)^2]^{\frac{3}{2}}}\,dM.dm'$$

ubi summatio per Σ' indicata ita intelligenda est, ut coordinatae ξ', η', ζ', nec non massa dm' ad omnia corpora m', m'', etc., excepto corpore m, deinceps referantur, coordinatae vero ξ, η, ζ in terminis omnibus immutatae remaneant, habemus

$$\left(\frac{d\Lambda}{d\Xi}\right)+\left(\frac{d\Lambda}{d\xi}\right)=-\left(\frac{d\Gamma}{d\xi}\right),\left(\frac{d\Lambda}{d\Upsilon}\right)+\left(\frac{d\Lambda}{d\eta}\right)=-\left(\frac{d\Gamma}{d\eta}\right),\left(\frac{d\Lambda}{dZ_1}\right)+\left(\frac{d\Lambda}{d\zeta}\right)=-\left(\frac{d\Gamma}{d\zeta}\right)$$

itaque ponendo

$$W=\varkappa\frac{M+m}{Mm}\iint\frac{dM.dm}{\sqrt{(\xi-\Xi)^2+(\eta-\Upsilon)^2+(\zeta-Z_1)^2}}+\frac{\varkappa}{m}\Sigma\iint\frac{dm.dm'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}$$
$$-\frac{\varkappa}{M}\Sigma'\iint\frac{(\xi-\Xi)(\xi'-\Xi)+(\eta-\Upsilon)(\eta'-\Upsilon)+(\zeta-Z_1)(\zeta'-Z_1)}{[(\xi'-\Xi)^2+(\eta'-\Upsilon)^2+(\zeta'-Z_1)]^{\frac{3}{2}}}\,dM.dm$$

habetur

$$\frac{d^2x}{dt^2} = \left(\frac{dW}{d\xi}\right),\ \frac{d^2y}{dt^2} = \left(\frac{dW}{d\eta}\right),\ \frac{d^2z}{dt^2} = \left(\frac{dW}{d\zeta}\right)$$

quae motum relativum centri gravitatis corporis m determinant. Ponendo insuper

$$W' = \varkappa\,\frac{M+m'}{Mm'}\iint\frac{dM\,.\,dm'}{\sqrt{(\xi'-\Xi)^2+(\eta'-\Upsilon)^2+(\zeta'-Z_1)^2}} + \frac{\varkappa}{m'}\Sigma\iint\frac{dm\,.\,dm'}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}$$
$$-\frac{\varkappa}{M}\Sigma''\iint\frac{(\xi'-\Xi)(\xi''-\Xi)+(\eta'-\Upsilon)(\eta''-\Upsilon)+(\zeta'-Z_1)(\zeta''-Z_1)}{[(\xi''-\Xi)^2+(\eta''-\Upsilon)^2+(\zeta''-Z_1)^2]^{\frac{3}{2}}}\,dM.dm''$$

ubi Σ'' indicat coordinatas et massam corporis m'' deinceps in coordinatas massasque corporum omnium m, m'', m''', etc., excepto corpore m', mutandas esse, coordinatas vero et massam corporis m' in terminis omnibus immutatas remanere debere, habetur

$$\frac{d^2x'}{dt^2} = \left(\frac{dW'}{d\xi'}\right),\ \frac{d^2y'}{dt^2} = \left(\frac{dW'}{d\eta'}\right),\ \frac{d^2z'}{dt^2} = \left(\frac{dW'}{d\zeta'}\right)$$

quae motum relativum centri gravitatis corporis m' definiunt, et similes aequationes nanciscimur pro corporibus omnibus reliquis.

6.

Quum systematis nostri Solaris corporum dimensiones respectu distantiarum eorum reciprocarum minutissimae sint, quantitates W, W', etc. commodissime in partes duas dividuntur, quarum prior a figura corporum independens est, altera vero e parametris superficierum corporum pendet, quae altera pars illa semper multo minor est. Sint ξ_1, η_1, ζ_1 coordinatae puncti cuiusvis corporis m ad centrum gravitatis huius corporis relatae, ξ'_1, η'_1, ζ'_1 coordinatae puncti cuiusvis corporis m' ad centrum gravitatis huius corporis relatae, et sic porro. Hinc sequitur ut

$$\xi = x + \xi_1\ ,\ \eta = y + \eta_1\ ,\ \zeta = z + \zeta_1$$
$$\xi' = x' + \xi'_1\ ,\ \eta' = y' + \eta'_1\ ,\ \zeta' = z' + \zeta'_1$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

Substitutis his valoribus in expressionibus ipsarum W, W', etc. art. praec., quantitates quae sub signis integrationis continentur, in series infinitas maxime convergentes et secundum potestates productaque quantitatum $(\xi_1-\xi'_1)$, $(\eta_1-\eta'_1)$, $(\zeta_1-\zeta'_1)$ etc. et resp. quantitatum $(\xi_1-\Xi)$, $(\eta_1-\Upsilon)$, (ζ_1-Z), etc. progredientes, evolvi possunt, quarum serierum primi ex his

quantitatibus independentes termini ex x, y, z, x', y', z', etc. eodem modo compositi erunt, ut expressiones illae ex ξ, η, ζ, ξ', η', ζ', etc. constant. Habemus igitur

$$\frac{1}{\sqrt{(\xi-\Xi)^2+(\eta-\Upsilon)^2+(\zeta-Z_I)^2}}=\frac{1}{r}+V$$

$$\frac{(\xi-\Xi)(\xi'-\Xi)+(\eta-\Upsilon)(\eta'-\Upsilon)+(\zeta-Z_I)(\zeta'-Z_I)}{[(\xi'-\Xi)^2+(\eta'-\Upsilon)^2+(\zeta'-Z_I)^2]^{\frac{3}{2}}}=\frac{xx'+yy'+zz'}{r'^3}+U$$

$$\frac{1}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}}=\frac{1}{\Delta'}+v$$

ubi V, U et v series infinitas repraesentant secundum potestates productaque ipsarum $(\xi_I-\xi'_I)$, $(\xi_I-\Xi)$, etc. etc. progredientes, et ubi brevitatis causa posui

$$\Delta'^2 = (x-x')^2 + (y-y') + (z-z')^2$$
$$r^2 = x^2 + y^2 + z^2$$
$$r'^2 = x'^2 + y'^2 + z'^2$$

ita ut generaliter sit $\Delta^{(\mu)}_{(\nu)}$ sive $\Delta^{(\nu)}_{(\mu)}$ distantia reciproca corporum $m^{(\mu)}$ et $m^{(\nu)}$, nec non $r^{(\mu)}$ distantia corporis $m^{(\mu)}$ a corpore M. Formulae igitur art. praec. ipsas W, W', etc. suppeditantes abeunt in has

$$W=\varkappa\frac{M+m}{r}+\varkappa\Sigma\frac{m'}{\Delta}-\varkappa\Sigma' m'\frac{xx'+yy'+zz'}{r'^3}+\varkappa\frac{M+m}{Mm}\iint V dM.dm$$
$$+\frac{\varkappa}{m}\Sigma\iint v dm.dm'-\frac{\varkappa}{M}\Sigma'\iint U dM.dm'$$

$$W'=\varkappa\frac{M+m'}{r'}+\varkappa\Sigma\frac{m''}{\Delta'_{\prime}}-\varkappa\Sigma' m''\frac{x'x''+y'y''+z'z''}{r''^3}+\varkappa\frac{M+m'}{Mm'}\iint V' dM.dm'$$
$$+\frac{\varkappa}{m'}\Sigma\iint v dm.dm'-\frac{\varkappa}{M}\Sigma''\iint U' dM.dm''$$

W'' = etc.

et quum nunc quotientes differentiales ipsarum W, W', etc. respectu ipsarum ξ, η, ζ, ξ', η', ζ', etc. resp. in quotientes differentiales respectu ipsarum x, y, z, x', y', z', etc. transmutare nobis liceat, habemus loco aequationum respectivarum art. praec. has

$$(14)\ldots\left\{\begin{array}{lll}\frac{d^2x}{dt^2}=\left(\frac{dW}{dx}\right), & \frac{d^2y}{dt^2}=\left(\frac{dW}{dy}\right), & \frac{d^2z}{dt^2}=\left(\frac{dW}{dz}\right)\\ \frac{d^2x'}{dt^2}=\left(\frac{dW'}{dx'}\right), & \frac{d^2y'}{dt^2}=\left(\frac{dW'}{dy'}\right), & \frac{d^2z'}{dt^2}=\left(\frac{dW'}{dz'}\right)\\ \text{etc.} & & \text{etc.}\end{array}\right.$$

quae aequationes differentiales motui relativo corporum m, m', etc. investigando inserviunt.

7.

Restat nobis ut quantitates V, U et v definiantur. Si in quantitate $[(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2]^{-\frac{1}{2}}$ substituuntur valores $x+\xi_1$, $y+\eta_1$, $z+\zeta_1$, etc. ipsarum ξ, η, ζ, etc. et si tum quantitas haec per methodos notas in seriem evolvitur, facili opera nanciscimur usque ad quantitates tertii ordinis,

$$v=-\frac{(x-x')(\xi_1-\xi'_1)+(y-y')(\eta_1-\eta'_1)+(z-z')(\zeta_1-\zeta'_1)}{\Delta^3}$$
$$+\frac{3}{2}\frac{[(x-x')(\xi_1-\xi'_1)+(y-y')(\eta_1-\eta'_1)+(z-z')(\zeta_1-\zeta'_1)]^2}{\Delta^5}-\frac{1}{2}\frac{(\xi_1-\xi'_1)^2+(\eta_1-\eta'_1)^2+(\zeta_1-\zeta'_1)^2}{\Delta^3}+\text{etc.}$$

et eodem modo obtinentur

$$V=-\frac{x(\xi_1-\Xi)+y(\eta_1-\Upsilon)+z(\zeta_1-Z_1)}{r^3}$$
$$+\frac{3}{2}\frac{[x(\xi_1-\Xi)+y(\eta_1-\Upsilon)+z(\zeta_1-Z_1)]^2}{r^5}-\frac{1}{2}\frac{(\xi_1-\Xi)^2+(\eta_1-\Upsilon)^2+(\zeta_1-Z_1)^2}{r^3}+\text{etc.}$$

$$U=-3\frac{xx'+yy'+zz'}{r'^5}[x'(\xi'_1-\Xi)+y'(\eta'_1-\Upsilon)+z'(\zeta'_1-Z_1)]+\frac{x(\xi'_1-\Xi)+x'(\xi_1-\Xi)+y(\eta'_1-\Upsilon)+y'(\eta_1-\Upsilon)+z(\zeta'_1-Z_1)+z'(\zeta_1-Z_1)}{r'^3}$$
$$+\frac{15}{2}\frac{xx'+yy'+zz'}{r'^7}[x'(\xi'_1-\Xi)+y'(\eta'_1-\Upsilon)+z'(\zeta'_1-Z_1)]^2-\frac{3}{2}\frac{xx'+yy'+zz'}{r'^5}[(\xi'_1-\Xi)^2+(\eta'_1-\Upsilon)^2+(\zeta'_1-Z_1)^2]$$
$$-3\frac{x'(\xi'_1-\Xi)+y'(\eta'_1-\Upsilon)+z'(\zeta'_1-Z_1)}{r'^5}[x(\xi'_1-\Xi)+x'(\xi_1-\Xi)+y(\eta'_1-\Upsilon)+y'(\eta_1-\Upsilon)+z(\zeta'_1-Z_1)+z'(\zeta_1-Z_1)]$$
$$+\frac{(\xi_1-\Xi)(\xi'_1-\Xi)+(\eta_1-\Upsilon)(\eta'_1-\Upsilon)+(\zeta_1-Z_1)(\zeta'_1-Z_1)}{r'^3}+\text{etc.}$$

Quum quantitates hae per elementa massarum multiplicandae et tum respectu massarum integrarum integrandae sint, animadverto proprietatem notam centri gravitatis nobis suppeditare aequationes has

$$\begin{array}{lll}\int \Xi\, dM = o & \int \Upsilon\, dM = o & \int Z_1\, dM = o \\ \int \xi_1\, dm = o & \int \eta_1\, dm = o & \int \zeta_1\, dm = o \\ \int \xi'_1\, dm' = o & \int \eta'_1\, dm' = o & \int \zeta'_1\, dm' = o \\ & \text{etc.} & \text{etc.}\end{array}$$

et quum integrationes duplices requisitae a se invicem independentes sint, haberi quoque pro binis corporibus

$$\iint \xi_1 \xi'_1 \, dm\, dm' = o, \quad \iint \xi_1 \eta'_1 \, dm\, dm' = o, \text{ etc.}$$
$$\iint \eta_1 \xi'_1 \, dm\, dm' = o, \quad \text{etc. etc.}$$
$$\iint \Xi \xi \, dM\, dm = o, \quad \text{etc. etc.}$$

Porro quum quantitates W, W', etc. ab origine directioneque coordinatarum independentes sint, directiones has ad lubitum, et in diversis ipsarum W, W', etc. terminis diversas accipere nobis licet. Electis vero axibus coordinatarum axibus principalibus corporis unius e. g. corporis M parallelis, habemus etiam

$$\int \Xi \Upsilon dM = o\ , \quad \int \Xi Z_1 dM = o\ , \quad \int \Upsilon Z_1 dM = o$$

Quibus aequationibus omnibus adiuvantibus, expressiones praecedentes ipsarum V, v et U, postquam per elementa debita massarum multiplicatae et integratae sunt, subministrant

$$\iint v\, dm\, dm' = \frac{3m'}{2\Delta'^5}\int\left\{(x-x')^2\xi_1{}^2+(y-y')^2\eta_1{}^2+(z-z')^2\zeta_1{}^2\right\}dm - \frac{m'}{2\Delta'^3}\int\left\{\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2\right\}dm$$
$$+\frac{3m}{2\Delta'^5}\int\left\{(x-x')\,\xi'_1+(y-y')\,\eta'_1+(z-z')\,\zeta'_1\right\}^2 dm' - \frac{m}{2\Delta'^3}\int\left\{\xi'_1{}^2+\eta'_1{}^2+\zeta'_1{}^2\right\}dm'$$

ubi ξ_1, η_1, ζ_1 axibus principalibus corporis m parallelas esse supposui, porro

$$\iint V\, dM\, dm = \frac{3m}{2r^5}\int\left\{x^2\,\Xi^2+y^2\,\Upsilon^2+z^2\,Z_1{}^2\right\}dM - \frac{m}{2r^3}\int\left\{\Xi^2+\Upsilon^2+Z_1{}^2\right\}dM$$
$$+\frac{3M}{2r^5}\int\left\{x\,\xi_1+y\,\eta_1+z\,\zeta_1\right\}^2 dm - \frac{M}{2r^3}\int\left\{\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2\right\}dm$$

$$\iint U\, dM\, dm' = \frac{15}{2}m'\frac{xx'+yy'+zz'}{r'^7}\int\left\{x'^2\,\Xi^2+y'^2\,\Upsilon^2+z'^2\,Z_1{}^2\right\}dM - \frac{3}{2}m'\frac{xx'+yy'+zz'}{r'^5}\int\left\{\Xi^2+\Upsilon^2+Z_1{}^2\right\}dM$$
$$-3\frac{m'}{r'^5}\int\left\{x'(x+x')\,\Xi^2+y'(y+y')\,\Upsilon^2+z'(z+z')\,Z_1{}^2\right\}dM + \frac{m'}{r'^3}\int\left\{\Xi^2+\Upsilon^2+Z_1{}^2\right\}dM$$
$$+\frac{15}{2}M\frac{xx'+yy'+zz'}{r'^7}\int\left\{x'\,\xi'_1+y'\,\eta'_1+z'\,\zeta'_1\right\}^2 dm' - \frac{3}{2}M\frac{xx'+yy'+zz'}{r'^5}\int\left\{\xi'_1{}^2+\eta'_1{}^2+\zeta'_1{}^2\right\}dm'$$
$$-3\frac{M}{r'^5}\int\left\{x'\,\xi'_1+y'\,\eta'_1+z'\,\zeta'_1\right\}\left\{x\,\xi'_1+y\,\eta'_1+z\,\zeta'_1\right\}dm'$$

ubi Ξ, Υ, Z_1 axibus principalibus corporis M parallelas supposui. Quae expressiones monstrant quantitates $\iint V dM dm$, $\iint v\, dm\, dm'$ et $\iint U dM dm'$ esse quantitates secundi ordinis respectu dimensionum corporum M, m, m', etc., itaque quantitates perparvulas, quia corpora respectu distantiarum eorum reciprocarum parvula sunt. Systema igitur corporum M, m, m', etc., si a paucis discesseris, movetur ac si massae in corporum centris gravitatis tantum continerentur. Integralia vero praecedentia, si ad corpora systematis nostri Solaris adhibentur, alia quoque ratione perparvula sunt.

Notum est theorema, secundum quod punctum quodcunque exterius a Sphaera ita attrahitur, ac si massa Sphaerae huius in centro eius gravitatis tantum contineretur. Itaque quum corporum systematis nostri Solaris superficies a figura Sphaerae paullulum tantum differant: hac sola ratione iam integralia praecedentia perparvula esse debent. Ceterum quum post integrationes peractas terminus $\frac{M+m}{Mm}\iint VdMdm$ per $M+m$, et termini $\frac{1}{m}\iint vdmdm'$ atque $\frac{1}{M}\iint UdMdm'$ per m' multiplicati sese praestent, et quum vis attractiva corporis primarii M vi attractiva corporum secundariorum seu perturbantium m, m', etc. semper multo maior sit, primus horum terminorum reliquis semper multo maior est.

8.

Expressiones ipsarum $\iint VdMdm$, $\iint vdmdm'$ et $\iint UdMdm'$ art. praec. reductionem adhuc admittunt. Consideremus terminum

$$(x'\xi'_1+y'\eta'_1+z'\zeta'_1)\ (x\xi'_1+y\eta'_1+z\zeta'_1)$$

ubi secundum praecedentia coordinatae omnes axibus principalibus corporis M resp. parallelae esse debent. Sint a, b, c coordinatae puncti cuiusvis corporis m' ad centrum eius gravitatis relatae et axibus eius principalibus resp. parallelae. Hinc habemus

$$\begin{aligned}\xi'_1 &= \alpha a+\beta b+\gamma c\\ \eta'_1 &= \alpha' a+\beta' b+\gamma' c\\ \zeta'_1 &= \alpha'' a+\beta'' b+\gamma'' c\end{aligned}$$

ubi quantitatum α, β, γ, α', etc. significatio ex theoria transformationis coordinatarum nota est. Substitutis his expressionibus ipsarum ξ'_1, η'_1, ζ'_1 in expressione praecedente, invenitur, postquam termini per ab, ac et bc multiplicati, qui in integralibus evanescent, deleti erunt,

$$\begin{aligned}(x'\xi'_1+y'\eta'_1+z'\zeta'_1)\ (x\xi'_1+y\eta'_1+z\zeta'_1) = &\ (\alpha x+\alpha' y+\alpha'' z)(\alpha x'+\alpha' y'+\alpha'' z')\,a^2\\ &+(\beta x+\beta' y+\beta'' z)(\beta x'+\beta' y'+\beta'' z')\,b^2\\ &+(\gamma x+\gamma' y+\gamma'' z)(\gamma x'+\gamma' y'+\gamma'' z')\,c^2\end{aligned}$$

Ex theoria vero transformationis coordinatarum sequitur, $(\alpha x+\alpha' y+\alpha'' z)$ esse expressionem transformatae coordinatae x in aliam axi principali ad quem a spectat parallelam; $(\beta x+\beta' y+\beta'' z)$ esse expressionem transformatae coordinatae y in aliam axi principali ad quem b spe-

ctat parallelam, et $(\gamma x + \gamma' y + \gamma'' z)$ esse expressionem transformatae coordinatae z in aliam axi principali ad quem c spectat parallelam; nec non quantitates $(\alpha x' + \alpha' y' + \alpha'' z')$, $(\beta x' + \beta' y' + \beta'' z')$ atque $(\gamma x' + \gamma' y' + \gamma'' z')$ respectu ipsarum x', y', z' eandem significationem habere. Hinc concluditur coordinatas omnes quae in expressione

$$(x' \xi'_1 + y' \eta'_1 + z' \zeta'_1)\ (x \xi'_1 + y \eta'_1 + z \zeta'_1)$$

continentur statim ad axes principales corporis m' referri posse, unde expressio haec, quatenus in integrali nostro vim habet, abit in

$$x x' \xi'^2_1 + y y' \eta'^2_1 + z z' \zeta'^2_1$$

Quum expressio $(x \xi_1 + y \eta_1 + z \zeta_1)^2$ sit casus specialis expressionis praecedentis, statim concludere licet loco eius expressionem hanc

$$x^2 \xi_1^2 + y^2 \eta_1^2 + z^2 \zeta_1^2$$

substitui posse, ubi coordinatae omnes axibus principalibus corporis m parallelae sunt, et eodem modo demonstratur loco quantitatis

$$[(x - x') \xi'_1 + (y - y') \eta'_1 + (z - z') \zeta'_1]^2$$

quantitatem

$$(x - x')^2 \xi'^2_1 + (y - y')^2 \eta'^2_1 + (z - z')^2 \zeta'^2_1$$

substitui posse, ubi coordinatae omnes axibus principalibus corporis m' parallelae esse debent. Hinc et quum quantitas $\xi'^2_1 + \eta'^2_1 + \zeta'^2_1$ semper eundem valorem habeat, quaecunque est directio coordinatarum orthogoniarum, sequitur

$$\begin{aligned}
\iint V\, dM\, dm = {} & \frac{3m}{2r^5} \int \left\{ x^2 \Xi^2 + y^2 \Upsilon^2 + z^2 Z_1^2 \right\} dM - \frac{m}{2r^3} \int \left\{ \Xi^2 + \Upsilon^2 + Z_1^2 \right\} dM \\
& + \frac{3M}{2r^5} \int \left\{ x^2 \xi_1^2 + y^2 \eta_1^2 + z^2 \zeta_1^2 \right\} dm - \frac{M}{2r^3} \int \left\{ \xi_1^2 + \eta_1^2 + \zeta_1^2 \right\} dm \\
\iint v\, dm\, dm' = {} & \frac{3m'}{2\Delta^5} \int \left\{ (x-x')^2 \xi_1^2 + (y-y')^2 \eta_1^2 + (z-z')^2 \zeta_1^2 \right\} dm - \frac{m'}{2\Delta^3} \int \left\{ \xi_1^2 + \eta_1^2 + \zeta_1^2 \right\} dm \\
& + \frac{3m}{2\Delta^5} \int \left\{ (x-x')^2 \xi'^2_1 + (y-y')^2 \eta'^2_1 + (z-z')^2 \zeta'^2_1 \right\} dm' - \frac{m}{2\Delta^3} \int \left\{ \xi'^2_1 + \eta'^2_1 + \zeta'^2_1 \right\} dm' \\
\iint U\, dM\, dm' = {} & \frac{15}{2} m' \frac{xx' + yy' + zz'}{r'^7} \int \left\{ x'^2 \Xi^2 + y'^2 \Upsilon^2 + z'^2 Z'^2_1 \right\} dM - \frac{3}{2} m' \frac{xx' + yy' + zz'}{r'^5} \int \left\{ \Xi^2 + \Upsilon^2 + Z_1^2 \right\} dM \\
& - 3 \frac{m'}{r'^5} \int \left\{ x'(x + x') \Xi^2 + y'(y + y') \Upsilon^2 + z'(z + z') Z_1^2 \right\} dM + \frac{m'}{r'^3} \int \left\{ \Xi^2 + \Upsilon^2 + Z_1^2 \right\} dM \\
& + \frac{15}{2} M \frac{xx' + yy' + zz'}{r'^7} \int \left\{ x'^2 \xi'^2_1 + y'^2 \eta'^2_1 + z'^2 \zeta'^2_1 \right\} dm' - \frac{3}{2} M \frac{xx' + yy' + zz'}{r'^5} \int \left\{ \xi'^2_1 + \eta'^2_1 + \zeta'^2_1 \right\} dm' \\
& - 3 \frac{M}{r'^5} \int \left\{ xx' \xi'^2_1 + yy' \eta'^2_1 + zz' \zeta'^2_1 \right\} dm'
\end{aligned}$$

ubi, in terminis per dM multiplicatis coordinatae omnes axibus principalibus corporis M, in terminis per dm multiplicatis coordinatae omnes axibus principalibus corporis m, et in terminis per dm' multiplicatis coordinatae omnes axibus principalibus corporis m' parallelae esse debent.

9.

Statuamus corporum M, m et m' superficies quaslibet, eorum tamen dimensiones respectu distantiarum mutuarum parvulas esse, et consideremus integralis $\iint V\,dM\,dm$ primam partem hanc

$$\frac{3m}{2r^5}\int\left\{x^2\Xi^2+y^2\Upsilon^2+z^2Z_1^2\right\}dM-\frac{m}{2r^3}\int\left\{\Xi^2+\Upsilon^2+Z_1^2\right\}dM$$

Denotemus corporis M momenta inertiae respectu axium principalium Z_1, Υ, Ξ resp. per A, B, C, tum erit

$$\int\Xi^2dM=\tfrac{1}{2}(A+B-C);\ \int\Upsilon^2dM=\tfrac{1}{2}(C+A-B);\ \int Z_1^2dM=\tfrac{1}{2}(B+C-A)$$

et, positis $B=A+\Theta$ atque $C=A+\Lambda$, expressio praecedens abit in hanc

$$-\frac{3m}{2r^5}(x^2\Lambda+y^2\Theta)+\frac{m}{2r^3}(\Lambda+\Theta) \quad \text{......(15)}$$

quae respectu corporum figurae non nisi e differentiis Λ et Θ momentorum inertiae pendet; eodem modo reliqua art. praec. integralia non nisi e differentiis Λ et Θ, λ et θ atque λ' et θ' momentorum inertiae corporum M, m atque m' pendere, confirmatur. Quum vero evolutae longitudinis latitudinisque corporis m perturbationes, quas expressio (15) et eius similes ad reliqua art. praec. integralia pertinentes expressiones proferent, non minus quam istae expressiones functiones harum differentiarum futurae sint, manifestum est, has differentias ad minimum observationibus astronomicis a posteriori determinari posse, etiamsi corporum figurae et lex eorum densitatis ita sint, ut momenta inertiae a priori computari non possint.

Hinc sequitur, perturbationes e corporum figuris provenientes semper accurate determinari posse, qualiscunque sit eorum lex densitatis et superficierum figura, modo eorum dimensiones respectu eorum distantiarum mutuarum parvulae sint. Annotandum est pro omnibus superficiebus revolutionis fieri $\Lambda=\Theta$, etc., quare in hoc casu perturbationes pro quoque corpore unam tantum ex eius figura et densitate pendentem constantem continent. Denotantibus μ sinum an-

guli, quem radius vector cum plano principali Ξ, Y, et quem planum ipsarum Ξ et Z_1 cum plano ipsarum Z_1 et corporis centrum gravitatis transeunti facit, habemus

$$x = r\sqrt{1-\mu^2}\cos\omega,\quad y = r\sqrt{1-\mu^2}\sin\omega,$$

quibus valoribus expressio (15) abit in

$$\frac{3m}{4r^3}\{(\Theta+\Lambda)(\mu^2-1)+(\Theta-\Lambda)(1-\mu^2)\cos 2\omega\},$$

quae notae approximatae in „Mechanica coelesti" ab ill. Laplace inventae formulae similis est.

Haec de indole integralium nostrorum explicasse hoc loco sufficiat; evolutiones eorum ulteriores suscipiam, quando de perturbationibus Lunae a figura sphaeroidica Terrae ortis sermo erit.

10.

Ad aequationum (14) integrationes sublevandas, conducet methodus variationis constantium arbitrariarum, quare talem methodum hanc, qualis ad has aequationes adhibenda est, explicabo.

Suppone vim integram W, quae corpus m urget, in partes duas tali modo distributam esse, ut recepta priore tantum harum partium aequationes (14) tres primae lineae rigorose integrari possint, vim integram W', quae corpus m' urget, in partes duas distributam esse, ut recepta priore tantum harum partium aequationes (14) tres secundae lineae rigorose integrari possint, et sic porro. Integrationes hae $2n$ constantes arbitrarias in valoribus ipsarum x, y, z, x', y', z', etc. introducent, quoties numerus corporum m, m', etc. est n, et ad partes nimirum W, W' etc. in hoc calculo neglectas supponendis constantibus his arbitrariis variabilibus respicitur.

Quum formulae ad corpora m', m'', etc. spectantes illis ad corpus m pertinentibus plane similes evadere debeant, in sequentibus aequationes motus corporis m has

(16)...... $$\frac{d^2x}{dt^2} = \left(\frac{dW}{dx}\right),\quad \frac{d^2y}{dt^2} = \left(\frac{dW}{dy}\right),\quad \frac{d^2z}{dt^2} = \left(\frac{dW}{dz}\right)$$

solummodo considerabo. Distributa quantitate W in partes duas T et R, ita ut habeatur $W = T + R$, unde

(17)...... $$\frac{d^2x}{dt^2} = \left(\frac{dT}{dx}\right) + \left(\frac{dR}{dx}\right),\quad \frac{d^2y}{dt^2} = \left(\frac{dT}{dy}\right) + \left(\frac{dR}{dy}\right),\quad \frac{d^2z}{dt^2} = \left(\frac{dT}{dz}\right) + \left(\frac{dR}{dz}\right)$$

suppono integralia rigorosa aequationum

$$\frac{d^2x}{dt^2} = \left(\frac{dT}{dx}\right), \quad \frac{d^2y}{dt^2} = \left(\frac{dT}{dy}\right), \quad \frac{d^2z}{dt^2} = \left(\frac{dT}{dz}\right) \qquad \text{......(18)}$$

esse haec

$$x = X, \ y = Y, \ z = Z \qquad \text{......(19)}$$

ubi igitur X, Y, Z functiones notae constantium arbitrariarum sex

$$a, \ b, \ c, \ e, \ f, \ g$$

his integrationibus introductarum et temporis t sunt. In casu, quem proprie tractamus, X, Y, Z functiones finitae harum quantitatum sunt, conclusiones vero sequentes nullo modo laeduntur, si X, Y, Z series infinitae sint. Quae aequationes (19) bis differentiatae, dum tempus, quatenus explicite in iis continetur, variabile tantum habetur, aequationibus (18) satisfacere debent, iisdem vero bis differentiatis, dum constantes illae arbitrariae variabiles et quidem functiones temporis spectantur, aequationes nanciscimur, quibus aequationes (16) aequari licet. Quum vero hoc modo ad constantes sex variabiles factas determinandas non nisi aequationes tres nacti simus, tres alias conditiones ad lubitum assumere licet. Ut formulae, quibus constantes hae variabiles factae determinandae sunt, aequationes differentiales primi ordinis fiant, geometrae aequationes conditionales semper adhibuerunt has

$$\left.\begin{aligned} o &= \left(\frac{dx}{da}\right) da + \left(\frac{dx}{db}\right) db + \text{etc.} + \left(\frac{dx}{dg}\right) dg \\ o &= \left(\frac{dy}{da}\right) da + \left(\frac{dy}{db}\right) db + \text{etc.} + \left(\frac{dy}{dg}\right) dg \\ o &= \left(\frac{dz}{da}\right) da + \left(\frac{dz}{db}\right) db + \text{etc.} + \left(\frac{dz}{dg}\right) dg \end{aligned}\right\} \text{......(20)}$$

quibus praeterea efficitur, ut differentialia prima aequationum (19) eiusdem formae sint, sive constantes arbitrariae variabiles sive constantes spectantur. Aequationibus igitur (19) semel differentiatis, emergunt

$$\frac{dx}{dt} = \left(\frac{dX}{dt}\right); \ \frac{dy}{dt} = \left(\frac{dY}{dt}\right); \ \frac{dz}{dt} = \left(\frac{dZ}{dt}\right)$$

quae iterum differentiatae suppeditant

$$(21)\ldots\ldots\begin{cases}\dfrac{d^2x}{dt^2}=\left(\dfrac{d^2X}{dt^2}\right)+\left(\dfrac{dx_,}{da}\right)\dfrac{da}{dt}+\left(\dfrac{dx_,}{db}\right)\dfrac{db}{dt}+\text{etc.}+\left(\dfrac{dx_,}{dg}\right)\dfrac{dg}{dt}\\[2mm] \dfrac{d^2y}{dt^2}=\left(\dfrac{d^2Y}{dt^2}\right)+\left(\dfrac{dy_,}{da}\right)\dfrac{da}{dt}+\left(\dfrac{dy_,}{db}\right)\dfrac{db}{dt}+\text{etc.}+\left(\dfrac{dy_,}{dg}\right)\dfrac{dg}{dt}\\[2mm] \dfrac{d^2z}{dt^2}=\left(\dfrac{d^2Z}{dt^2}\right)+\left(\dfrac{dz_,}{da}\right)\dfrac{da}{dt}+\left(\dfrac{dz_,}{db}\right)\dfrac{db}{dt}+\text{etc.}+\left(\dfrac{dz_,}{dg}\right)\dfrac{dg}{dt}\end{cases}$$

ubi signorum $x_,$, $y_,$, $z_,$ introductorum haec est significatio

$$x_,=\frac{dx}{dt}=\left(\frac{dX}{dt}\right);\ y_,=\frac{dy}{dt}=\left(\frac{dY}{dt}\right);\ z_,=\frac{dz}{dt}=\left(\frac{dZ}{dt}\right)$$

Quum per hypothesin habeatur

$$\left(\frac{d^2X}{dt^2}\right)=\left(\frac{dT}{dx}\right);\ \left(\frac{d^2Y}{dt^2}\right)=\left(\frac{dT}{dy}\right);\ \left(\frac{d^2Z}{dt^2}\right)=\left(\frac{dT}{dz}\right)$$

aequationes praecedentes, substitutis valoribus ipsarum $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ ex (17) desumendis, abeunt in has

$$(22)\ldots\ldots\begin{cases}\left(\dfrac{dR}{dx}\right)=\left(\dfrac{dx_,}{da}\right)\dfrac{da}{dt}+\left(\dfrac{dx_,}{db}\right)\dfrac{db}{dt}+\text{etc.}+\left(\dfrac{dx_,}{dg}\right)\dfrac{dg}{dt}\\[2mm] \left(\dfrac{dR}{dy}\right)=\left(\dfrac{dy_,}{da}\right)\dfrac{da}{dt}+\left(\dfrac{dy_,}{db}\right)\dfrac{db}{dt}+\text{etc.}+\left(\dfrac{dy_,}{dg}\right)\dfrac{dg}{dt}\\[2mm] \left(\dfrac{dR}{dz}\right)=\left(\dfrac{dz_,}{da}\right)\dfrac{da}{dt}+\left(\dfrac{dz_,}{db}\right)\dfrac{db}{dt}+\text{etc.}+\left(\dfrac{dz_,}{dg}\right)\dfrac{dg}{dt}\end{cases}$$

quae cum aequationibus (20) iunctae problema solvunt.

11.

Ut aequationes simpliciores nanciscamur, habeatur R pro functione ipsarum a, b, c, etc. Itaque quum quantitas W in praecedentibus explicata quantitates $x_,$, $y_,$, $z_,$ non contineat, habemus

$$\left(\frac{dR}{da}\right)=\left(\frac{dR}{dx}\right)\left(\frac{dx}{da}\right)+\left(\frac{dR}{dy}\right)\left(\frac{dy}{da}\right)+\left(\frac{dR}{dz}\right)\left(\frac{dz}{da}\right)$$
$$\left(\frac{dR}{db}\right)=\left(\frac{dR}{dx}\right)\left(\frac{dx}{db}\right)+\left(\frac{dR}{dy}\right)\left(\frac{dy}{db}\right)+\left(\frac{dR}{dz}\right)\left(\frac{dz}{db}\right)$$
$$\text{etc.}\qquad\qquad\text{etc.}$$

Iam si aequationes (22) adduntur, postquam resp. per $\left(\frac{dx}{da}\right)$, $\left(\frac{dy}{da}\right)$, $\left(\frac{dz}{da}\right)$ multiplicatae erunt, obtinetur ope primae aequationum praecedentium

$$\left(\frac{dR}{da}\right) dt = \left\{\left(\frac{dx_,}{da}\right)\left(\frac{dx}{da}\right) + \left(\frac{dy_,}{da}\right)\left(\frac{dy}{da}\right) + \left(\frac{dz_,}{da}\right)\left(\frac{dz}{da}\right)\right\} da$$
$$+ \left\{\left(\frac{dx_,}{db}\right)\left(\frac{dx}{da}\right) + \left(\frac{dy_,}{db}\right)\left(\frac{dy}{da}\right) + \left(\frac{dz_,}{db}\right)\left(\frac{dz}{da}\right)\right\} db$$
$$+ \text{ etc.}$$
$$+ \left\{\left(\frac{dx_,}{dg}\right)\left(\frac{dx}{da}\right) + \left(\frac{dy_,}{dg}\right)\left(\frac{dy}{da}\right) + \left(\frac{dz_,}{dg}\right)\left(\frac{dz}{da}\right)\right\} dg$$

Multiplicata vero prima aequationum (20) per $\left(\frac{dx_,}{da}\right)$, secunda per $\left(\frac{dy_,}{da}\right)$ et tertia per $\left(\frac{dz_,}{da}\right)$, nanciscimur addendo

$$o = \left\{\left(\frac{dx}{da}\right)\left(\frac{dx_,}{da}\right) + \left(\frac{dy}{da}\right)\left(\frac{dy_,}{da}\right) + \left(\frac{dz}{da}\right)\left(\frac{dz_,}{da}\right)\right\} da$$
$$+ \left\{\left(\frac{dx}{db}\right)\left(\frac{dx_,}{da}\right) + \left(\frac{dy}{db}\right)\left(\frac{dy_,}{da}\right) + \left(\frac{dz}{db}\right)\left(\frac{dz_,}{da}\right)\right\} db$$
$$+ \text{ etc.}$$
$$+ \left\{\left(\frac{dx}{dg}\right)\left(\frac{dx_,}{da}\right) + \left(\frac{dy}{dg}\right)\left(\frac{dy_,}{da}\right) + \left(\frac{dz}{dg}\right)\left(\frac{dz_,}{da}\right)\right\} dg$$

Subtracta hac aequatione a praecedente, emergit

$$\left(\frac{dR}{da}\right) dt = (a, b)\, db + (a, c)\, dc + (a, e)\, de + (a, f)\, df + (a, g)\, dg$$

ubi da eliminata est, et ubi

$$\left.\begin{array}{l}
(a, b) = \left(\frac{dx}{da}\right)\left(\frac{dx_,}{db}\right) + \left(\frac{dy}{da}\right)\left(\frac{dy_,}{db}\right) + \left(\frac{dz}{da}\right)\left(\frac{dz_,}{db}\right) \\
\qquad - \left(\frac{dx_,}{da}\right)\left(\frac{dx}{db}\right) - \left(\frac{dy_,}{da}\right)\left(\frac{dy}{db}\right) - \left(\frac{dz_,}{da}\right)\left(\frac{dz}{db}\right) \\
(a, c) = \left(\frac{dx}{da}\right)\left(\frac{dx_,}{dc}\right) + \left(\frac{dy}{da}\right)\left(\frac{dy_,}{dc}\right) + \left(\frac{dz}{da}\right)\left(\frac{dz_,}{dc}\right) \\
\qquad - \left(\frac{dx_,}{da}\right)\left(\frac{dx}{dc}\right) - \left(\frac{dy_,}{da}\right)\left(\frac{dy}{dc}\right) - \left(\frac{dz_,}{da}\right)\left(\frac{dz}{dc}\right) \\
\qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.}
\end{array}\right\} \text{......(23)}$$

Eodem modo emergunt aequationes pro $\left(\frac{dR}{db}\right)$, $\left(\frac{dR}{dc}\right)$, etc. ubi coefficientes ipsarum da, db, etc. omnes mutatis mutandis ex aequationibus (23) petendi sunt, unde habetur

$$(b, a) = -(a, b)\,;\ (c, a) = -(a, c)\,;\ \text{etc.}$$

Quibus aequationibus congestis, habemus

$$(24)\ldots\ldots\left\{\begin{array}{l}
\left(\frac{dR}{da}\right) dt = (a, b)\, db + (a, c)\, dc + (a, e)\, de + (a, f)\, df + (a, g)\, dg \\
\left(\frac{dR}{db}\right) dt = -(a, b)\, da + (b, c)\, dc + (b, e)\, de + (b, f)\, df + (b, g)\, dg \\
\left(\frac{dR}{dc}\right) dt = -(a, c)\, da - (b, c)\, db + (c, e)\, de + (c, f)\, df + (c, g)\, dg \\
\left(\frac{dR}{de}\right) dt = -(a, e)\, da - (b, e)\, db - (c, e)\, dc + (e, f)\, df + (e, g)\, dg \\
\left(\frac{dR}{df}\right) dt = -(a, f)\, da - (b, f)\, db - (c, f)\, dc - (e, f)\, de + (f, g)\, dg \\
\left(\frac{dR}{dg}\right) dt = -(a, g)\, da - (b, g)\, db - (c, g)\, dc - (e, g)\, de - (f, g)\, df
\end{array}\right.$$

ubi coefficientes ipsarum da, db, etc. omnes mutatis mutandis ex aequationibus (23) petendi sunt. Quibus coefficientibus in casu quoque speciali computatis, differentialia singula da, db, etc. facili eliminatione per quotientes differentiales ipsius R exprimuntur, ita ut denique habeatur

$$(25)\ldots\ldots\left\{\begin{array}{l}
da = [a, b] \left(\frac{dR}{db}\right) dt + [a, c] \left(\frac{dR}{dc}\right) dt + \ldots + [a, g] \left(\frac{dR}{dg}\right) dt \\
db = -[a, b] \left(\frac{dR}{da}\right) dt + [b, c] \left(\frac{dR}{dc}\right) dt + \ldots + [b, g] \left(\frac{dR}{dg}\right) dt \\
\qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.} \\
dg = -[a, g] \left(\frac{dR}{da}\right) dt - [b, g] \left(\frac{dR}{db}\right) dt - \ldots - [f, g] \left(\frac{dR}{df}\right) dt
\end{array}\right.$$

designantibus $[a, b]$, $[a, c]$, etc. coefficientes hac eliminatione ortos.

12.

Coefficientes (a, b), (a, c), etc. nec non $[a, b]$, $[a, c]$, etc. proprietate hac insigni gaudent, quod tempus explicite non continent, quod theorema sequenti modo demonstrari potest. Differentiata prima aequatione (23) respectu temporis, habetur, quia $x_{\prime} = \frac{dx}{dt}$, etc.

$$\frac{d\,(a, b)}{dt} = \left(\frac{dx}{da}\right)\left(\frac{d^2x_{\prime}}{db\, dt}\right) + \left(\frac{dy}{da}\right)\left(\frac{d^2y_{\prime}}{db\, dt}\right) + \left(\frac{dz}{da}\right)\left(\frac{d^2z_{\prime}}{db\, dt}\right)$$
$$-\left(\frac{d^2x_{\prime}}{da\, dt}\right)\left(\frac{dx}{db}\right) - \left(\frac{d^2y_{\prime}}{da\, dt}\right)\left(\frac{dy}{db}\right) - \left(\frac{d^2z_{\prime}}{da\, dt}\right)\left(\frac{dz}{db}\right).$$

Aequationes vero (16) sunt

$$\frac{dx_{\prime}}{dt} = \left(\frac{dW}{dx}\right);\quad \frac{dy_{\prime}}{dt} = \left(\frac{dW}{dy}\right);\quad \frac{dz_{\prime}}{dt} = \left(\frac{dW}{dz}\right)$$

ubi notandum est quantitatem W ipsas $x_,$, $y_,$, $z_,$ non continere. Hinc differentiando inveniuntur

$$\frac{d^2 x_,}{da\,dt} = \left(\frac{d^2 W}{dx^2}\right)\left(\frac{dx}{da}\right) + \left(\frac{d^2 W}{dx\,dy}\right)\left(\frac{dy}{da}\right) + \left(\frac{d^2 W}{dx\,dz}\right)\left(\frac{dz}{da}\right)$$
$$\frac{d^2 x_,}{db\,dt} = \left(\frac{d^2 W}{dx^2}\right)\left(\frac{dx}{db}\right) + \left(\frac{d^2 W}{dx\,dy}\right)\left(\frac{dy}{db}\right) + \left(\frac{d^2 W}{dx\,dz}\right)\left(\frac{dz}{db}\right)$$

etc. etc.

Quodsi aequationes hae in valore praecedenti ipsius $\frac{d(a,b)}{dt}$ substituuntur, termini omnes evanescunt, itaque fit

$$\frac{d\,(a,\,b)}{dt} = o$$

et eodem modo demonstratur esse

$$\frac{d\,(a,\,c)}{dt} = o\,,\ \frac{d\,(b,\,c)}{dt} = o\,,\ \text{etc. etc.}$$

unde sequitur quantitates $(a,\,b)$, etc. tempus explicite non continere. Quum vero quantitates hae tempus non contineant, ipsae $[a,\,b]$, etc. etiam tempus continere non possunt.

13.

Quae sunt momenta praecipua methodi variationis constantium arbitrariarum, qualis ab ill. Lagrange expolita est. Quum problema nostrum, sicut problemata complura Mechanicae, in eo consistat, ut aut coordinatae x, y, z, x', y', z', etc. aut coordinatae corporum alio quodam modo in spatio sitae in functione temporis exhibeantur: primum integratione aequationum (18) coordinatae in functione temporis et quantitatum a, b, c, etc. eruuntur, et deinde integratione aequationum (25) quantitates a, b, c, etc. in functione temporis exhibentur, unde denique, ipsis a, b, c, etc. eliminatis, coordinatae in functione temporis explicita eliciuntur. Quae igitur methodus, praeterea quod in applicatione ad systema nostrum Solare incommodi quid, olim a me explicati, habet, per ambages ad solutionem problematis perducit. Nam coordinatae in functione explicita temporis exhibendae, revera in functione temporis et quantitatum a, b, etc., quae functiones implicitae temporis sunt, exhibentur. En vero huius methodi emen-

dationem, qua efficitur, ut coordinatae statim in functione explicita temporis determinentur, et qua simul incommoda illa amoveantur.

In praecedentibus supposuimus a, b, c, e, f, g esse constantes arbitrarias integratione aequationum (18) in expressionibus ipsarum x, y, z introductas. Sed perfacile intelligitur conclusiones praecedentes omnes, et inter has nominatim aequationes (24), locum quoque habere, si a, b, c, e, f, g functiones quaecunque essent constantium arbitrariarum, quas integratio illa revera introduxerat. Sit $\varDelta$ functio quaedam constantium a, b, etc., tum aequatione inter $\varDelta$ atque a, b, c, etc. respectu constantis unius e. g. respectu a resoluta, habetur

$$a = \text{funct.}\ (\varDelta,\ b,\ c,\ e,\ f,\ g)$$

qua aequatione constans a e functionibus X, Y, Z aequationum (19) eliminare licet, ita ut x, y, z functiones evadant constantium $\varDelta$, b, c, e, f, g. Quibus factis, methodus modo explicata aequationes suppeditat ipsis (23) et (24) plane similes, quibus vero ubique $\varDelta$ loco a inest. Quae conclusiones quum locum habeant, quomodocunque sunt constantes reliquae, quibus in functione

$$\varDelta = \text{funct.}\ (a,\ b,\ c,\ e,\ f,\ g)$$

quantitates a, b, c, etc. inter se coniunguntur; supponere nobis licet, huic functioni quantitatem quandam arbitrariam τ inesse, dummodo in formulis, in quibus a, b, c, etc. variabiles tractandae sunt, ipsa τ constans ponatur. Quantitas indeterminata haec τ, quia indeterminata est, post integrationes aequationum (25) peractas ad lubitum determinari potest; ponendo vero post integrationes has peractas $\tau = t$, novam methodi variationis constantium arbitrariarum applicationem assequemur, latioremque ei campum aperiemus. Quem in finem ante omnia hoc est demonstrandum, quod in nostra perturbationum theoria fundamentale dici potest

Theorema.

Quoties loco aliquot constantium arbitrariarum a, b, c, etc. eliguntur totidem constantes arbitrariae $\varDelta$, Γ, etc., quae ex constantibus illis et ex quantitate indeterminata τ tali modo compositae sunt, quali quantitates quaelibet L, G, etc. ex iisdem constantibus a, b, c, etc. et ex

tempore t constant: valores veri quantitatum L, G, etc. aequationibus (17) *respondentes, quos valores perturbatos appellabo, obtinentur, si in aequationibus* (25) *ad ipsas $\varDelta$, $\varGamma$, etc. applicatis post integrationes peractas τ in t mutata erit.*

Demonstratio.

Sint datae quantitates duae quaelibet L atque G, functiones temporis t et constantium arbitrariarum a, b, c, etc., ita ut habeatur

$$\left.\begin{aligned} L &= \Psi(t, a, b, c, e, f, g) \\ G &= \Pi(t, a, b, c, e, f, g) \end{aligned}\right\} \text{......(26)}$$

denotantibus Ψ atque Π functiones quasdam. Assumantur

$$\left.\begin{aligned} \varDelta &= \Psi(\tau, a, b, c, e, f, g) \\ \varGamma &= \Pi(\tau, a, b, c, e, f, g) \end{aligned}\right\} \text{......(27)}$$

Ex his expressionibus valores duarum constantium arbitrariarum, e. g. a et b elici possunt, quibus in functionibus X, Y, Z aequationum (19) a et b eliminare licet. Quo facto habetur

$$x = \chi(t, \tau, \varDelta, \varGamma, c, e, f, g)$$

denotante χ functionem quandam, et similes expressiones nanciscimur pro y et z, unde $x_{,}$, $y_{,}$, $z_{,}$ quoque functiones earundem quantitatum redditae sunt. Itaque ponendo $\varDelta$ loco a et $\varGamma$ loco b, pervenitur ope valorum praecedentium ipsarum x, y, etc. ad aequationes (25) pro $d\varDelta$, $d\varGamma$, dc, etc. quae in hoc casu indeterminatam τ quoque continent. Quae aequationes integratae valores perturbatos ipsarum $\varDelta$, $\varGamma$, c, e, f, g suppeditant. Quibus absolutis, valores ipsarum a et b ex (27) eliciti et in (26) substituti suppeditant

$$\left.\begin{aligned} L &= \varphi(t, \tau, \varDelta, \varGamma, c, e, f, g) \\ G &= F(t, \tau, \varDelta, \varGamma, c, e, f, g) \end{aligned}\right\} \text{......(28)}$$

ubi φ et F functiones quasdam designant. Quantitates L et G hoc modo exhibitae, si valores perturbati ipsarum $\varDelta$, $\varGamma$, c, etc. substituti erunt, functiones explicitae temporis redduntur, quae praeterea quantitatem indeterminatam τ duplici modo continent. Nam τ functionibus φ et F ante

valores perturbatos ipsarum Λ, Γ, c, etc. substitutos inest, et substitutione horum valorum denuo reproducitur. Iidem valores perturbati ipsarum L et G obtinentur, substituendo valores perturbatos ipsarum a, b, c, etc. in aequationibus (26), quae in hoc casu ipsam τ non continent. Quum vero valores hi ipsarum L et G cum illis congruere debeant, concluditur in aequationibus (28) post substitutiones ipsarum Λ, Γ, c, etc. peractas quantitatem τ sua sponte evanescere debere.

Inquiramus in indolem harum functionum. Quem in finem valores perturbati ipsarum L et G ex (28) eliciti et respectu temporis differentiati suppeditant

$$\frac{dL}{dt} = \frac{d\varphi}{dt} + \left(\frac{d\varphi}{d\Lambda}\right)\frac{d\Lambda}{dt} + \left(\frac{d\varphi}{d\Gamma}\right)\frac{d\Gamma}{dt} + \left(\frac{d\varphi}{dc}\right)\frac{dc}{dt} + \text{etc.}$$
$$\frac{dG}{dt} = \frac{dF}{dt} + \left(\frac{dF}{d\Lambda}\right)\frac{d\Lambda}{dt} + \left(\frac{dF}{d\Gamma}\right)\frac{d\Gamma}{dt} + \left(\frac{dF}{dc}\right)\frac{dc}{dt} + \text{etc.}$$

sed ex (27) emergunt

$$\frac{d\Lambda}{dt} = \left(\frac{d\psi}{da}\right)\frac{da}{dt} + \left(\frac{d\psi}{db}\right)\frac{db}{dt} + \left(\frac{d\psi}{dc}\right)\frac{dc}{dt} + \text{etc.}$$
$$\frac{d\Gamma}{dt} = \left(\frac{d\Pi}{da}\right)\frac{da}{dt} + \left(\frac{d\Pi}{db}\right)\frac{db}{dt} + \left(\frac{d\Pi}{dc}\right)\frac{dc}{dt} + \text{etc.}$$

unde

$$\frac{dL}{dt} = \frac{d\varphi}{dt} + \left[\left(\frac{d\varphi}{d\Lambda}\right)\left(\frac{d\psi}{da}\right) + \left(\frac{d\varphi}{d\Gamma}\right)\left(\frac{d\Pi}{da}\right)\right]\frac{da}{dt} + \left[\left(\frac{d\varphi}{d\Lambda}\right)\left(\frac{d\psi}{db}\right) + \left(\frac{d\varphi}{d\Gamma}\right)\left(\frac{d\Pi}{db}\right)\right]\frac{db}{dt}$$
$$+ \left[\left(\frac{d\varphi}{d\Lambda}\right)\left(\frac{d\psi}{dc}\right) + \left(\frac{d\varphi}{d\Gamma}\right)\left(\frac{d\Pi}{dc}\right) + \left(\frac{d\varphi}{dc}\right)\right]\frac{dc}{dt} + \text{etc.}$$
$$\frac{dG}{dt} = \frac{dF}{dt} + \left[\left(\frac{dF}{d\Lambda}\right)\left(\frac{d\psi}{da}\right) + \left(\frac{dF}{d\Gamma}\right)\left(\frac{d\Pi}{da}\right)\right]\frac{da}{dt} + \left[\left(\frac{dF}{d\Lambda}\right)\left(\frac{d\psi}{db}\right) + \left(\frac{dF}{d\Gamma}\right)\left(\frac{d\Pi}{db}\right)\right]\frac{db}{dt}$$
$$+ \left[\left(\frac{dF}{d\Lambda}\right)\left(\frac{d\psi}{dc}\right) + \left(\frac{dF}{d\Gamma}\right)\left(\frac{d\Pi}{dc}\right) + \left(\frac{dF}{dc}\right)\right]\frac{dc}{dt} + \text{etc.}$$

Aequationes vero (26) suppeditant

$$\frac{dL}{dt} = \frac{d\psi}{dt} + \left(\frac{d\psi}{da}\right)\frac{da}{dt} + \left(\frac{d\psi}{db}\right)\frac{db}{dt} + \left(\frac{d\psi}{dc}\right)\frac{dc}{dt} + \text{etc.}$$
$$\frac{dG}{dt} = \frac{d\Pi}{dt} + \left(\frac{d\Pi}{da}\right)\frac{da}{dt} + \left(\frac{d\Pi}{db}\right)\frac{db}{dt} + \left(\frac{d\Pi}{dc}\right)\frac{dc}{dt} + \text{etc.}$$

quae cum illis resp. identicae esse debent. Itaque

$$\frac{d\varphi}{dt} = \frac{d\psi}{dt}\,;\quad \frac{dF}{dt} = \frac{d\Pi}{dt}$$

$$\left(\frac{d\varphi}{d\Lambda}\right)\left(\frac{d\psi}{da}\right)+\left(\frac{d\varphi}{d\Gamma}\right)\left(\frac{d\Pi}{da}\right)=\left(\frac{d\psi}{da}\right);\ \left(\frac{dF}{d\Lambda}\right)\left(\frac{d\psi}{da}\right)+\left(\frac{dF}{d\Gamma}\right)\left(\frac{d\Pi}{da}\right)=\left(\frac{d\Pi}{da}\right)$$

$$\left(\frac{d\varphi}{d\Lambda}\right)\left(\frac{d\psi}{db}\right)+\left(\frac{d\varphi}{d\Gamma}\right)\left(\frac{d\Pi}{db}\right)=\left(\frac{d\psi}{db}\right);\ \left(\frac{dF}{d\Lambda}\right)\left(\frac{d\psi}{db}\right)+\left(\frac{dF}{d\Gamma}\right)\left(\frac{d\Pi}{db}\right)=\left(\frac{d\Pi}{db}\right)$$

$$\left(\frac{d\varphi}{d\Lambda}\right)\left(\frac{d\psi}{dc}\right)+\left(\frac{d\varphi}{d\Gamma}\right)\left(\frac{d\Pi}{dc}\right)+\left(\frac{d\varphi}{dc}\right)=\left(\frac{d\psi}{dc}\right);\ \left(\frac{dF}{d\Lambda}\right)\left(\frac{d\psi}{dc}\right)+\left(\frac{dF}{d\Gamma}\right)\left(\frac{d\Pi}{dc}\right)+\left(\frac{dF}{dc}\right)=\left(\frac{d\Pi}{dc}\right)$$

etc. etc.

quarum ultimae sex suppeditant

$$\left(\frac{d\varphi}{d\Lambda}\right)=1;\ \left(\frac{d\varphi}{d\Gamma}\right)=o;\ \left(\frac{d\varphi}{dc}\right)=o;\ \text{etc.}\ \left(\frac{dF}{d\Lambda}\right)=o;\ \left(\frac{dF}{d\Gamma}\right)=1;\ \left(\frac{dF}{dc}\right)=o;\ \text{etc.}$$

itaque $\varphi = \Lambda$, cui prima aequatio praecedens subiungit conditionem, ut tempus, quatenus ex constantibus arbitrariis variabiles factis non provenit, in functionibus φ et ψ sive in φ et Λ eodem modo contineatur; porro $F = \Gamma$, cui secunda aequatio praecedens subiungit conditionem, ut tempus, quatenus ex constantibus arbitrariis variabiles factis non provenit, in functionibus F et Π sive in F et Γ eodem modo contineatur, quibus conditionibus satisfit, si post integrationes in functionibus Λ et Γ τ in t mutatur. Functionibus igitur φ et F opus non est, statim enim obtinentur valores perturbati ipsarum L atque G, si in aequationibus (25), quae valores perturbatos ipsarum Λ atque Γ suppeditant, post integrationes peractas τ in t mutatur; quod est theorema nostrum, quoties duae functiones quaedam veluti G et L dantur, et eodem modo theorema demonstratur, quoties plures functiones huius generis adsunt *).

14.

Theorema art. praec. quamvis generaliter valeat, quomodocunque functiones L, G, etc. compositae sunt, sensu tamen stricto ad earum compositionem modo referendum est, quod vero ad earum significationem pertinet, respectu aequationum (18) et respectu aequationum (16) discrimen essentiale interesse potest. Quae huius theorematis conditio sequentibus illustratur.

*) Aliam huius theorematis demonstrationem theoremate Tayloriano fundatam invenies in „Astron. Nachr." No. 258.

Integralia aequationum (16) vel (17) in praecedentibus ita determinavimus, ut eandem formam habeant, sive R est cifrae aequalis sive non. Itaque coordinatae x, y, z et proinde functio quaelibet ipsarum x, y, z proprietate hac gaudent, quod eandem formam habent, sive ad aequationes (17) sive ad (18) referuntur. Item, aequationes (20) probant, quotientes differentiales primi ordinis ipsarum x, y, z respectu temporis in utroque casu eiusdem formae quoque esse. Aequationes vero (21) demonstrant, quotientes differentiales secundi ordinis $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, nec non quotientes differentiales ordinum altiorum omnes in utroque casu formam eandem non habere. Ergo quoties L, G, etc. sunt aut ipsae functiones ipsarum x, y, z, $x_{,}$, $y_{,}$, $z_{,}$, aut ad functiones harum quantitatum reducuntur, non modo earum compositio, sed etiam earum significatio respectu aequationum (18) et respectu aequationum (16) eadem est; igitur integralia $\int d\Lambda$, $\int d\Gamma$, etc. post τ in t mutatam, non modo respectu eorum compositionis sed etiam respectu eorum significationis valores perturbatos ipsarum L, G, etc. praebebunt. Quoties vero functiones L, G, etc. aut praeter illas quantitates insuper $\frac{d^2x}{dt^2}$, etc. $\frac{d^3x}{dt^3}$ etc. etc. continent, aut functiones modo horum quotientum sunt, integralia illa modo quantum ad earum compositionem valores perturbatos ipsarum L, G, etc., hoc est valores perturbatos functionum in quas L, G, etc. transibunt, si valores ipsarum $\frac{d^2x}{dt^2}$, $\frac{d^3x}{dt^3}$ etc. etc. per constantes arbitrarias a, b, c, etc. et per tempus expressi substituti fuerint, subministrabunt, quod attinet ad earum significationem, haec respectu aequationum (17) ab earum significatione respectu aequationum (18) necessario differt.

Si L denotat functionem ipsarum x, y, z, absque $x_{,}$, $y_{,}$, $z_{,}$, differentialibusque ordinum altiorum, aequationes (20) statim suppeditant

$$o = \left(\frac{dL}{da}\right) da + \left(\frac{dL}{db}\right) db + \text{etc.} + \left(\frac{dL}{dg}\right) dg$$

Si igitur et valor perturbatus ipsius L et valor perturbatus ipsius Λ differentiatur, habetur statim

$$\frac{dL}{dt} = \overline{\left(\frac{d\Lambda}{d\tau}\right)}$$

designante linea superposita post differentiationem τ in t mutandam esse.

15.

En eorum quae in praecedentibus exposui applicationem. Electis constantibus arbitrariis tribus A, Γ, etc. quae ex constantibus arbitrariis a, b, c, etc. et ex quantitate indeterminata τ eodem modo compositae sunt, quo coordinatae tres quaelibet, per quas situs corporis m in spatio determinatur; electisque constantibus arbitrariis analogis pro corporibus m', m'', etc.; si valores perturbati harum constantium methodo in praecedentibus explicata computati fuerint, nanciscemur statim valores perturbatos coordinatarum mutando τ in t in valoribus perturbatis constantium arbitrariarum A, Γ, etc. Problema igitur hoc: datis viribus quae systema corporum urgent, invenire coordinatas horum corporum in functione temporis, ex praecedentibus solvimus.

16.

Praecepta praecedentia ad motum corporum coelestium investigandum applicaturi ponimus esse

$$T = \varkappa\,\frac{M+m}{r}$$

tum, ponendo $R = \varkappa(M+m)\,\Omega$, elicitur ex artt. 6. atque 10.

$$\Omega = \frac{1}{M+m}\,\Sigma\,\frac{m'}{\Delta} - \frac{1}{M+m}\,\Sigma'\,m'\,\frac{xx'+yy'+zz'}{r'^3} + \frac{1}{Mm}\iint V dM\,.\,dm$$
$$+ \frac{1}{m(M+m)}\,\Sigma\iint v\,dm\,.\,dm' - \frac{1}{M(M+m)}\,\Sigma'\iint U dM\,.\,dm'$$

ita ut T repraesentet vim primariam et Ω vires secundarias seu perturbatrices quae corpus m urgent. Nota sunt integralia aequationum (18) haec

$$\begin{aligned} x &= r\cos(v-\theta)\cos\theta - r\sin(v-\theta)\sin\theta\cos i \\ y &= r\cos(v-\theta)\sin\theta + r\sin(v-\theta)\cos\theta\cos i \\ z &= r\sin(v-\theta)\sin i \end{aligned}$$

ubi r radium vectorem, v longitudinem veram in orbita, i inclinationem orbitae plani ad planum fixum ipsarum xy, quod in spatio ad lubitum situm esse potest, et θ longitudinem nodi ascendentis orbitae plani cum plano ipsarum xy denotat. Quum expressiones plane analogas pro corporibus m', m'', etc. nanciscamur, ad has non explicite respiciam.

Coordinatarum forma generalis est haec

$$(29)\ldots\ldots\left\{\begin{aligned} x &= \alpha\,\xi + \beta\,\eta + \gamma\,\zeta \\ y &= \alpha_{,}\xi + \beta_{,}\eta + \gamma_{,}\zeta \\ z &= \alpha_{,,}\xi + \beta_{,,}\eta + \gamma_{,,}\zeta \end{aligned}\right.$$

denotantibus ξ, η atque ζ coordinatas arbitrarias. Valores simplicissimos ipsis ξ, η, ζ attribuimus ponendo

$$\xi = r\cos f\ ,\ \eta = r\sin f\ ,\ \zeta = o$$

ubi f est anomalia vera. Si valores hi ipsarum x, y, z cum illis comparati erunt et si ad aequationes conditionales has respexeris

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1\ ,\ \alpha\alpha_{,} + \beta\beta_{,} + \gamma\gamma_{,} = o \\ \alpha_{,}^2 + \beta_{,}^2 + \gamma_{,}^2 &= 1\ ,\ \alpha\alpha_{,,} + \beta\beta_{,,} + \gamma\gamma_{,,} = o \\ \alpha_{,,}^2 + \beta_{,,}^2 + \gamma_{,,}^2 &= 1\ ,\ \alpha_{,}\alpha_{,,} + \beta_{,}\beta_{,,} + \gamma_{,}\gamma_{,,} = o \end{aligned}$$

inveniuntur

$$(30)\ldots\ldots\left\{\begin{aligned} \alpha &= \cos\varpi\cos\theta - \sin\varpi\sin\theta\cos i \\ \beta &= -\sin\varpi\cos\theta - \cos\varpi\sin\theta\cos i \\ \gamma &= \sin\theta\sin i \\ \alpha_{,} &= \cos\varpi\sin\theta + \sin\varpi\cos\theta\cos i \\ \beta_{,} &= -\sin\varpi\sin\theta + \cos\varpi\cos\theta\cos i \\ \gamma_{,} &= -\cos\theta\sin i \\ \alpha_{,,} &= \sin\varpi\sin i \\ \beta_{,,} &= \cos\varpi\sin i \\ \gamma_{,,} &= \cos i \end{aligned}\right.$$

ubi ϖ est arcus inter locum perihelii et nodum θ interceptus. Radius vector et anomalia vera ex elementis a semiaxi maiore, n motu medio in unitate temporis absoluto, ae excentricitate et c anomalia media certae determinataeque epochae respondente aequationibus pendent his

$$\begin{aligned} r\cos f &= a\cos u - ae \\ r\sin f &= a\sqrt{1-e^2}\,.\,\sin u \\ u - e\sin u &= nt + c \\ a^3n^2 &= \varkappa\,(M+m) \end{aligned}$$

ubi angulus auxiliaris u est anomalia excentrica. Denique aequationes conditionales supra allatae reciproce suppeditant

$$\left.\begin{array}{lll} \alpha^2 + \alpha_{,}^2 + \alpha_{,,}^2 = 1 & , & \alpha\beta + \alpha_{,}\beta_{,} + \alpha_{,,}\beta_{,,} = o \\ \beta^2 + \beta_{,}^2 + \beta_{,,}^2 = 1 & , & \alpha\gamma + \alpha_{,}\gamma_{,} + \alpha_{,,}\gamma_{,,} = o \\ \gamma^2 + \gamma_{,}^2 + \gamma_{,,}^2 = 1 & , & \beta\gamma + \beta_{,}\gamma_{,} + \beta_{,,}\gamma_{,,} = o \end{array}\right\} \ldots\ldots(31)$$

17.

In integralibus aequationum (18), quae in art. praecedente allata sunt, elementa sex a, e, c, ω, i, θ, constantes arbitrariae integratione introductae sunt. Loco horum elementorum a, e, c introducam constantes arbitrarias sive elementa φ, ϱ, $\varphi_{,}$, quae ex elementis a, e, c et ex quantitate indeterminata τ tali modo composita sint, quali f, r et $\frac{df}{dt}$ ex iisdem elementis et ex tempore t constant. Erunt nobis igitur

$$\left.\begin{array}{r} \varrho \cos\varphi = a \cos v - ae \\ \varrho \sin\varphi = a\sqrt{1-e^2} \,.\, \sin v \\ v - e \sin v = n\tau + c \\ \varphi_{,} = \frac{a^2 n}{\varrho^2} \sqrt{1-e^2} \end{array}\right\} \ldots(31^*)$$

ubi v est angulus auxiliaris ipsi u respondens. Loco elementorum ω, i, θ introducam alia tria χ, ψ, σ, quae infra explicabuntur.

Elementa igitur a, c, e, ω, i, θ functiones sunt elementorum φ, ϱ, $\varphi_{,}$, χ, ψ, σ, et reciproce elementa haec functiones sunt illorum. Coordinatae x, y, z, quae supra functiones elementorum illorum exhibitae sunt, non minus quam Ω, quae in art. praec. functio coordinatarum exhibita est, functiones elementorum horum spectari possunt. Quum in expressionibus (29) quantitates α, β, etc. tempus non contineant, habemus

$$\begin{aligned} x_{,} &= \alpha\, \xi_{,} + \beta\, \eta_{,} \\ y_{,} &= \alpha_{,} \xi_{,} + \beta_{,} \eta_{,} \\ z_{,} &= \alpha_{,,} \xi_{,} + \beta_{,,} \eta_{,} \end{aligned}$$

ubi

$$\xi_{,} = \frac{d\xi}{dt} \;,\; \eta_{,} = \frac{d\eta}{dt}$$

Praeterea quum φ, $\varphi_{,}$ et ϱ functiones sint elementorum a, c, e, quae in α, β etc. non continentur, emergunt

$$\left(\frac{dx}{d\varphi}\right) = \alpha\left(\frac{d\xi}{d\varphi}\right) + \beta\left(\frac{d\eta}{d\varphi}\right) \;;\; \left(\frac{dx_{,}}{d\varphi}\right) = \alpha\left(\frac{d\xi_{,}}{d\varphi}\right) + \beta\left(\frac{d\eta_{,}}{d\varphi}\right)$$

et similes expressiones nanciscimur pro quotientibus differentialibus ipsarum x, $x_,$, y, $y_,$, z, $z_,$ et respectu ipsarum $\varphi_,$ et ϱ, et respectu ipsius φ. Substitutis his expressionibus in prima aequatione (23), postquam in ea φ loco a et $\varphi_,$ loco b scripta est, invenitur propter aequationes conditionales (31)

$$(32)\ldots\ldots \quad (\varphi, \varphi_,) = \left(\frac{d\xi}{d\varphi}\right)\left(\frac{d\xi_,}{d\varphi_,}\right) + \left(\frac{d\eta}{d\varphi}\right)\left(\frac{d\eta_,}{d\varphi_,}\right) - \left(\frac{d\xi_,}{d\varphi}\right)\left(\frac{d\xi}{d\varphi_,}\right) - \left(\frac{d\eta_,}{d\varphi}\right)\left(\frac{d\eta}{d\varphi_,}\right)$$

e qua aequatione mutatis mutandis valores quantitatum (φ, ϱ) et $(\varphi_,, \varrho)$ obtinemus.

Porro quum χ, ψ, σ functiones sint elementorum ω, i, θ, quae in ξ et η non continentur, elicitur

$$\left(\frac{dx}{d\chi}\right) = \left(\frac{d\alpha}{d\chi}\right)\xi + \left(\frac{d\beta}{d\chi}\right)\eta \; ; \; \left(\frac{dx_,}{d\chi}\right) = \left(\frac{d\alpha}{d\chi}\right)\xi_, + \left(\frac{d\beta}{d\chi}\right)\eta_,$$

quibus plane similes sunt expressiones pro quotientibus differentialibus ipsarum x, $x_,$, y, $y_,$, z, $z_,$ et respectu ipsarum ψ et σ, et respectu ipsius χ. Substitutis his expressionibus nec non expressionibus praecedentibus pro quotientibus differentialibus respectu φ in prima aequatione (23), postquam φ loco a et χ loco b scripta est, obtinemus, quia aequationes conditionales (31) praebent

$$\alpha d\alpha + \alpha_, d\alpha_, + \alpha_{,,} d\alpha_{,,} = o \;,\; \beta d\beta + \beta_, d\beta_, + \gamma_, d\gamma_, = o$$
$$\alpha d\beta + \alpha_, d\beta_, + \alpha_{,,} d\beta_{,,} = -\; \beta d\alpha - \beta_, d\alpha_, - \beta_{,,} d\alpha_{,,}$$

expressionem hanc

$$(33)\ldots\ldots \quad (\varphi, \chi) = -\left\{\beta\left(\frac{d\alpha}{d\chi}\right) + \beta_,\left(\frac{d\alpha_,}{d\chi}\right) + \beta_{,,}\left(\frac{d\alpha_{,,}}{d\chi}\right)\right\}\frac{d\,.\,(\xi\eta_, - \xi_,\eta)}{d\varphi}$$

e qua mutatis mutandis expressiones ipsarum (φ, ψ), (φ, σ), $(\varphi_,, \chi)$, $(\varphi_,, \psi)$, $(\varphi_,, \sigma)$, (ϱ, χ), (ϱ, ψ) et (ϱ, σ) emergunt.

Denique quotientibus differentialibus coordinatarum respectu χ et ψ in prima aequatione (23), postquam χ loco a et ψ loco b scripta est, substitutis, emergit

$$(\chi, \psi) = (\xi\eta_, - \xi_,\eta)\left\{\begin{array}{l} \left(\frac{d\alpha}{d\chi}\right)\left(\frac{d\beta}{d\psi}\right) + \left(\frac{d\alpha_,}{d\chi}\right)\left(\frac{d\beta_,}{d\psi}\right) + \left(\frac{d\alpha_{,,}}{d\chi}\right)\left(\frac{d\beta_{,,}}{d\psi}\right) \\ - \left(\frac{d\alpha}{d\psi}\right)\left(\frac{d\beta}{d\chi}\right) - \left(\frac{d\alpha_,}{d\psi}\right)\left(\frac{d\beta_,}{d\chi}\right) - \left(\frac{d\alpha_{,,}}{d\psi}\right)\left(\frac{d\beta_{,,}}{d\chi}\right) \end{array}\right\}$$

Adiumento aequationum conditionalium art. praec. facile reperitur identicas esse aequationes has

$$\begin{aligned}
d\alpha &= \beta\,(\beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}) + \gamma\,(\gamma d\alpha + \gamma_{,} d\alpha_{,} + \gamma_{,,} d\alpha_{,,})\\
d\beta &= -\alpha\,(\beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}) + \gamma\,(\gamma d\beta + \gamma_{,} d\beta_{,} + \gamma_{,,} d\beta_{,,})\\
d\alpha_{,} &= \beta_{,}\,(\beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}) + \gamma_{,}\,(\gamma d\alpha + \gamma_{,} d\alpha_{,} + \gamma_{,,} d\alpha_{,,})\\
d\beta_{,} &= -\alpha_{,}\,(\beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}) + \gamma_{,}\,(\gamma d\beta + \gamma_{,} d\beta_{,} + \gamma_{,,} d\beta_{,,})\\
d\alpha_{,,} &= \beta_{,,}\,(\beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}) + \gamma_{,,}\,(\gamma d\alpha + \gamma_{,} d\alpha_{,} + \gamma_{,,} d\alpha_{,,})\\
d\beta_{,,} &= -\alpha_{,,}\,(\beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}) + \gamma_{,,}\,(\gamma d\beta + \gamma_{,} d\beta_{,} + \gamma_{,,} d\beta_{,,})
\end{aligned}$$

Substitutis his valoribus ipsarum $d\alpha$, $d\beta$, $d\alpha_{,}$, etc. in aequatione praecedenti, elicitur

$$(\chi,\psi) = (\xi\eta_{,} - \xi_{,}\eta)\left\{\begin{array}{l} \left\{\gamma\left(\frac{d\alpha}{d\chi}\right) + \gamma_{,}\left(\frac{d\alpha_{,}}{d\chi}\right) + \gamma_{,,}\left(\frac{d\alpha_{,,}}{d\chi}\right)\right\}\left\{\gamma\left(\frac{d\beta}{d\psi}\right) + \gamma_{,}\left(\frac{d\beta_{,}}{d\psi}\right) + \gamma_{,,}\left(\frac{d\beta_{,,}}{d\psi}\right)\right\} \\ -\left\{\gamma\left(\frac{d\alpha}{d\psi}\right) + \gamma_{,}\left(\frac{d\alpha_{,}}{d\psi}\right) + \gamma_{,,}\left(\frac{d\alpha_{,,}}{d\psi}\right)\right\}\left\{\gamma\left(\frac{d\beta}{d\chi}\right) + \gamma_{,}\left(\frac{d\beta_{,}}{d\chi}\right) + \gamma_{,,}\left(\frac{d\beta_{,,}}{d\chi}\right)\right\} \end{array}\right\} \quad \ldots\ldots(34)$$

e qua mutatis mutandis expressiones ipsarum $(\chi,\ \sigma)$ atque $(\psi,\ \sigma)$ nanciscimur.

18.

Ad quotientes differentiales, quos formulae art. praec. requirunt, obtinendos, habetur primum $\xi = r\cos f$, $\eta = r\sin f$. Quum vero secundum art. 12. in evolutis quantitatibus $(\varphi,\ \varphi_{,})$, $(\varphi,\ \varrho)$ etc. tempus evanescere debeat, ad libitum loco t valorem quemlibet substituere nobis licet, quare, ut earum computatio tantum, quantum fieri potest, contrahatur, ponam τ loco t. Hinc emergunt

$$\xi = \varrho\cos\varphi\ ;\ \eta = \varrho\sin\varphi$$

quae ipsae per se functiones ipsarum ϱ et φ absque elementis a, c, e, etc. sese praestant. Habemus igitur statim

$$\begin{aligned}
\left(\frac{d\xi}{d\varphi}\right) &= -\varrho\sin\varphi\ ;\ \left(\frac{d\xi}{d\varphi_{,}}\right) = o\ ;\ \left(\frac{d\xi}{d\varrho}\right) = \cos\varphi\\
\left(\frac{d\eta}{d\varphi}\right) &= \varrho\cos\varphi\ ;\ \left(\frac{d\eta}{d\varphi_{,}}\right) = o\ ;\ \left(\frac{d\eta}{d\varrho}\right) = \sin\varphi
\end{aligned}$$

Porro scripta τ loco t, ipsae $\xi_{,}$ et $\eta_{,}$ abeunt in quotientes differentiales ipsarum ξ et η respectu τ, unde, differentiatione secundum regulas notas instituta, elicitur

$$\xi_, = -\frac{an \sin \varphi}{\sqrt{1-e^2}}$$

$$\eta_, = \frac{ane}{\sqrt{1-e^2}} + \frac{an \cos \varphi}{\sqrt{1-e^2}}$$

Aequatio vero $\varphi_, = \frac{a^2 n}{\varrho^2} \sqrt{1-e^2}$ praebet

$$\frac{1}{\sqrt{1-e^2}} = \frac{a^2 n}{\varrho^2 \varphi_,}$$

unde

$$\xi_, = -\frac{a^3 n^2 \sin \varphi}{\varrho^2 \varphi_,} = -\frac{\varkappa (M+m) \sin \varphi}{\varrho^2 \varphi_,}$$

Quantitas igitur $\xi_,$ functio reddita est ipsarum φ, ϱ et $\varphi_,$ absque elementis a, c, e, etc. quare statim ex aequatione praecedente invenitur

$$\left(\frac{d\xi_,}{d\varphi}\right) = -\frac{an}{\sqrt{1-e^2}} \cos \varphi\,;\ \left(\frac{d\xi_,}{d\varphi_,}\right) = \frac{\varrho^2 \sin \varphi}{a(1-e^2)}\,;\ \left(\frac{d\xi_,}{d\varrho}\right) = 2\frac{an}{\varrho\sqrt{1-e^2}} \sin \varphi$$

Quum in valore ipsius $\eta_,$ supra allato elementa a atque e haud aeque facile eliminari possint, ad eius differentialia partialia respectu φ, $\varphi_,$ et ϱ obtinenda viam aliam ingrediemur. Si ex valoribus ipsarum ξ, η, $\xi_,$, $\eta_,$ modo datis computatur quantitas $\xi\eta_, - \xi_,\eta$, qua praeterea opus est, facile invenitur

$$\xi\eta_, - \xi_,\eta = a^2 n \sqrt{1-e^2} = \varrho^2 \varphi_,$$

quae quantitas igitur functionem ipsarum ϱ et $\varphi_,$ absque elementis a, c, e, etc. sese praestat. Ideo differentiata praebet

$$d\eta_, = 2\frac{\varrho\varphi_,}{\xi} d\varrho + \frac{\varrho^2}{\xi} d\varphi_, - \frac{\eta_,}{\xi} d\xi + \frac{\eta}{\xi} d\xi_, + \frac{\xi_,}{\xi} d\eta$$

unde

$$\left(\frac{d\eta_,}{d\varphi}\right) = -\frac{\eta_,}{\xi}\left(\frac{d\xi}{d\varphi}\right) + \frac{\eta}{\xi}\left(\frac{d\xi_,}{d\varphi}\right) + \frac{\xi_,}{\xi}\left(\frac{d\eta}{d\varphi}\right)$$

$$\left(\frac{d\eta_,}{d\varphi_,}\right) = \frac{\varrho^2}{\xi} - \frac{\eta_,}{\xi}\left(\frac{d\xi}{d\varphi_,}\right) + \frac{\eta}{\xi}\left(\frac{d\xi_,}{d\varphi_,}\right) + \frac{\xi_,}{\xi}\left(\frac{d\eta}{d\varphi_,}\right)$$

$$\left(\frac{d\eta_,}{d\varrho}\right) = 2\frac{\varrho\varphi_,}{\xi} - \frac{\eta_,}{\xi}\left(\frac{d\xi}{d\varrho}\right) + \frac{\eta}{\xi}\left(\frac{d\xi_,}{d\varrho}\right) + \frac{\xi_,}{\xi}\left(\frac{d\eta}{d\varrho}\right)$$

substitutis valoribus ipsarum $\left(\frac{d\xi}{d\varphi}\right)$, etc. ex praecedentibus desumendis, habetur denique

$$\left(\frac{d\eta_{,}}{d\varphi}\right)=\frac{an}{\sqrt{1-e^2}}\cdot\frac{\sin\varphi(e-\cos\varphi)}{\cos\varphi};\ \left(\frac{d\eta_{,}}{d\varphi_{,}}\right)=\frac{\varrho^2(2+e\cos\varphi-\cos^2\varphi)}{a\cos\varphi\,(1-e^2)};\ \left(\frac{d\eta_{,}}{d\varrho}\right)=\frac{an}{\sqrt{1-e^2}}\cdot\frac{3+e\cos\varphi-2\cos^2\varphi}{\varrho\cos\varphi}.$$

Iam omnia praesto sunt quae ad computationem ipsarum $(\varphi,\ \varphi_{,})$, $(\varphi,\ \varrho)$ atque $(\varphi_{,},\ \varrho)$ requiruntur, et substitutis quotientibus differentialibus in hoc articulo inventis in aequatione (32) et in eius similibus, emergunt

$$(\varphi,\varphi_{,}) = \varrho^2\ ;\ (\varphi,\varrho) = 2\,\frac{a^2 n\sqrt{1-e^2}}{\varrho} - \frac{an}{\sqrt{1-e^2}}\cdot\frac{e}{\cos\varphi}$$

$$(\varphi_{,},\varrho) = -\,\frac{\varrho^2\sin\varphi\,(2+e\cos\varphi)}{a\cos\varphi\,(1-e^2)}$$

Ad reliquas quantitates $(\varphi,\ \chi)$, etc. evolvendas necesse est, elementa adhuc indeterminata χ, ψ et σ determinentur. Quem in finem, et ut quantitates $(\varphi,\ \chi)$, etc. simplicissimae evadant, pono

$$\left.\begin{aligned}
\beta\left(\frac{d\alpha}{d\chi}\right) + \beta_{,}\left(\frac{d\alpha_{,}}{d\chi}\right) + \beta_{,,}\left(\frac{d\alpha_{,,}}{d\chi}\right) &= 1\\
\gamma\left(\frac{d\alpha}{d\psi}\right) + \gamma_{,}\left(\frac{d\alpha_{,}}{d\psi}\right) + \gamma_{,,}\left(\frac{d\alpha_{,,}}{d\psi}\right) &= 1\\
\gamma\left(\frac{d\beta}{d\sigma}\right) + \gamma_{,}\left(\frac{d\beta_{,}}{d\sigma}\right) + \gamma_{,,}\left(\frac{d\beta_{,,}}{d\sigma}\right) &= 1
\end{aligned}\right\}\text{......(35)}$$

quibus aequationibus propter aequationes conditionales (31) satisfaciunt valores hi

$$\left(\frac{d\alpha}{d\chi}\right)=\beta\ ;\ \left(\frac{d\alpha_{,}}{d\chi}\right)=\beta_{,}\ ;\ \left(\frac{d\alpha_{,,}}{d\chi}\right)=\beta_{,,}$$
$$\left(\frac{d\alpha}{d\psi}\right)=\gamma\ ;\ \left(\frac{d\alpha_{,}}{d\psi}\right)=\gamma_{,}\ ;\ \left(\frac{d\alpha_{,,}}{d\psi}\right)=\gamma_{,,}$$
$$\left(\frac{d\beta}{d\sigma}\right)=\gamma\ ;\ \left(\frac{d\beta_{,}}{d\sigma}\right)=\gamma_{,}\ ;\ \left(\frac{d\beta_{,,}}{d\sigma}\right)=\gamma_{,,}$$

ipsarum $\left(\frac{d\alpha}{d\chi}\right)$, etc. Adiumento harum aequationum et aequationum conditionalium memoratarum invenitur statim non modo

$$\beta\left(\frac{d\alpha}{d\psi}\right) + \beta_{,}\left(\frac{d\alpha_{,}}{d\psi}\right) + \beta_{,,}\left(\frac{d\alpha_{,,}}{d\psi}\right) = 0$$

sed etiam omnes quantitates reliquae huius generis cifrae aequales inveniuntur.

Expressio ipsius $\xi\eta_{,} - \xi_{,}\eta$ supra inventa differentiata suppeditat

$$\frac{d.(\xi\eta_{,}-\xi_{,}\eta)}{d\varphi} = 0\ ;\ \frac{d.(\xi\eta_{,}-\xi_{,}\eta)}{d\varphi_{,}} = \varrho^2\ ;\ \frac{d.(\xi\eta_{,}-\xi_{,}\eta)}{d\varrho} = 2\,\frac{a^2 n\sqrt{1-e^2}}{\varrho}.$$

Quae omnia sunt quae ad computationem ipsarum (φ, χ), etc. requiruntur, et facili opera invenitur per aequationes (33) et (34) et per earum similes

$$(\varphi, \chi) = o;\ (\varphi, \psi) = o;\ (\varphi, \sigma) = o;\ (\varphi_{,}, \chi) = -\varrho^2;\ (\varphi_{,}, \psi) = o;$$
$$(\varphi_{,}, \sigma) = o;\ (\varrho, \chi) = -2\frac{a^2 n\sqrt{1-e^2}}{\varrho};\ (\varrho, \psi) = o;\ (\varrho, \sigma) = o;$$
$$(\chi, \psi) = o;\ (\chi, \sigma) = o;\ (\psi, \sigma) = a^2 n\sqrt{1-e^2}.$$

Denique expressiones ipsarum $d\chi$, $d\psi$ et $d\sigma$ in $d\omega$, di et $d\theta$ obtinentur, si expressiones (30) differentiatae et in (35) substitutae erunt, tales

$$(36)\ldots\ldots\left\{\begin{aligned} d\chi &= d\omega + \cos i\, d\theta \\ d\psi &= \sin\omega\, di - \cos\omega \sin i\, d\theta \\ d\sigma &= \cos\omega\, di + \sin\omega\sin i\, d\theta \end{aligned}\right.$$

unde evidens est, χ, ψ atque σ esse rotationes plani orbitae circum axes ipsarum x, y, z.

19.

Ope valorum ipsarum $(\varphi, \varphi_{,})$, etc. in art. praec. inventorum aequationes (24), postquam φ, $\varphi_{,}$, ϱ, χ, ψ, σ resp. loco a, b, c, e, f, g scriptae sunt, suppeditant, si insuper loco R ponitur eius valor $\varkappa(M+m)\,\Omega$ sive $a^3 n^2 \Omega$,

$$a^3n^2\left(\frac{d\Omega}{d\varphi}\right)dt = \varrho^2 d\varphi_{,} + \left\{2\frac{a^2n\sqrt{1-e^2}}{\varrho} - \frac{an}{\sqrt{1-e^2}}\cdot\frac{e}{\cos\varphi}\right\}d\varrho$$

$$a^3n^2\left(\frac{d\Omega}{d\varphi_{,}}\right)dt = -\varrho^2 d\varphi - \frac{\varrho^2\sin\varphi\,(2+e\cos\varphi)}{a\cos\varphi\,(1-e^2)}\,d\varrho - \varrho^2 d\chi$$

$$a^3n^2\left(\frac{d\Omega}{d\varrho}\right)dt = -\left\{2\frac{a^2n\sqrt{1-e^2}}{\varrho} - \frac{an}{\sqrt{1-e^2}}\cdot\frac{e}{\cos\varphi}\right\}d\varphi + \frac{\varrho^2\sin\varphi\,(2+e\cos\varphi)}{a\cos\varphi\,(1-e^2)}\,d\varphi_{,} - 2\frac{a^2n\sqrt{1-e^2}}{\varrho}\,d\chi$$

$$a^3n^2\left(\frac{d\Omega}{d\chi}\right)dt = \varrho^2 d\varphi_{,} + 2\frac{a^2n\sqrt{1-e^2}}{\varrho}\,d\varrho$$

$$a^3n^2\left(\frac{d\Omega}{d\psi}\right)dt = a^2n\sqrt{1-e^2}\,d\sigma$$

$$a^3n^2\left(\frac{d\Omega}{d\sigma}\right)dt = -a^2n\sqrt{1-e^2}\,d\psi$$

Hinc eliminatione perfacili nanciscimur

$$\left.\begin{aligned}
d\varphi &= \frac{a^2 n \cos\varphi\sqrt{1-e^2}}{e}\left(\frac{d\Omega}{d\varrho}\right)dt - \frac{an\sin\varphi\,(2+e\cos\varphi)}{e\sqrt{1-e^2}}\left(\frac{d\Omega}{d\chi}\right)dt - 2\,\frac{a^4n^2(1-e^2)\cos\varphi}{e\varrho^3}\left(\frac{d\Omega}{d\varphi_{,}}\right)dt \\
d\varphi_{,} &= 2\,\frac{a^4n^2(1-e^2)\cos\varphi}{e\varrho^3}\left(\frac{d\Omega}{d\varphi}\right)dt + \left\{\frac{a^3n^2}{\varrho^2} - 2\,\frac{a^4n^2(1-e^2)\cos\varphi}{e\varrho^3}\right\}\left(\frac{d\Omega}{d\chi}\right)dt \\
d\varrho &= \frac{a^2n\cos\varphi\sqrt{1-e^2}}{e}\left\{\left(\frac{d\Omega}{d\chi}\right) - \left(\frac{d\Omega}{d\varphi}\right)\right\}dt \\
d\chi &= \frac{an\sin\varphi\,(2+e\cos\varphi)}{e\sqrt{1-e^2}}\left(\frac{d\Omega}{d\varphi}\right)dt - \frac{a^2n\cos\varphi\sqrt{1-e^2}}{e}\left(\frac{d\Omega}{d\varrho}\right)dt + \left\{2\frac{a^4n^2(1-e^2)\cos\varphi}{e\varrho^3} - \frac{a^3n^2}{\varrho^2}\right\}\left(\frac{d\Omega}{d\varphi_{,}}\right)dt \\
d\psi &= -\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{d\sigma}\right)dt \\
d\sigma &= \frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{d\psi}\right)dt
\end{aligned}\right\}\ldots\ldots(37)$$

Quibus aequationibus integratis, secundum theorema art. 13., valor perturbatus anomaliae verae obtinetur, si in valore ipsius φ, et valor perturbatus radii vectoris, si in valore ipsius ϱ erit τ in t mutata. Valores ipsarum χ, ψ et σ ex posterioribus tribus harum aequationum prodituri situm orbitae in spatio determinant, aequatio vero pro $\varphi_{,}$ perficiendis integrationibus reliquarum aequationum tantum inservit. Hoc igitur modo integralia aequationum (16) quae ad corpus m spectant nanciscimur, et his plane similia integralia aequationum differentialium ad corpora m', m'', etc. spectantium obtinentur, sed hae aequationes omnes in problemate generali trium quatuorve, etc. corporum simultanee integrandae sunt.

Elementa in praecedentibus electa paullulum mutari placet. Ad situm corporis coelestis determinandum quam aptissima sunt elementa λ, $l\varrho$, p atque q, quae cum illis ita coniuncta sunt, ut sit

$$\lambda = \varphi + \chi \text{ unde } \varphi_{,} = \lambda_{,}$$

denotante $\lambda_{,}$ differentiale ipsius λ respectu τ per $d\tau$ divisum

$$\begin{aligned}
l\varrho &= \text{logarithmo hyperbolico ipsius } \varrho \\
p &= \sin i \sin(\chi - \omega) \\
q &= \sin i \cos(\chi - \omega)
\end{aligned}$$

Quae aequationes differentiatae subministrant

$$d\lambda = d\varphi + d\chi\,,\ d\varphi_{,} = d\lambda_{,}\,,\ dl\varrho = \frac{d\varrho}{\varrho}$$
$$dp = \cos i \sin(\chi - \omega)\, di + \sin i \cos(\chi - \omega)\,(d\chi - d\omega)$$
$$dq = \cos i \cos(\chi - \omega)\, di - \sin i \sin(\chi - \omega)\,(d\chi - d\omega)$$

sive adiumento aequationum (36)

$$dp = -\cos i \cos \chi\, d\psi + \cos i \sin \chi\, d\sigma$$
$$dq = \cos i \sin \chi\, d\psi + \cos i \cos \chi\, d\sigma$$

Iam habita Ω et pro functione ipsarum φ, $\varphi_{,}$, χ, ψ, σ, et pro functione ipsarum λ, $\lambda_{,}$, χ, p, q, algorithmus notus differentialium partialium suppeditat

$$\left(\frac{d\Omega}{d\varphi}\right) = \left(\frac{d\Omega}{d\lambda}\right) ; \left(\frac{d\Omega}{d\varphi_{,}}\right) = \left(\frac{d\Omega}{d\lambda_{,}}\right) ; \left(\frac{d\Omega}{d\chi}\right) = \left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right)$$
$$\left(\frac{d\Omega}{d\psi}\right) = -\cos i \cos \chi \left(\frac{d\Omega}{dp}\right) + \cos i \sin \chi \left(\frac{d\Omega}{dq}\right)$$
$$\left(\frac{d\Omega}{d\sigma}\right) = \cos i \sin \chi \left(\frac{d\Omega}{dp}\right) + \cos i \cos \chi \left(\frac{d\Omega}{dq}\right)$$

quarum sinistrae partes pro functionibus ipsarum φ, $\varphi_{,}$, etc., dextrae vero partes pro functionibus ipsarum λ, $\lambda_{,}$, etc. habendae sunt. Quibus aequationibus aequationes (37) facile transmutantur in has

$$(38)\ldots\ldots \left\{ \begin{aligned}
d\lambda &= -\frac{an \sin \varphi\, (2 + e \cos \varphi)}{e\sqrt{1-e^2}} \left(\frac{d\Omega}{d\chi}\right) dt - \frac{a^3 n^2}{\varrho^2} \left(\frac{d\Omega}{d\lambda_{,}}\right) dt \\
d\lambda_{,} &= \left\{\frac{a^3 n^2}{\varrho^2} - 2\,\frac{a^4 n^2 (1-e^2) \cos \varphi}{e\varrho^3}\right\} \left(\frac{d\Omega}{d\chi}\right) dt + \frac{a^3 n^2}{\varrho^2} \left(\frac{d\Omega}{d\lambda}\right) dt \\
dl\varrho &= \frac{a^2 n \cos \varphi \sqrt{1-e^2}}{e\varrho} \left(\frac{d\Omega}{d\chi}\right) dt \\
dp &= \frac{an \cos^2 i}{\sqrt{1-e^2}} \left(\frac{d\Omega}{dq}\right) dt \\
dq &= -\frac{an \cos^2 i}{\sqrt{1-e^2}} \left(\frac{d\Omega}{dp}\right) dt
\end{aligned} \right.$$

ubi aequationem pro $d\chi$ omisi, quia hac opus non est. Quinque enim aequationes praecedentes ad locum corporis coelestis in spatio per methodum nostram determinandum sufficiunt, quamquam locus corporis coelestis alicuius non minus quam aequationes praecedentes ipsae ex elementis sex pendet, quod paradoxon expendendum lectori perito relinquo.

Denotantibus v longitudinem veram in orbita, s sinum latitudinis supra planum fixum ipsarum xy et l longitudinem veram ad idem planum reductam, habemus per trigonometriam sphaericam

$$s = \sin i \sin (v - \theta)$$
$$tg\,(l - \theta) = tg\,(v - \theta) \cos i$$

Sed posito $v_, = \int d\lambda$, quoties in dextro huius aequationis membro post integrationem τ in t mutata erit, habetur

$$v_, = f + \chi$$

quum vero sit $v = f + \varpi + \theta$, emergit

$$v = v_, + \varpi + \theta - \chi$$

unde aequationes praecedentes transeunt in has

$$\left.\begin{aligned} s &= \sin i \sin (v_, - \chi + \varpi) \\ tg\ (l - \theta) &= tg\ (v_, - \chi + \varpi) \cos i \end{aligned}\right\} \ldots\ldots (39)$$

In his aequationes i, θ, ϖ et χ ope quantitatum p et q eliminari possunt, id quod infra explicabitur. Quantitatibus igitur $v_,$, p atque q latitudo corporis supra planum ipsarum xy in spatio ad lubitum collocatum, et longitudo ad idem planum reducta determinantur.

SECTIO II.

DISQVISITIONES DE AEQVATIONIBVS MOTVI LVNAE INVESTIGANDO INSERVIENTIBVS.

1.

In theoria Lunae motus Lunae relativus respectu Terrae investigandus est. Denotat igitur secundum ea, quae in Sectione prima protulimus, M massam terrae; m massam Lunae; a semiaxem maiorem orbitae Lunae; ae eius excentricitatem; c eius anomaliam mediam certae cuidam temporis epochae respondentem; n eius motum medium in unitate temporis, quam annum Iulianum statuemus, absolutum; ω arcum inter locum eius perigaei et nodum ascendentem plani eius orbitae cum plano quolibet fixo, centrum gravitatis Terrae transiente, interceptum; i inclinationem orbitae eius ad idem planum fixum; θ longitudinem nodi ascendentis orbitae eius cum eodem plano; etc.

Denotat porro m' massam Solis, et referuntur elementa explicata ad Solem, quoties lineola superne ad dextram iis affigitur. Quibus praemissis aequationes (38) Sect. I., quae generaliter motui corporum coelestium inveniendo inserviunt, hae

$$\left.\begin{aligned}
d\lambda &= -\frac{an\sin\varphi\,(2+e\cos\varphi)}{e\sqrt{1-e^2}}\left(\frac{d\Omega}{d\chi}\right)dt - \frac{a^2n^2}{\varrho^2}\left(\frac{d\Omega}{d\lambda_{,}}\right)dt \\
d\lambda_{,} &= \left\{\frac{a^2n^2}{\varrho^2} - 2\,\frac{a^4n^2\,(1-e^2)\cos\varphi}{e\varrho^3}\right\}\left(\frac{d\Omega}{d\chi}\right)dt + \frac{a^2n^2}{\varrho^2}\left(\frac{d\Omega}{d\lambda}\right)dt \\
dl\varrho &= \frac{a^2n\cos\varphi\sqrt{1-e^2}}{e\varrho}\left(\frac{d\Omega}{d\chi}\right)dt \\
dp &= \frac{an}{\sqrt{1-e^2}}\cos^2 i\left(\frac{d\Omega}{dq}\right)dt \\
dq &= -\frac{an}{\sqrt{1-e^2}}\cos^2 i\left(\frac{d\Omega}{dp}\right)dt
\end{aligned}\right\}\ldots\ldots(1)$$

motum quoque Lunae definiunt, et quum hoc loco ad perturbationes a figura sphaeroidica Terrae ortas nec non ad perturbationes a planetis prolatas non respiciamus, habemus ex art. 16. Sect. I.

$$\Omega = \frac{m'}{M+m}\left\{\frac{1}{\varDelta} - \frac{xx'+yy'+zz'}{r'^3}\right\}$$

ubi

$$\varDelta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

Quantitates λ, $\lambda_{,}$, ϱ, χ, p et q nec non quantitates analogae ad corpus perturbans, quod hoc loco est Sol ipse, spectantes in his formulis pro variabilibus independentibus habendae sunt; quantitates igitur a, n, e, c, i et θ pro functionibus illarum quantitatum habendae sunt, et functio perturbatrix, quam supra in functione coordinatarum orthogoniarum exhibui, in evolvendis formulis (1) pro functione illarum variabilium independentium censenda est.

2.

Ad aequationes (1) integrandas posui in theoria planetarum $\lambda = \Pi\zeta$ et $l\varrho = \Gamma\zeta + \beta$, ubi ζ et β novae variabiles functiones ipsarum τ et t, atque Π et Γ signa functionum sunt. Hoc vero loco ponam formulas generaliores has

$$\left.\begin{aligned}
\lambda &= \Pi(\zeta, t) \\
l\varrho &= \Gamma(\zeta, t) + \beta
\end{aligned}\right\}\ldots\ldots(2)$$

unde mutata τ in t evadunt

$$
(3)\ldots\ldots\left\{\begin{aligned} v_, &= \Pi(z,\ t) \\ lr &= \Gamma(z,\ t) + w \end{aligned}\right.
$$

ubi $v_,$, r, z et w quantitates designant, in quas mutata τ in t resp. λ, ϱ, ζ et β abeunt. Ideo $v_,$ est quasi longitudo vera in orbita et r radius vector temporis t, indoles vero quantitatum z et w adhuc definita non est. Differentiatis aequationibus (2), tum secundum τ, tum secundum t elicitur

$$
\begin{aligned}
\frac{d\lambda}{d\tau} &= \Pi'(\zeta,\ t)\ \frac{d\zeta}{d\tau} \\
\frac{d\lambda}{dt} &= \Pi'(\zeta,\ t)\ \frac{d\zeta}{dt} + \Pi_,(\zeta,\ t) \\
\frac{dl\varrho}{d\tau} &= \Gamma\ (\zeta,\ t)\ \frac{d\zeta}{d\tau} + \frac{d\beta}{d\tau} \\
\frac{dl\varrho}{dt} &= \Gamma'(\zeta,\ t)\ \frac{d\zeta}{dt} + \Gamma_,(\zeta,\ t) + \frac{d\beta}{dt}
\end{aligned}
$$

ubi Π' et Γ' resp. quotientes differentiales functionum Π atque Γ secundum ζ, et $\Pi_,$ atque $\Gamma_,$ resp. quotientes differentiales earundem functionum secundum t denotant. Eliminatis $\Pi'(\zeta,\ t)$ atque $\Gamma'(\zeta,\ t)$ ex aequationibus praecedentibus nanciscimur

$$
\begin{aligned}
&\frac{d\lambda}{d\tau}\cdot\frac{d\zeta}{dt} - \frac{d\lambda}{dt}\cdot\frac{d\zeta}{d\tau} = -\ \Pi_,(\zeta,\ t)\ \frac{d\zeta}{d\tau} \\
&\frac{dl\varrho}{d\tau}\cdot\frac{d\zeta}{dt} - \frac{dl\varrho}{dt}\cdot\frac{d\zeta}{d\tau} = \frac{d\beta}{d\tau}\cdot\frac{d\zeta}{dt} - \frac{d\beta}{dt}\cdot\frac{d\zeta}{d\tau} - \Gamma_,(\zeta,\ t)\ \frac{d\zeta}{d\tau}
\end{aligned}
$$

unde

$$
\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)} = \frac{\left(\frac{d\lambda}{dt}\right)}{\left(\frac{d\lambda}{d\tau}\right)} - \Pi_,(\zeta,t)\frac{1}{\left(\frac{d\lambda}{d\tau}\right)}
$$

$$
(4)\ldots\ldots\ \frac{d\beta}{dt} - \left(\frac{d\beta}{d\tau}\right)\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)} = \frac{\left(\frac{dl\varrho}{dt}\right)\left(\frac{d\lambda}{d\tau}\right) - \left(\frac{dl\varrho}{d\tau}\right)\left(\frac{d\lambda}{dt}\right)}{\left(\frac{d\lambda}{d\tau}\right)} + \Pi_,(\zeta,t)\frac{\left(\frac{dl\varrho}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)} - \Gamma_,(\zeta,t)
$$

attamen loco prioris harum aequationum differentiale eius, ut semper feci, adhibebo hoc

$$
(5)\ldots\ldots\ \frac{d.\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}}{d\tau} = \frac{\left(\frac{d^2\lambda}{dt\,d\tau}\right)\left(\frac{d\lambda}{d\tau}\right) - \left(\frac{d^2\lambda}{d\tau^2}\right)\left(\frac{d\lambda}{dt}\right)}{\left(\frac{d\lambda}{d\tau}\right)^2} + \Pi_,(\zeta,t)\frac{\left(\frac{d^2\lambda}{d\tau^2}\right)}{\left(\frac{d\lambda}{d\tau}\right)^2} - \Pi_,'(\zeta,t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)}
$$

ubi $\Pi_{\prime}'$ est quotiens differentialis functionis Π et secundum ζ et secundum t differentiatae.

3.

Ex formulis notis ad motum ellipticum pertinentibus et in Sectione I. allatis nanciscimur

$$\frac{d^2\lambda}{d\tau^2} = -2\frac{a^3n^2}{\varrho^3}e\sin\varphi$$

$$\frac{d\lambda}{d\tau} = \frac{a^2n}{\varrho^2}\sqrt{1-e^2}$$

$$\frac{dl\varrho}{d\tau} = \frac{ane\sin\varphi}{\varrho\sqrt{1-e^2}}$$

Substitutis his valoribus nec non valoribus ipsarum $\left(\frac{d^2\lambda}{d\tau\, dt}\right)$, $\left(\frac{d\lambda}{dt}\right)$ et $\left(\frac{dl\varrho}{dt}\right)$ ex aequationibus (1) petendis in aequationibus (4) et (5), nanciscimur

$$T = \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{d\chi}\right)+\left(\frac{d\Omega}{d\lambda}\right)\right\} - 2\frac{n\varrho(2e+\cos\varphi+e^2\cos\varphi)}{e(1-e^2)^{\frac{1}{2}}}\left(\frac{d\Omega}{d\chi}\right) - 2\frac{a^2n^2e\sin\varphi}{\varrho(1-e^2)}\left(\frac{d\Omega}{d\lambda_{\prime}}\right)$$
$$-2\Pi_{\prime}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)} - \Pi_{\prime}'(\zeta,t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)}$$

$$R = \frac{n\varrho(2e+\cos\varphi+e^2\cos\varphi)}{e(1-e^2)^{\frac{1}{2}}}\left(\frac{d\Omega}{d\chi}\right) + \frac{a^2n^2e\sin\varphi}{\varrho(1-e^2)}\left(\frac{d\Omega}{d\lambda_{\prime}}\right) + \Pi_{\prime}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)} - \Gamma_{\prime}(\zeta,t)$$

ubi T quantitatem $\dfrac{d.\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}}{d\tau}$, et R quantitatem $\dfrac{d\beta}{dt} - \left(\dfrac{d\beta}{d\tau}\right)\dfrac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}$ repraesentat.

Habita vero Ω tum pro functione ipsarum $v_{\prime}$, r, i et θ, tum pro functione ipsarum λ, ϱ, $\lambda_{\prime}$, χ, p et q, erit

$$d\Omega = \left(\frac{d\Omega}{dv_{\prime}}\right)dv_{\prime} + \left(\frac{d\Omega}{dr}\right)dr + \left(\frac{d\Omega}{di}\right)di + \left(\frac{d\Omega}{d\theta}\right)d\theta$$

$$d\Omega = \left(\frac{d\Omega}{d\lambda}\right)d\lambda + \left(\frac{d\Omega}{d\varrho}\right)d\varrho + \left(\frac{d\Omega}{d\lambda_{\prime}}\right)d\lambda_{\prime} + \left(\frac{d\Omega}{d\chi}\right)d\chi + \left(\frac{d\Omega}{dp}\right)dp + \left(\frac{d\Omega}{dq}\right)dq$$

sed propter aequationes $\lambda = \varphi + \chi$ et $d\chi = d\omega + \cos i\, d\theta$ aequationes (31*) Sect. I. differentiatae praebent

$$d\lambda = \left(\frac{d\lambda}{da}\right) da + \frac{a^2}{\varrho^2}\sqrt{1-e^2}\,.\,dc + \left(\frac{1}{1-e^2} + \frac{a}{\varrho}\right)\sin\varphi\, de + d\omega + \cos i\, d\theta$$

$$d\varrho = \left(\frac{d\varrho}{da}\right) da + ae\frac{\sin\varphi}{\sqrt{1-e^2}}\, dc - a\cos\varphi\, de$$

$$d\lambda_{,} = \left(\frac{d\lambda_{,}}{da}\right) da - 2\frac{a^3 n}{\varrho^3}\, e\sin\varphi\, dc + \left\{2\frac{a^3 n}{\varrho^3}\cos\varphi\sqrt{1-e^2} - \frac{a^2 ne}{\varrho^2\sqrt{1-e^2}}\right\} de$$

e quibus mutata τ in t evadunt

$$dv_{,} = \left(\frac{dv_{,}}{da}\right) da + \frac{a^2}{r^2}\sqrt{1-e^2}\,.\,dc + \left(\frac{1}{1-e^2} + \frac{a}{r}\right)\sin f\, de + d\omega + \cos i\, d\theta$$

$$dr = \left(\frac{dr}{da}\right) da + ae\frac{\sin f}{\sqrt{1-e^2}}\, dc - a\cos f\, de$$

Porro habemus e Sectione prima

$$d\chi = d\omega + \cos i\, d\theta$$
$$dp = \cos i \sin(\chi-\omega)\, di + \sin i\cos i\cos(\chi-\omega)\, d\theta$$
$$dq = \cos i\cos(\chi-\omega)\, di - \sin i\cos i\sin(\chi-\omega)\, d\theta$$

Substitutis his differentialium valoribus in expressionibus praecedentibus ipsius Ω, nanciscimur post comparatos terminos per idem differentiale multiplicatos

$$\left(\frac{d\Omega}{dv_{,}}\right)\frac{a^2}{r^2}\sqrt{1-e^2} + \left(\frac{d\Omega}{dr}\right)\frac{ae\sin f}{\sqrt{1-e^2}} = \left(\frac{d\Omega}{d\lambda}\right)\frac{a^2}{\varrho^2}\sqrt{1-e^2} + \left(\frac{d\Omega}{d\varrho}\right)\frac{ae\sin\varphi}{\sqrt{1-e^2}} - \left(\frac{d\Omega}{d\lambda_{,}}\right) 2\frac{a^3 ne\sin\varphi}{\varrho^3}$$

$$\left(\frac{d\Omega}{dv_{,}}\right)\left(\frac{1}{1-e^2} + \frac{a}{r}\right)\sin f - \left(\frac{d\Omega}{dr}\right) a\cos f =$$
$$\left(\frac{d\Omega}{d\lambda}\right)\left(\frac{1}{1-e^2} + \frac{a}{\varrho}\right)\sin\varphi - \left(\frac{d\Omega}{d\varrho}\right) a\cos\varphi + \left(\frac{d\Omega}{d\lambda_{,}}\right)\left\{2\frac{a^3 n}{\varrho^3}\cos\varphi\sqrt{1-e^2} - \frac{a^2 ne}{\varrho^2\sqrt{1-e^2}}\right\}$$

$$\left(\frac{d\Omega}{dv_{,}}\right) = \left(\frac{d\Omega}{d\lambda}\right) + \left(\frac{d\Omega}{d\chi}\right)$$

$$\left(\frac{d\Omega}{dv_{,}}\right)\cos i + \left(\frac{d\Omega}{d\theta}\right) = \left(\frac{d\Omega}{d\lambda}\right)\cos i + \left(\frac{d\Omega}{d\chi}\right)\cos i$$
$$+ \left(\frac{d\Omega}{dp}\right)\sin i\cos i\cos(\chi-\omega) - \left(\frac{d\Omega}{dq}\right)\sin i\cos i\sin(\chi-\omega)$$

$$\left(\frac{d\Omega}{di}\right) = \left(\frac{d\Omega}{dp}\right)\cos i\sin(\chi-\omega) + \left(\frac{d\Omega}{dq}\right)\cos i\cos(\chi-\omega)$$

Multiplicata prima harum aequationum per $\cos\varphi$, secunda per $\frac{e\sin\varphi}{\sqrt{1-e^2}}$, additisque productis, facili opera invenitur propter tertiam aequationem, et quia $1 = \frac{\varrho}{a(1-e^2)} + \frac{\varrho e\cos\varphi}{a(1-e^2)}$ atque $f - \varphi = v_{,} - \lambda_{,}$

$$\left(\frac{d\Omega}{dv_{,}}\right)\left\{\frac{\cos(v_{,}-\lambda)}{(1-e^2)^{\frac{3}{2}}}+\frac{a\cos(v_{,}-\lambda)}{r\sqrt{1-e^2}}-\frac{1}{(1-e^2)^{\frac{3}{2}}}-\frac{a}{\varrho\sqrt{1-e^2}}\right\}+r\left(\frac{d\Omega}{dr}\right)\frac{a\sin(v_{,}-\lambda)}{r\sqrt{1-e^2}}=$$
$$-\frac{1}{\varrho n}\left\{\frac{n\varrho\,(2e+\cos\varphi+e^2\cos\varphi)}{e\,(1-e^2)^{\frac{3}{2}}}\left(\frac{d\Omega}{d\chi}\right)+\frac{a^2n^2e\sin\varphi}{\varrho\,(1-e^2)}\left(\frac{d\Omega}{d\lambda_{,}}\right)\right\}$$

unde statim obtinetur

$$T=\left\{2\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+2\frac{\varrho}{a\,(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$+\;2\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}\,r\left(\frac{d\Omega}{dr}\right)-2\,\Pi_{,}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)}-\Pi'_{,}(\zeta,t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)}$$

$$R=-\left\{\frac{\varrho}{r}\cos(v'_{,}-\lambda)-1+\frac{\varrho}{a\,(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$-\;\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}\,r\left(\frac{d\Omega}{dr}\right)+\Pi_{,}(\zeta,t)\frac{\varrho e\sin\varphi}{a\,(1-e^2)}-\Gamma_{,}(\zeta,t)$$

ubi Ω pro functione quantitatum $v_{,}$, r, p, q, et quantitatum analogarum ad Solem spectantium habenda est. Functiones arbitrariae $\Pi_{,}(\zeta, t)$, $\Pi'_{,}(\zeta, t)$ atque $\Gamma_{,}(\zeta, t)$, quae in theoria nostra planetarum non adsunt, in theoria Lunae ita determinandae sunt, ut termini in integrali aequationum praecedentium et per tempus ipsum et per sinus et cosinus multiplicium anomaliae mediae multiplicati evanescant.

Terminus generalis quantitatis Ω in seriem infinitam secundum multiplicia cosinuum anomaliarum mediarum progredientem evolutae, ut notum, huius est formae

$$X\cos(ig+i'g'+i''\pi+i'''\pi'+i^{iv}\theta+i^{v}\theta')$$

ubi g anomaliam mediam Lunae, g' anomaliam mediam Solis, π longitudinem perigaei Lunae, π' longitudinem perigaei Solis et i, i', i'' etc. numeros integros positivos aut negativos, cifra inclusa, denotant, et X functionem excentricitatum, inclinationum semiaxium maiorum et numerorum i, i', etc. repraesentat. Hinc facile concluditur, terminum generalem in evoluta expressione ipsius T esse huius formae

$$Z\sin(\varkappa\gamma+ig+i'g'+i''\pi+i'''\pi'+i^{iv}\theta+i^{v}\theta')$$

ubi Z functio est eiusdem generis atque X, $\varkappa$ numerus integer positivus aut negativus aut cifra ipsa, et γ quantitas quae ex elementis ellipticis et

ex quantitate indeterminata τ eodem modo composita est, quo anomalia media Lunae ex elementis iisdem et ex tempore t.

Iam constat, terminos illos in expressione longitudinis et radii vectoris Lunae evitandos, qui huius formae sunt

$$\alpha\, t \sin(\varkappa g + i''\pi + \text{etc.}) + \beta\, t^2 \sin(\varkappa g + i''\pi + \text{etc.})$$
$$\text{et } \alpha' t \cos(\varkappa g + i''\pi + \text{etc.}) + \beta' t^2 \cos(\varkappa g + i''\pi + \text{etc.})$$

ubi α, α', β et β' coefficientes numerici sunt, solummodo ex terminis iis ipsarum T atque R nasci posse, in quibus simul et $i = o$ et $i' = o$, hoc est ex terminis huius formae

$$Z \sin(\varkappa\gamma + i''\pi + i'''\pi' + i^{v}\theta + i^{v}\theta')$$

functiones igitur arbitrariae $\Pi_{,}(\zeta, t)$, $\Pi_{,}'(\zeta, t)$ et $\Gamma_{,}(\zeta, t)$ ita determinandae sunt, ut termini hi non adsint.

Quum ea quae ad inclinationes nodosque spectant infra allaturi simus, nunc quidem inclinationes in expressione praecedenti negligemus; ideo considerabimus terminos ipsarum T atque R eos, qui huius formae sunt

(6)...... $$Z \sin(\varkappa\gamma + i''\pi + i'''\pi')$$

Inter varios terminos quos expressio haec continet deligamus eos, in quibus $i'' = o$ et simul $i''' = o$, hoc est terminos

$$Z \sin \varkappa\gamma$$

Iam demonstrabo functiones arbitrarias illas introductas ita determinari posse, ut termini formae praecedentis omnino evanescant, et simul determinationem hanc subsidia suppeditare, quibus efficitur, ut termini sub (6) repraesentati in expressione longitudinis radiique vectoris terminos per tempus ipsum multiplicatos proferre non possint.

4.

Quum $\left(\frac{d\Omega}{dv_{,}}\right) = \left(\frac{d\Omega}{d\pi}\right)$, pars ea expressionum ipsarum T atque R, quae per $\left(\frac{d\Omega}{dv_{,}}\right)$ multiplicata est, terminos formae $Z \sin \varkappa\gamma$ producere nequit, quare termini hi solummodo ex ea harum expressionum parte, quae per $\left(\frac{d\Omega}{dr}\right)$ multiplicata est, nasci possunt.

Habemus igitur, si non nisi ad terminos formae $Z \sin \varkappa\gamma$ respicimus

$$T = 2\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right) - 2\Pi_{,}(\zeta,t)\frac{\varrho e \sin\varphi}{a(1-e^2)} - \Pi_{,}'(\zeta,t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)}$$

$$R = -\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right) + \Pi_{,}(\zeta,t)\frac{\varrho e \sin\varphi}{a(1-e^2)} - \Gamma_{,}(\zeta,t)$$

quae, quum sit $v_{,}-\lambda = f-\varphi$, abeunt in has

$$T = 2\frac{\varrho}{r}\cos\varphi\sin f\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right) - 2\frac{\varrho}{r}\sin\varphi\cos f\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right)$$
$$- 2\Pi_{,}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)} - \Pi_{,}'(\zeta,t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)}$$

$$R = -\frac{\varrho}{r}\cos\varphi\sin f\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right) + \frac{\varrho}{r}\sin\varphi\cos f\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right)$$
$$+ \Pi_{,}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)} - \Gamma_{,}(\zeta,t)$$

Quantitas $\sin f\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right)$ in seriem infinitam evoluta secundum sinus multiplicium anomaliarum mediarum procedit, itaque termini expressionum praecedentium per quantitatem hanc multiplicati terminos formae $Z\sin\varkappa\gamma$ proferre nequeunt; quantitas vero $\cos f\frac{an}{\sqrt{1-e^2}}r\left(\frac{d\Omega}{dr}\right)$ evoluta, quum secundum cosinus multiplicium anomaliarum mediarum procedat, in T atque R terminos tales proferet. Facile igitur perspicitur, denotante V terminum constantem in evoluta quantitate $\frac{\cos f\, a^2 n\sqrt{1-e^2}}{re}r\left(\frac{d\Omega}{dr}\right)$, quantitatem $V\varrho\sin\varphi$ post evolutionem solummodo terminos formae $Z\sin\varkappa\gamma$ producere posse. Habemus igitur ad terminos hos tollendos aequationes has

$$\left.\begin{aligned} o &= -2\frac{e}{a(1-e^2)}V\varrho\sin\varphi - 2\Pi_{,}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)} - \Pi_{,}'(\zeta,t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)} \\ o &= \frac{e}{a(1-e^2)}V\varrho\sin\varphi + \Pi_{,}(\zeta,t)\frac{\varrho e\sin\varphi}{a(1-e^2)} - \Gamma_{,}(\zeta,t) \end{aligned}\right\}\ldots\ldots(7)$$

e quibus, eliminata $\Pi_{,}(\zeta,t)$, elicitur

$$o = 2\,\Gamma_{,}(\zeta, t) + \Pi'_{,}(\zeta, t)\frac{\left(\frac{d\zeta}{d\tau}\right)}{\left(\frac{d\lambda}{d\tau}\right)}$$

cui aequationi satisfactum erit, si posueris

(8)...... $$o = \Pi'_{,}(\zeta, t) \text{ atque } o = \Gamma_{,}(\zeta, t)$$

unde aequationes (7) suppeditant

(9)...... $$\Pi_{,}(\zeta, t) = -V$$

quae igitur aequatio terminos omnes formae Z sin $\varkappa\gamma$ tollit. Praeterea aequationes (7) monstrant, non modo in approximatione prima ad valores veros perturbationum obtinendos instituenda terminos formae Z sin $\varkappa\gamma$ per valorem ipsius $\Pi_{,}(\zeta, t)$ sub (9) datum evanescere, sed eandem rem etiam locum habere in approximationibus omnibus subsequentibus. Nam τ solummodo in quantitatibus ϱ atque φ continetur; itaque aequationes (7), (8) et (9) non modo locum habent, si valores approximati variabilium in expressionibus ipsarum T et R, sed etiam si valores earum accurati substituti erunt.

5.

Aequationes (8) alio modo exhibitae sunt hae

$$\frac{d^2\,\Pi(\zeta, t)}{d\zeta\, dt} = o\,,\quad \frac{d\Gamma(\zeta, t)}{dt} = o$$

quae integratae suppeditant

$$\Pi(\zeta, t) = \Lambda\zeta + \psi t$$
$$\Gamma(\zeta, t) = \Xi\zeta$$

denotantibus Λ, ψ et Ξ functiones arbitrarias quantitatum quibus praefixae sunt.

Hinc evadit

$$\frac{d\Pi(\zeta, t)}{dt} = \Pi_{,}(\zeta, t) = \psi t$$

denotante $\psi_{,} t$ quotientem differentialem functionis ψt. Aequatio vero (9) monstrat $\Pi_{,}(\zeta, t)$ constantem esse, itaque $\psi_{,} t$ quoque constanti aequalis est, quam $(n)\, y$ appellabo.

Quibus statutis, expressiones ipsarum T et R supra datae abeunt in has

$$T=\left\{2\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+2\frac{1}{a(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$+\;2\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}\,r\left(\frac{d\Omega}{dr}\right)-2(n)y\frac{\varrho e\sin\varphi}{a(1-e^2)}$$
$$R=-\left\{\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+\frac{\varrho}{a(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$-\;\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}\,r\left(\frac{d\Omega}{dr}\right)+(n)y\frac{\varrho e\sin\varphi}{a(1-e^2)}$$

et aequationes (2) et (3) transeunt in has

$$\begin{aligned}\lambda &= \Lambda\zeta+(n)yt\\ l\varrho &= \Xi\zeta+\beta\\ v_{,} &= \Lambda z+(n)yt\\ lr &= \Xi z+w\end{aligned}$$

Simili modo supponitur in motu Solis esse

$$\begin{aligned}v_{,}' &= \Lambda z'+(n)y't\\ lr' &= \Xi z'+w'\end{aligned}$$

Expressio finita eius ipsius Ω partis, quae terminos profert huius formae

$$\mathrm{X}\cos(ig+i'g'+i''\pi+i'''\pi')$$

est haec

$$\Omega=\frac{m'}{M+m}\left\{\frac{1}{(r^2+r'^2-2rr'\cos(v_{,}-v_{,}'))^{\frac{1}{2}}}-\frac{r}{r'^2}\cos(v_{,}-v_{,}')\right\}$$

quae, factis inclinationibus cifrae aequalibus, ex expressione ipsius Ω in art. 1. data perfacile derivatur, et quum in approximatione prima ad valores veros perturbationum obtinendos valores approximati coordinatarum in hac expressione substitui debeant, statim $f+\pi+(n)yt$ loco $v_{,}$, $f'+\pi'+(n)y't$ loco $v_{,}'$ et valores pure ellipticos radiorum vectorum loco r et r' substituere nobis licet. Itaque habemus in approximatione prima

$$\Omega=\frac{m'}{M+m}\left\{\frac{1}{[r^2+r'^2-2rr'\cos(f-f'+\pi+(n)yt-\pi'-(n)y't)]^{\frac{1}{2}}}-\frac{r}{r'^2}\cos(f-f'+\pi+(n)yt-\pi'-(n)y't)\right\}$$

cuius terminus generalis post evolutionem in seriem infinitam est huius formae

$$\mathrm{X}\cos(ig+i'g'+i''(\pi+(n)yt)+i'''(\pi'+(n)y't))$$

Expressio igitur ipsius T evoluta, si praeterea $\varphi + \varkappa + (\varkappa) yt$ loco λ et valor pure ellipticus ipsius ϱ substitutus fuerit, evadet

$$T = A \sin \gamma + B \sin 2\gamma + \text{etc.} - ya \sin \gamma - yb \sin 2\gamma - \text{etc.}$$
$$+ \Sigma Z \sin [\varkappa\gamma + ig + i'g' + i'' (\pi + (n) yt) + i''' (\pi' + (n) y't)]$$

ubi A, B, etc. a, b, etc. et Z coefficientes determinati sunt, quarum valores numerici semper computari possunt, et ubi in termino sub signo summationis casus in quo simul $i = o$, $i' = o$, $i'' = o$ et $i''' = o$ sunt, excludendus est, quia terminos ad hunc pertinentes iam separatim adscripsi. Sed aequationes (7) monstrant necessario esse debere

$$\frac{A}{a} = \frac{B}{b} = \text{etc.}$$

unde posita aequatione hac

$$y = \frac{A}{a}$$

termini omnes formae $Z \sin \varkappa\gamma$ sublati erunt.

Integrata igitur expressio praecedens praebet, quia $g = nt + c$ et $g' = n't + c'$,

$$\int Tdt = -\Sigma \frac{Z}{in + i'n' + i''(n)y + i'''(n)y'} \cos [\varkappa\gamma + ig + i'g' + i'' (\pi + (n)yt) + i''' (\pi' + (n)y't)]$$

ubi casus in quo simul $i = o$, $i' = o$, $i'' = o$, et $i''' = o$ sit, non adest; et similem expressionem nanciscimur pro $\int Rdt$.

Manifestum igitur est valorem praecedentem ipsius $\int Tdt$ nec non valorem analogum ipsius $\int Rdt$ terminos per tempus ipsum et per sinus aut cosinus multiplicium anomaliarum mediarum multiplicatos continere non posse, nisi forte pro valoribus specialibus ipsarum i, i', etc. haberetur

$$in + i'n' + i''(n) y + i'''(n) y' = o$$

quem vero casum, quum in motu Lunae locum non habeat, hoc loco non tractabo.

6.

Ut ex expressionibus ipsarum $\int Tdt$ et $\int Rdt$ ipsae ζ et β indagentur, necesse est, functiones adhuc arbitrariae $\Delta\zeta$ et $\Xi\zeta$ determinentur; sed cum functionibus his iam inhaereat conditio, ut t et τ explicite non contineant,

(quia functiones solius variabilis ζ sunt,) integrationes ad ζ et β ex $\int Tdt$ et $\int Rdt$ derivandas requisitae ipsas τ et t extra signa sinuum et cosinuum gignere non possunt; itaque pro ζ et β obtinentur series pure periodicae eiusdem formae, ac series supra pro $\int Tdt$ allata.

Valoribus ipsarum ζ et β in approximatione prima computatis, valores magis approximati ipsarum λ, ϱ, $v_{,}$, r obtinebuntur, si substituas valores illos ipsarum ζ, z, in functionibus $\varLambda\zeta$ et $\Xi\zeta$. Iam quum in approximatione prima $\varphi + \pi + (n)yt$ loco λ, $f + \pi + (n)yt$ loco $v_{,}$, $f' + \pi' + (n)y't$ loco $v_{,}'$, et valores pure elliptici radiorum vectorum, quos resp. (ϱ), (r) et (r') denotabo, substituti fuerint, in approximatione secunda differentiae $\varLambda\zeta - \varphi$ ad λ, $\varLambda z - f$ ad $v_{,}$, $\varLambda z' - f'$ ad $v_{,}'$, $\Xi\zeta + \beta - l(\varrho)$ ad $l\varrho$, $\Xi z + w - l(r)$ ad lr et $\Xi z' + w' - l(r')$ ad lr' addendae sunt. Quod idem est ac si in approximatione secunda quantitatibus illis incrementa $\delta\lambda$, $\delta v_{,}$, etc. attribuas, et ponas

$$\begin{aligned}
\delta\lambda &= \varLambda\zeta - \varphi\ , & \delta l\varrho &= \Xi\zeta + \beta - l(\varrho)\\
\delta v_{,} &= \varLambda z - f\ , & \delta lr &= \Xi z + w - l(r)\\
\delta v_{,}' &= \varLambda z' - f'\ , & \delta lr' &= \Xi z' + w' - l(r')
\end{aligned}$$

ubi in functionibus $\varLambda$ et Ξ valores ipsarum ζ et z substituendi sunt, qui in approximatione prima inveniebantur. Quo facto $\delta\lambda$, $\delta v_{,}$, etc. in T substituendae sunt, id quod commodissime ope theorematis Tayloriani perficitur, unde necessario termini eiusdem formae ac termini approximationis primae prodire debent, ita ut habeatur

$$\begin{aligned}
\delta T = A' \sin\gamma + B' \sin 2\gamma + \text{etc.} - \delta y\, a' \sin\gamma - \delta y\, b' \sin 2\gamma - \text{etc.}\\
+ \Sigma Z' \cos\left[\varkappa\gamma + ig + i'g' + i''(\pi + (n)yt) + i'''(\pi' + (n)y't)\right]
\end{aligned}$$

denotantibus δT et δy incrementa, quae T et y in approximatione secunda capiunt; et similem expressionem nanciscimur pro δR. Aequationes vero (7) probant semper esse debere

$$\frac{A'}{a'} = \frac{B'}{b'} = \text{etc.}$$

unde

$$\delta y = \frac{A'}{a'}$$

et quum valor approximatus ipsius y in approximatione prima inventus ipsi $\frac{A}{a}$ aequalis sit, habetur valor accuratior hic

$$y = \frac{A}{a} + \frac{A'}{a'}$$

Expressio vero praecedens pro δT praebet instituta integratione

$$\int \delta T dt = -\frac{Z'}{in + i'n' + i''(n)y + i'''(n)y'} \cos\left[\varkappa\gamma + ig + i'g' + i''(\pi + (n)yt) + i'''(\pi' + (n)y't)\right]$$

et similem expressionem obtinebimus pro $\int \delta R dt$, unde approximatio quoque secunda valores ipsarum ζ et β nec non ipsarum z et w pure periodicos suppeditat, et eodem modo demonstrabitur approximationes subsequentes terminos per tempus ipsum multiplicatos in valoribus ipsarum ζ et β introducere non posse.

7.

Analysis praecedens supponit perturbationes Solis (sive terrae) eodem modo ac perturbationes Lunae computari aut computatas esse, id quod semper fieri potest. Nam in motu quoque planetarum quantitatem hoc loco y denotatam introducere nobis liceret, unde perturbationes eorum a terminis per tempus ipsum multiplicatis liberae obtinerentur. Quum vero hinc inde in longitudinibus radiisque vectoribus planetarum termini permagni, quorum periodus paullulum a revolutione planetae integra differret, orirentur, quorum coefficientes periodique propter incertitudinem, quae massis planetarum semper inhaeret, non accurati evaderent, quumque termini in motu planetarum et per tempus ipsum et per sinus aut cosinus anomaliarum mediarum multiplicati series infinitas rapidissime convergentes constituant: praestat in theoria quoque Lunae perturbationes Solis sub forma ea admitti, quam perturbationibus planetarum in disquisitionibus meis prioribus de hoc argumento attribui.

Quibus positis, approximatio quidem prima in motu Lunae terminos per tempus ipsum multiplicatos non profert, sed approximatio secunda et subsequentes in parte ea, quae ex coordinatis Solis originem ducit, terminos procreabunt, qui quidem et per tempus et per sinus aut cosinus multiplicium anomaliarum mediarum multiplicati, sed iidem sunt minutissimi, variationes quasi saeculares coefficientium perturbationum Lunae constituentes. Praeterea coordinatarum Solis valores accuratiores in $r\left(\frac{d\Omega}{dr}\right)$ substituti terminos suppeditant huius formae

$$\alpha t + \beta t^2 + \text{etc.}$$

qui ipsi V in art. 4. introductae adiungendi sunt, ita ut V omnino constans non sit, sed huius formae

$$V_{,} + V_{,,} t + V_{,,,} t^2 + \text{etc.}$$

ubi $V_{,}$, $V_{,,}$, $V_{,,,}$ etc. constantes. Hinc sequitur ut forma ipsius y futura sit haec

$$y_{,} + (n) y_{,,} t + (n)^2 y_{,,,} t^2 + \text{etc.}$$

ubi $y_{,}$, $y_{,,}$, $y_{,,,}$, etc. non minus quam (n) constantes sunt. Ergo, quum secundum art. 5. sit

$$\psi t = (n) \int y dt$$

loco $(n) yt$ in formulis praecedentibus substitui debet

$$(n) y_{,} t + \tfrac{1}{2} (n)^2 y_{,,} t^2 + \tfrac{1}{3} (n)^3 y_{,,,} t^3 \text{ etc.}$$

Porro in evoluta quantitate T adstabunt termini hi

$$k + k_{,} t + k_{,,} t^2 + \text{etc.}$$

qui ex coordinatis Solis in $\left(\frac{d\Omega}{dv_{,}}\right)$ substitutis proveniunt, atque in z terminos hos

$$kt + \tfrac{1}{2} k_{,} t^2 + \tfrac{1}{3} k_{,,} t^3 + \text{etc.}$$

gignent, ubi k, $k_{,}$, etc. constantes, qui termini omnes perturbationes constituunt sub nominibus variationis saecularis motus medii et perigaei Lunaris notas.

Ergo, licet terminos omnes per tempus ipsum et per temporis potestates multiplicatos amovere possemus, tamen terminos modo descriptos admittemus, quia series rapidissime convergentes constituunt, et quia introductio functionum pure periodicarum motus Solis, a quibus series hae originem trahunt, molesta et manca foret.

8.

Per analysin praecedentem terminos evitandos per tempus ipsum multiplicatos ab integralibus expressionum ipsarum T atque R, e quibus perturbationes longitudinis atque radii vectoris pendent, amovimus, et quidem calculum ita instituimus, ut functiones $\Delta\zeta$ et $\Xi\zeta$ arbitrariae manerent. Antequam has functiones aptissime determinandas aggredimur, considerationes nonnullae de integratione illarum expressionum generales adiiciantur necesse est. Posito

$$\frac{an}{\sqrt{1-e^2}} = h$$

erit, propter $n^2 a^3 = \varkappa\,(M+m)$

$$\frac{1}{a\,(1-e^2)} = \frac{h^2}{\varkappa\,(M+m)}$$

et expressiones art. 5. ipsarum T et R, si primo momento coefficientis ipsius y rationem non habemus, functiones sunt et variabilium independentium quinque λ, ϱ, h, p, q, ($v_{,}$ enim et r, quum e λ et ϱ tali modo pendeant, ut mutata τ in t ex his prodeant, pro variabilibus independentibus habendae non sunt,) ad Lunam ipsam pertinentium, et variabilium independentium quatuor $v_{,}'$, r', p', q' ad Solem spectantium. Itaque, ut expressiones ipsarum T et R complete integrari possint, necesse est valores magis magisque approximati quantitatum harum omnium λ, ϱ, h, p, q, $v_{,}$, r, $v_{,}'$, r', p', q' obtineantur. Iam valoribus eiusmodi ipsarum λ, ϱ, $v_{,}$ et r obtinendis expressiones ipsae ipsarum T et R inserviunt, quae integratae ζ et β praebent, e quibus illae derivantur; valores accurati ipsarum $v_{,}'$, r', p' et q' ex theoria motus terrae noti supponuntur; de valoribus eiusmodi ipsarum p et q infra loquar; restat vero, ut expressio derivetur, quae valorem accuratum ipsius h suppeditet. Habetur

$$\lambda_{,} = \frac{d\lambda}{d\tau} = \frac{na^2\sqrt{1-e^2}}{\varrho^2} = \frac{\varkappa\,(M+m)}{h\,\varrho^2}$$

hinc nanciscimur

$$lh = l.\varkappa(M+m) - l\lambda_{,} - 2\,l\varrho$$

denotante l logarithmum hyperbolicum quantitatis cui praefixa est. Differentiata hac aequatione respectu temporis, prodit

$$d\,lh = -\frac{\varrho^2}{na^2\sqrt{1-e^2}}\,d\lambda_{,} - 2\,dl\varrho$$

unde, substitutis valoribus ipsarum $d\lambda_{,}$ et $dl\varrho$ ex aequ. (1) desumendis, evadit

$$d\,lh = -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right)\right\}dt$$

quae per aequationem hanc

$$\left(\frac{d\Omega}{dv_{,}}\right) = \left(\frac{d\Omega}{d\chi}\right) + \left(\frac{d\Omega}{d\lambda}\right)$$

in art. 3. inventam transfertur in hanc

$$dlh = -\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)dt$$

Si igitur statuimus

$$S = \int\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)dt$$

denotabit — S perturbationes ipsius lh, quibus ad expressiones ipsarum T et R complete integrandas opus est.

Si nunc ad coefficientes ipsius y in expressionibus ipsarum T et R respicimus, manifestum est, hos coefficientes quantitates e et φ continere, quae pro functionibus ipsarum λ, ϱ et h habendae sint, et ope harum quantitatum eliminari debeant. Quae eliminatio perfacile peragitur, nam

$$\frac{d.\varrho^2}{d\tau} = 2\,\frac{na\varrho e\sin\varphi}{\sqrt{1-e^2}}$$

unde coefficientes de quibus agitur hanc induunt formam

$$-\frac{(n)}{na^2\sqrt{1-e^2}}\,\frac{d.\varrho^2}{d\tau}\ \text{et}\ +\frac{(n)}{2na^2\sqrt{1-e^2}}\,\frac{d.\varrho^2}{d\tau}$$

quae propter aequationem

$$\frac{1}{na^2\sqrt{1-e^2}} = \frac{h}{\varkappa\,(M+m)}$$

functiones variabilium independentium ϱ et h, absque variabilibus dependentibus, sese praestant. Expressiones igitur nostrae ipsarum T et R denique ita se habent,

$$\begin{aligned}T = &\left\{2\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+2\frac{\varrho}{a(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)\\ &+2\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}\,r\left(\frac{d\Omega}{dr}\right)-\frac{(n)y}{na^2\sqrt{1-e^2}}\cdot\frac{d.\varrho^2}{d\tau}\\ R = -&\left\{\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+\frac{\varrho}{a(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)\\ &-\frac{\varrho}{r}\sin(v_{,}-\lambda)\frac{an}{\sqrt{1-e^2}}\,r\left(\frac{d\Omega}{dr}\right)+\frac{(n)y}{2na^2\sqrt{1-e^2}}\cdot\frac{d.\varrho^2}{d\tau}\end{aligned}$$

et praeterea erit nobis

$$\frac{d.\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}}{d\tau}=T$$

$$\frac{d\beta}{dt}-\left(\frac{d\beta}{d\tau}\right)\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}=R\,.$$

9.

Manentibus functionibus $\varDelta\zeta$ et $\Xi\zeta$ arbitrariis, relationes nonnullas generales, quibus postea utemur, assignare licet. Facile perspicitur relationem antea a me datam hanc

$$2R+T=\frac{dS}{dt}$$

quae propter aequationes ultimas art. praec. abit in hanc

$$\frac{d.\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}}{d\tau}+2\left(\frac{d\beta}{dt}\right)-2\left(\frac{d\beta}{d\tau}\right)\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}=\frac{dS}{dt}$$

in hac quoque theoria locum habere, cuius igitur integrale hoc *)

$$2\beta+l\left(\frac{d\zeta}{d\tau}\right)-S=f\zeta$$

ubi l logarithmum hyperbolicum et $f\zeta$ functionem arbitrariam ipsius ζ denotat, etiam locum habet. Sed aequationes hae

$$\overline{\left(\int T d\tau\right)}=o\;,\quad\overline{\left(\frac{d\zeta}{d\tau}\right)}=\frac{dz}{dt}\;,\quad\overline{\left(\frac{d\beta}{d\tau}\right)}=\frac{dw}{dt}$$

ubi linea superposita τ in t mutandam esse denotat, quas in theoria motus planetarum inveneram, in hac Lunae theoria locum non habent. Ut aequationes eruantur, quae earum vice funguntur, aequationes has

*) Vide: Unters. über die gegenseitigen Störungen des Jupiters und Saturns art. 27.

$$\begin{aligned}
\lambda &= \Lambda\zeta + \psi t \\
l\varrho &= \Xi\zeta + \beta \\
v_{,} &= \Lambda z + \psi t \\
lr &= \Xi z + w
\end{aligned}$$

resumo, quae differentiatae, et postquam τ in t mutata erit, suppeditant

$$\left.\begin{aligned}
\overline{\left(\frac{d\lambda}{d\tau}\right)} &= \left(\frac{d\,.\,\Lambda z}{dz}\right)\overline{\left(\frac{d\zeta}{d\tau}\right)} \\
\frac{dv_{,}}{dt} &= \left(\frac{d\,.\,\Lambda z}{dz}\right)\left(\frac{dz}{dt}\right) + (n)y \\
\overline{\left(\frac{dl\varrho}{d\tau}\right)} &= \left(\frac{d\,.\,\Xi z}{dz}\right)\overline{\left(\frac{d\zeta}{d\tau}\right)} + \overline{\left(\frac{d\beta}{d\tau}\right)} \\
\frac{dlr}{dt} &= \left(\frac{d\,.\,\Xi z}{dz}\right)\left(\frac{dz}{dt}\right) + \frac{dw}{dt}
\end{aligned}\right\}\text{......}(9^*)$$

Sed aequatio $r^2 = x^2 + y^2 + z^2$ monstrat r nec non lr esse functionem coordinatarum x, y, z, absque quotientibus earum differentialibus respectu temporis $x_{,}$, $y_{,}$, $z_{,}$, et absque elementis ellipticis. Aequationes porro

$$r \cos f = \xi \;,\; r \sin f = \eta$$

in art. 16. Sect. I. introductae differentiatione subministrant

$$r^2 df = \xi d\eta - \eta d\xi$$

Ex theoria vero transformationis coordinatarum notum est, aequationes (29) Sect. I. reciproce suppeditare debere

$$\begin{aligned}
\xi &= \alpha x + \alpha_{,} y + \alpha_{,,} z \\
\eta &= \beta x + \beta_{,} y + \beta_{,,} z
\end{aligned}$$

unde

$$\begin{aligned}
d\xi &= \alpha dx + \alpha_{,} dy + \alpha_{,,} dz + x d\alpha + y d\alpha_{,} + z d\alpha_{,,} \\
d\eta &= \beta dx + \beta_{,} dy + \beta_{,,} dz + x d\beta + y d\beta_{,} + z d\beta_{,,}
\end{aligned}$$

Substitutis his ipsarum $d\xi$ atque $d\eta$ valoribus in aequatione pro df modo inventa, emergit

$$\begin{aligned}
r^2 df = {}& (\xi\beta - \eta\alpha)\, dx + (\xi\beta_{,} - \eta\alpha_{,})\, dy + (\xi\beta_{,,} - \eta\alpha_{,,})\, dz \\
& + \xi x d\beta + \xi y d\beta_{,} + \xi z d\beta_{,,} - \eta x d\alpha - \eta y d\alpha_{,} - \eta z d\alpha_{,,}
\end{aligned}$$

Quae aequatio, si ex terminis sex ultimis x, y, z ope aequationum (29) Sect. I. eliminatae fuerint, et si differentialia aequationum conditionalium

(31) Sect. I., aequationem $\zeta = o$ .ibi statutam et aequationem $r^2 = \xi^2 + \eta^2$ respexeris, abit in hanc

$$df = \frac{\xi\beta - \eta\alpha}{r^2}\,dx + \frac{\xi\beta_{,} - \eta\alpha_{,}}{r^2}\,dy + \frac{\xi\beta_{,,} - \eta\alpha_{,,}}{r^2}\,dz$$
$$- \beta d\alpha - \beta_{,} d\alpha_{,} - \beta_{,,} d\alpha_{,,}$$

Aequatio vero prima (35) Sect. I. suppeditat

$$d\chi = \beta d\alpha + \beta_{,} d\alpha_{,} + \beta_{,,} d\alpha_{,,}$$

unde

$$dv_{,} = df + d\chi = \frac{\xi\beta - \eta\alpha}{r^2}\,dx + \frac{\xi\beta_{,} - \eta\alpha_{,}}{r^2}\,dy + \frac{\xi\beta_{,,} - \eta\alpha_{,,}}{r^2}\,dz$$

Quae aequatio *) probat, quantitatem $v_{,}$ non minus quam lr functionem esse coordinatarum x, y, z, absque $x_{,}$, $y_{,}$, $z_{,}$ et elementis ellipticis, quare aequatio haec

$$\frac{dL}{dt} = \overline{\left(\frac{dA}{d\tau}\right)}$$

in fine art. 14. Sect. I. demonstrata et ad $v_{,}$ et ad lr applicari potest. Habemus igitur

$$\frac{dv_{,}}{dt} = \overline{\left(\frac{d\lambda}{d\tau}\right)}\,,\quad \frac{dlr}{dt} = \overline{\left(\frac{dl\varrho}{d\tau}\right)}$$

Quibus aequationibus adiuvantibus, aequationes (9*) statim suppeditant

(10)...... $$\frac{dz}{dt} = \overline{\left(\frac{d\zeta}{d\tau}\right)} - \frac{(n)y}{\left(\frac{d.\,Az}{dz}\right)}$$

$$\left(\frac{d.\,\Xi z}{dz}\right)\overline{\left(\frac{d\zeta}{d\tau}\right)} + \overline{\left(\frac{d\beta}{d\tau}\right)} = \left(\frac{d.\,\Xi z}{dz}\right)\left(\frac{dz}{dt}\right) + \frac{dw}{dt}$$

quarum posterior transit in hanc

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*) Si ope aequationum art. 16. Sect. I. quantitates ξ, η, α, β, $\alpha_{,}$, etc. ex hac aequatione eliminantur, facili opera invenitur aequatio haec

$$r^2\,dv_{,}^2 = dx^2 + dy^2 + dz^2$$

qua concluditur, $dv_{,}$ esse angulum inter radium vectorem tempori t et radium vectorem tempori $t + dt$ respondentem interceptum, id quod in Astr. Nachr. No. 244. iam alio modo demonstravi.

$$\frac{dw}{dt} = \overline{\left(\frac{d\beta}{d\tau}\right)} + (n)y\,\frac{\left(\frac{d\,.\,\mathfrak{B}z}{dz}\right)}{\left(\frac{d\,.\,\Lambda z}{dz}\right)} \qquad \ldots\ldots(11)$$

itaque ad $\frac{dz}{dt}$ ex $\frac{d\zeta}{d\tau}$ eliciendam mutetur τ in t et subtrahatur quantitas $\frac{(n)y}{\left(\frac{d\,.\,\Lambda z}{dz}\right)}$, et ad $\frac{dw}{dt}$ ex $\frac{d\beta}{d\tau}$ eliciendam mutetur τ in t et addatur quantitas $(n)y\,\frac{\left(\frac{d\,.\,\mathfrak{B}z}{dz}\right)}{\left(\frac{d\,.\,\Lambda z}{dz}\right)}$.

10.

Aequatio haec

$$d\,.\,\frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)} = T d\tau$$

integrata subministrat

$$\frac{d\zeta}{dt} = \left(\frac{d\zeta}{d\tau}\right) \int T d\tau + \left(\frac{d\zeta}{d\tau}\right).\,\text{const.}$$

ubi constans integrationi adiecta functio ipsius t esse potest. Integrale $\int T d\tau$ est functio aliqua ipsarum τ et t, quae semper pluribus deinceps approximationibus quam accuratissime determinari potest. Quam functionem per $\varphi(\tau, t)$ reddemus, unde

$$\frac{d\zeta}{dt} = \left(\frac{d\zeta}{d\tau}\right) \varphi(\tau, t) + \left(\frac{d\zeta}{d\tau}\right).\,\text{const.}$$

nec non

$$\overline{\left(\frac{d\zeta}{dt}\right)} = \overline{\left(\frac{d\zeta}{d\tau}\right)}\,\varphi(t, t) + \overline{\left(\frac{d\zeta}{d\tau}\right)}.\,\text{const.}$$

Statuamus, mutata τ in t, quotientem differentialem $\frac{d\zeta}{dt}$ in functionem aliquam ipsius t abire, quae per χt designetur; quaeritur indoles huius functionis?

Conditio introducta subministrat

$$\overline{\left(\frac{d\zeta}{dt}\right)} = \chi t$$

quae aequatio cum praecedente comparata suppeditat

$$\text{const.} = \frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)} - \varphi(t, t)$$

unde nanciscimur

$$\frac{d\zeta}{dt} = \left(\frac{d\zeta}{d\tau}\right)\varphi(\tau, t) - \left(\frac{d\zeta}{d\tau}\right)\varphi(t, t) + \left(\frac{d\zeta}{d\tau}\right)\frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}$$

hinc integrando elicitur

$$\zeta = \int\left(\frac{d\zeta}{d\tau}\right)\varphi(\tau, t)\,dt - \int\left(\frac{d\zeta}{d\tau}\right)\varphi(t, t)\,dt + \int\left(\frac{d\zeta}{d\tau}\right)\frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}\,dt$$

ubi, constantem adiiciendam sub signis integrationis contentam esse, censeri potest. Ex hac aequatione mutata τ in t evadit

$$z = \int\left(\frac{d\zeta}{d\bar{\tau}}\right)\varphi(\bar{\tau}, t)\,dt - \int\left(\frac{d\zeta}{d\bar{\tau}}\right)\varphi(t, t)\,dt + \int\left(\frac{d\zeta}{d\bar{\tau}}\right)\frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}\,dt$$

ubi $\bar{\tau}$ significat, post integrationem respectu ipsius t peractam, τ in t mutandam esse. Itaque $\left(\frac{d\zeta}{d\bar{\tau}}\right)$ quidem aequalis est ipsi $\overline{\left(\frac{d\zeta}{d\tau}\right)}$ et $\varphi(\bar{\tau}, t)$ ipsi $\varphi(t, t)$, sed $\int\left(\frac{d\zeta}{d\bar{\tau}}\right)dt$ ipsi $\int\overline{\left(\frac{d\zeta}{d\tau}\right)}dt$ et $\int\varphi(\bar{\tau}, t)\,dt$ ipsi $\int\varphi(t, t)\,dt$ aequales esse cave credas.

Aequatio praecedens valorem ipsius ζ suppeditans praebet differentiata respectu ipsius τ hanc

$$\begin{aligned}\frac{d\zeta}{d\tau} = {} & \int\left(\frac{d^2\zeta}{d\tau^2}\right)\varphi(\tau, t)\,dt + \int\left(\frac{d\zeta}{d\tau}\right)\frac{d.\varphi(\tau, t)}{d\tau}\,dt \\ & - \int\left(\frac{d^2\zeta}{d\tau^2}\right)\varphi(t, t)\,dt + \int\left(\frac{d^2\zeta}{d\tau^2}\right)\frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}\,dt\end{aligned}$$

unde mutata τ in t evadit

$$\begin{aligned}\overline{\left(\frac{d\zeta}{d\tau}\right)} = {} & \int\left(\frac{d^2\zeta}{d\bar{\tau}^2}\right)\varphi(\bar{\tau}, t)\,dt + \int\left(\frac{d\zeta}{d\bar{\tau}}\right)\frac{d.\varphi(\bar{\tau}, t)}{d\bar{\tau}}\,dt \\ & - \int\left(\frac{d^2\zeta}{d\bar{\tau}^2}\right)\varphi(t, t)\,dt + \int\left(\frac{d^2\zeta}{d\bar{\tau}^2}\right)\frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}\,dt\end{aligned}$$

aequatio vero praecedens valorem ipsius z suppeditans praebet differentiata hanc

$$\frac{dz}{dt} = \left(\frac{d\zeta}{d\bar{\tau}}\right) \varphi(\bar{\tau}, t) + \int \left(\frac{d^2\zeta}{d\bar{\tau}^2}\right) \varphi(\bar{\tau}, t)\, dt + \int \left(\frac{d\zeta}{d\bar{\tau}}\right) \frac{d.\varphi(\bar{\tau}, t)}{d\bar{\tau}}\, dt$$
$$- \left(\frac{d\zeta}{d\bar{\tau}}\right) \varphi(t, t) - \int \left(\frac{d^2\zeta}{d\bar{\tau}^2}\right) \varphi(t, t)\, dt$$
$$+ \left(\frac{d\zeta}{d\bar{\tau}}\right) \frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)} + \int \left(\frac{d^2\zeta}{d\bar{\tau}^2}\right) \frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}\, dt$$

quae propter aequationes

$$\left(\frac{d\zeta}{d\bar{\tau}}\right) = \overline{\left(\frac{d\zeta}{d\tau}\right)} \text{ atque } \varphi(\bar{\tau}, t) = \varphi(t, t)$$

transit in hanc

$$\frac{dz}{dt} = \int \left(\frac{d^2\zeta}{d\bar{\tau}^2}\right) \varphi(\bar{\tau}, t)\, dt + \int \left(\frac{d\zeta}{d\bar{\tau}}\right) \frac{d.\varphi(\bar{\tau}, t)}{d\bar{\tau}}\, dt$$
$$- \int \left(\frac{d^2\zeta}{d\bar{\tau}^2}\right) \varphi(t, t)\, dt + \int \left(\frac{d^2\zeta}{d\bar{\tau}^2}\right) \frac{\chi t}{\left(\frac{d\zeta}{d\tau}\right)}\, dt + \chi t$$

quae cum valore praecedenti ipsius $\overline{\left(\frac{d\zeta}{d\tau}\right)}$ comparata monstrat esse

$$\frac{dz}{dt} = \overline{\left(\frac{d\zeta}{d\tau}\right)} + \chi t$$

aequatio vero (10) suppeditavit

$$\frac{dz}{dt} = \overline{\left(\frac{d\zeta}{d\tau}\right)} - \frac{(n)y}{\left(\frac{d.\Delta z}{dz}\right)}$$

quare esse debet

$$\chi t = - \frac{(n)y}{\left(\frac{d.\Delta z}{dz}\right)}$$

Q. E. I.

Hinc sequitur, constantem integrali $\int T d\tau$ addendam ita determinandam esse, ut quotiens differentialis $\frac{d\zeta}{dt}$, qui aequalis est producto $\left(\frac{d\zeta}{d\tau}\right) \int T d\tau$, mutata τ in t, aequalis fiat ipsi $- \frac{(n)y}{\left(\frac{d.\Delta z}{dz}\right)}$, quae conditio ita exprimitur

$$\overline{\left(\frac{d\zeta}{dt}\right)} = -\frac{(n)y}{\left(\frac{d.\Delta z}{dz}\right)}$$

Aequatio haec in theoria nostra Lunae aequationis huius $\overline{(\int T d\tau)} = 0$, quae ad theoriam planetarum pertinet, vice fungitur, nec nisi casus specialis illius est.

11.

Aequatio haec

$$2\beta + l\left(\frac{d\zeta}{d\tau}\right) - S = f\zeta$$

in art. 9. inventa praebet differentiata respectu ipsius τ et mutata post differentiationem τ in t, hanc

$$\overline{\left(\frac{d\beta}{d\tau}\right)} = \tfrac{1}{2}\frac{d.fz}{dz}\overline{\left(\frac{d\zeta}{d\tau}\right)} - \tfrac{1}{2}\overline{\left(\frac{\left(\frac{d^2\zeta}{d\tau^2}\right)}{\left(\frac{d\zeta}{d\tau}\right)}\right)}$$

unde ope aequationum (10) et (11) nanciscimur

$$dw = \frac{\left(\frac{d.\Xi z}{dz}\right)}{\left(\frac{d.\Delta z}{dz}\right)}(n)ydt + \tfrac{1}{2}\frac{\left(\frac{d.fz}{dz}\right)}{\left(\frac{d.\Delta z}{dz}\right)}(n)ydt + \tfrac{1}{2}\frac{d.fz}{dz}dz - \tfrac{1}{2}\overline{\left(\frac{\left(\frac{d^2\zeta}{d\tau^2}\right)}{\left(\frac{d\zeta}{d\tau}\right)}\right)}dt$$

itaque

(12)...... $$w = \text{const.} + \int\frac{\Xi_{,}z}{\Delta_{,}z}(n)ydt + \tfrac{1}{2}\int\frac{f_{,}z}{\Delta_{,}z}(n)ydt + \tfrac{1}{2}fz - \tfrac{1}{2}\int\overline{\left(\frac{\left(\frac{d^2\zeta}{d\tau^2}\right)}{\left(\frac{d\zeta}{d\tau}\right)}\right)}dt$$

ubi constans adiecta vera constans est, et ubi $\Xi_{,}$, $\Delta_{,}$ et $f_{,}$ quotientes differentiales harum functionum respectu ipsius z denotant. Constans vero adiecta adiumento aequationis huius

$$w = \tfrac{1}{2}S - \tfrac{1}{2}l\overline{\left(\frac{d\zeta}{d\tau}\right)} + \tfrac{1}{2}fz$$

determinanda est.

12.

Functiones $\varDelta\zeta$ et $\Xi\zeta$ multis modis diversis determinari possunt, et quidem a se invicem independentes sunt, functionem vero arbitrariam $f\zeta$ ad libitum determinare nobis non licet; haec enim ex illis pendet.

Facile ostenditur formam simplicissimam, quam functionibus $\varDelta\zeta$ et $\Xi\zeta$ attribuere nobis licet, computationem perturbationum simplicissimam sibi non conciliare. Positis

$$\varDelta\zeta = \zeta$$
$$\Xi\zeta = o$$

nihil simplicius assumi potest. Hinc habetur

$$\lambda = \zeta + (n)yt$$
$$l\varrho = \beta$$

unde manifestum est assumtionem hanc efficere, ut perturbationes immediate ad longitudinem veram et ad logarithmum radii vectoris applicandae sint. Aequationes vero praecedentes suppeditant

$$\frac{d\zeta}{d\tau} = \frac{d\lambda}{d\tau}$$
$$\frac{d\beta}{d\tau} = \frac{dl\varrho}{d\tau}$$

In approximatione igitur prima ad valores perturbationum computandos instituenda loco $\frac{d\zeta}{d\tau}$ valor pure ellipticus ipsius $\frac{d\lambda}{d\tau}$, et loco $\frac{d\beta}{d\tau}$ valor pure ellipticus ipsius $\frac{dl\varrho}{d\tau}$ in formulis

$$\zeta = \int dt \left(\frac{d\zeta}{d\tau}\right) \int T d\tau$$
$$\beta = \int dt \left(\frac{d\beta}{d\tau}\right) \int T d\tau + \int R dt$$

sive loco posterioris in aequatione (12) substituendus erit, et in approximationibus subsequentibus ii valores accuratiores ipsarum $\frac{d\lambda}{d\tau}$ et $\frac{dl\varrho}{d\tau}$, quos approximatio praecedens suppeditaverit, substitui debebunt. Constantes arbitrariae integralibus praecedentibus addendae erunt valores ii, quos λ et $l\varrho$ accipiunt, quoties vires perturbantes evanescunt, etc. etc.

Quum vero in casu eo, ubi functiones $\Lambda\zeta$ et $\Xi\zeta$ ita determinantur, ut perturbationes ad longitudinem mediam addendae sint, in approximatione prima nobis statuendam sit

$$\frac{d\zeta}{d\tau} = 1\ ,\ \frac{d\beta}{d\tau} = o$$

sicut ex theoria nostra planetarum iam notum est: assumtio haec non modo computationem perturbationum primi órdinis, sed etiam computationem perturbationum ordinum altiorum simpliciorem reddit, quare assumtionem huius articuli non amplius evolvam.

13.

Perturbationum ad longitudinem mediam applicandarum computationem evolvens, quaedam iis, quae antea de hoc argumento divulgavi, generaliora afferam, quae ad accuratissimam perturbationum computationem consequendam maximi momenti sunt.

Determinentur quantitates $\bar{\varrho}$ et $\bar{\varphi}$ per aequationes has

$$(13)\ldots\ldots\left\{\begin{aligned}(n)\,\zeta &= v - (e)\sin v\\ \bar{\varrho}\cos\bar{\varphi} &= (a)\cos v - (a)(e)\\ \bar{\varrho}\sin\bar{\varphi} &= (a)\sqrt{1-(e)^2}\,.\sin v\end{aligned}\right.$$

ubi lineola superposita quantitates $\bar{\varrho}$ et $\bar{\varphi}$ a quantitatibus simpliciter ϱ et φ denotatis discernit, itaque τ in t mutandam esse non designat, et ubi v angulus auxiliaris et (a), (e) atque (n) constantes quidem sunt, sed cum valoribus iis quos elementa a, e et n habent, si vis perturbans evanescit, non congruunt. Sint porro constantes (a) et (n) aequatione hac

$$(a)^3(n)^2 = \varkappa(M+m)$$

iunctae, ubi $\varkappa$, uti in Sectione prima, intensitatem vis attractivae, quoties et distantia et massa unitati aequales sunt, denotat. Pono nunc

$$\begin{aligned}\Lambda\zeta &= \bar{\varphi} + \pi\\ \Xi\zeta &= l\bar{\varrho}\end{aligned}$$

ubi π quoque constans est, et quasi longitudinem perigaei designat. Hinc nobis erit

$$\left.\begin{aligned} \lambda &= \bar{\varphi} + (n)\,yt + \pi \\ l\varrho &= l\bar{\varrho} + \beta \end{aligned}\right\} \quad \ldots\ldots(14)$$

quibus assumtionibus effecimus, ut perturbationes longitudinis ad longitudinem mediam applicandae sint, sicut iam in theoria planetarum protuli. Aequationes (14) praebent differentiatae

$$\frac{d\zeta}{d\tau} = \frac{\left(\frac{d\lambda}{d\tau}\right)}{\left(\frac{d\bar{\varphi}}{d\zeta}\right)}\,,\quad \frac{d\beta}{d\tau} = \frac{dl\varrho}{d\tau} - \frac{dl\bar{\varrho}}{d\zeta}\left(\frac{d\zeta}{d\tau}\right)$$

Sunt vero notae aequationes hae

$$\frac{d\lambda}{d\tau} = \frac{a^2 n\sqrt{1-e^2}}{\varrho^2}\,,\quad \frac{dl\varrho}{d\tau} = \frac{ane\sin\varphi}{\varrho\sqrt{1-e^2}}$$

Quae quum ita sint, aequationes (13) necessario suppeditare debent aequationes illis similes has

$$\frac{d\bar{\varphi}}{d\zeta} = \frac{(a)^2(n)\sqrt{1-(e)^2}}{\bar{\varrho}^2}\,,\quad \frac{dl\bar{\varrho}}{d\zeta} = \frac{(a)(n)(e)\sin\bar{\varphi}}{\bar{\varrho}\sqrt{1-(e)^2}}$$

Substitutis his valoribus ipsarum $\frac{d\lambda}{d\tau}$, $\frac{dl\varrho}{d\tau}$, $\frac{d\varphi}{d\zeta}$ et $\frac{dl\bar{\varrho}}{d\zeta}$ in aequationibus praecedentibus pro $\frac{d\zeta}{d\tau}$ et $\frac{d\beta}{d\tau}$, nanciscimur

$$\frac{d\zeta}{d\tau} = \frac{\bar{\varrho}^2 a^2 n\sqrt{1-e^2}}{\varrho^2 (a)^2 (n)\sqrt{1-(e)^2}} \quad \ldots\ldots(15)$$

$$\frac{d\beta}{d\tau} = \frac{ane\sin\varphi}{\varrho\sqrt{1-e^2}} - \frac{\bar{\varrho}\, a^2 n\sqrt{1-e^2}\,.\,(e)\sin\bar{\varphi}}{\varrho^2 (a)(1-(e)^2)} \quad \ldots\ldots(16)$$

ubi elementa a, e et n pro variabilibus et quidem pro functionibus ipsarum λ, $l\varrho$ et h habenda sunt.

14.

Si vis perturbans evanescit, ζ accipit valorem quendam, quem (ζ) appellabo. Sint $(\bar{\varphi})$, $(\bar{\varrho})$ et (v) valores ipsarum $\bar{\varphi}$, $\bar{\varrho}$ et v huic casui respondentes, ita ut habeatur

$$\left.\begin{aligned} (n)(\zeta) &= (v) - (e)\sin(v) \\ (\bar{\varrho})\cos(\bar{\varphi}) &= (a)\cos(v) - (a)(e) \\ (\bar{\varrho})\sin(\bar{\varphi}) &= (a)\sqrt{1-(e)^2}\,.\sin(v) \end{aligned}\right\} \quad \ldots\ldots(17)$$

Si porro λ_0 denotat valorem ipsius λ, qui locum habet, quoties vis perturbans evanescit, et ϱ_0 valorem eiusmodi ipsius ϱ, esse debet

$$(18)\ldots\ldots\left\{\begin{aligned} \lambda_0 &= (\overline{\varphi}) + \pi \\ l\varrho_0 &= l(\overline{\varrho}) + (\beta) \end{aligned}\right.$$

ubi (β) denotat valorem ipsius β eidem casui respondentem, et terminus $(n)yt$ omissus est, quia in hoc casu necessario esse debet $y = o$. Ut indoles quantitatum (ζ) et (β) indagetur, formulae praecedentes cum formulis ad motum pure ellipticum spectantibus comparandae sunt. Sunt autem hae

$$(19)\ldots\ldots\left\{\begin{aligned} a_0^3 n_0^2 &= \varkappa(M+m) \\ n_0\tau + c_0 &= v_0 - e_0 \sin v_0 \\ \varrho_0 \cos \varphi_0 &= a_0 \cos v_0 - a_0 e_0 \\ \varrho_0 \sin \varphi_0 &= a_0\sqrt{1-e^2} \,.\, \sin v_0 \end{aligned}\right.$$

$$(20)\ldots\ldots\left\{\begin{aligned} \lambda_0 &= \varphi_0 + \pi_0 \\ l\varrho_0 &= l\varrho_0 \end{aligned}\right.$$

designantibus a_0, e_0, n_0, c_0, π_0 valores elementorum a, e, n, c, π, qui locum habent, si vis perturbans evanescit, et ϱ_0, φ_0 atque v_0 valores ipsarum $\overline{\varrho}$, $\overline{\varphi}$ atque v huic casui respondentes.

Prior aequatio (18) suppeditat

$$\frac{d(\zeta)}{d\tau} = \frac{\left(\frac{d\lambda_0}{d\tau}\right)}{\left(\frac{d(\overline{\varphi})}{d(\zeta)}\right)}$$

sed ex (17) et ex (19) eliciuntur

$$\frac{d(\overline{\varphi})}{d(\zeta)} = \frac{(a)^2(n)\sqrt{1-(e)^2}}{(\overline{\varrho})^2}\,,\ \frac{d\lambda_0}{d\tau} = \frac{a_0^2 n_0\sqrt{1-e_0^2}}{\varrho_0^2}$$

hinc adipiscimur

$$(21)\ldots\ldots\qquad \frac{d(\zeta)}{d\tau} = \frac{n_0(\overline{\varrho})^2 a_0^2\sqrt{1-e_0^2}}{(n)\varrho_0^2(a)^2\sqrt{1-(e)^2}}$$

aequatio vero altera (18) suppeditat

$$(22)\ldots\ldots\qquad (\beta) = l\varrho_0 - l(\overline{\varrho})$$

quae quidem aequationes evolvendae sunt. Quem in finem animadverto aequationes (19) suppeditare debere

$$\frac{\varrho_0}{a_0} = \frac{1-e_0^2}{1+e_0\cos\varphi_0}$$

unde

$$\frac{\overline{(\varrho)}\,a_0}{\varrho_0\,(a)} = \frac{\overline{(\varrho)}}{(a)}\cdot\frac{1+e_0\cos\varphi_0}{1-e_0^2}$$

sed ex prioribus aequationibus (18) et (20) emergit

$$\varphi_0 = \overline{(\varphi)} + \pi - \pi_0$$

qua aequatio praecedens transit in hanc

$$\frac{\overline{(\varrho)}\,a_0}{\varrho_0\,(a)} = \frac{\frac{\overline{(\varrho)}}{(a)} + e_0\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)}\cos(\pi-\pi_0) - e_0\frac{\overline{(\varrho)}}{(a)}\sin\overline{(\varphi)}\sin(\pi-\pi_0)}{1-e_0^2}$$

Ex aequationibus vero (17) nanciscimur hanc

$$\frac{\overline{(\varrho)}}{(a)} = \frac{1-(e)^2}{1+(e)\cos\overline{(\varphi)}}$$

unde identica est haec

$$\frac{\overline{(\varrho)}}{(a)} = 1-(e)^2-(e)\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)}$$

Substituto hoc ipsius $\frac{\overline{(\varrho)}}{(a)}$ valore in primo membro ad dextram aequationis praecedentis, nanciscimur

$$\frac{\overline{(\varrho)}\,a_0}{\varrho_0\,(a)} = \frac{1-(e)^2+[e_0\cos(\pi-\pi_0)-(e)]\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)} - e_0\sin(\pi-\pi_0)\frac{\overline{(\varrho)}}{(a)}\sin\overline{(\varphi)}}{1-e_0{}^2}$$

Positis

$$\left.\begin{aligned} e_0\sin(\pi-\pi_0) &= \eta\,(1-(e)^2) \\ e_0\cos(\pi-\pi_0) &= (e)+\xi\,(1-(e)^2) \end{aligned}\right\}\text{....(22*)}$$

unde

$$e_0^2 = (e)^2+2(e)(1-(e)^2)\xi+(1-(e)^2)^2\xi^2+(1-(e)^2)^2\eta^2$$

aequatio praecedens abit in

$$\frac{\overline{(\varrho)}\,a_0}{\varrho_0\,(a)} = \frac{1+\xi\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)}-\eta\frac{\overline{(\varrho)}}{(a)}\sin\overline{(\varphi)}}{1-2(e)\xi-(1-(e)^2)\xi^2-(1-(e)^2)\eta^2} \qquad\text{......(23)}$$

Adiumento harum aequationum aequatio (21) transfertur in hanc

$$(24) \ldots\ldots \qquad \frac{d(\zeta)}{d\tau} = \frac{n_0}{(n)} \cdot \frac{\left\{1 + \xi \frac{\overline{(\varrho)}}{(a)} \cos \overline{(\varphi)} - \eta \frac{\overline{(\varrho)}}{(a)} \sin \overline{(\varphi)}\right\}^2}{\left\{1 - 2(e)\xi - (1-(e)^2)\xi^2 - (1-(e)^2)\eta^2\right\}^{\frac{3}{2}}}$$

Porro aequatio (23) una cum aequationibus

$$(a)^3 (n)^2 = \varkappa (M+m)$$
$$a_0^3 \; n_0^2 \;\; = \varkappa (M+m)$$

subministrat

$$l\varrho_0 - l\overline{(\varrho)} = \tfrac{2}{3} l \frac{(n)}{n_0} + l\left\{1 - 2(e)\xi - (1-(e)^2)\xi^2 - (1-(e)^2)\eta^2\right\}$$
$$- l\left\{1 + \xi \frac{\overline{(\varrho)}}{(a)} \cos \overline{(\varphi)} - \eta \frac{\overline{(\varrho)}}{(a)} \sin \overline{(\varphi)}\right\}$$

unde aequatio (22) abit in

$$(25) \ldots\ldots \left\{ \begin{aligned} (\beta) &= -\tfrac{2}{3} l(1-b) + l\left\{1 - 2(e)\xi - (1-(e)^2)\xi^2 - (1-(e)^2)\eta^2\right\} \\ &\quad - l\left\{1 + \xi \frac{\overline{(\varrho)}}{(a)} \cos \overline{(\varphi)} - \eta \frac{\overline{(\varrho)}}{(a)} \sin \overline{(\varphi)}\right\} \end{aligned} \right.$$

ubi posui

$$n_0 = (n)(1-b)$$

quae aequationes constantibus arbitrariis, quae integralibus $\int d\zeta$ et $\int d\beta$ addendae sunt, determinandis inserviunt.

15.

Aequationes (24) et (25) valores ipsarum $d(\zeta)$ et (β) terminis finitis expressos suppeditant. Quum vero suppono ξ, η atque b esse quantitates perparvulas ordinis vis perturbantis, aequationes illas in series infinitas secundum potestates productaque ipsarum ξ, η atque b progredientes evolvere nobis licebit. Neglectis cubis nec non productis trium dimensionum harum quantitatum, ex aequatione (24) facili opera elicitur

$$d(\zeta) = (1-b)\,d\tau + (1-b)\,\xi\left\{2\frac{\overline{(\varrho)}}{(a)} \cos \overline{(\varphi)} + 3(e)\right\} d\tau - 2(1-b)\,\eta \frac{\overline{(\varrho)}}{(a)} \sin \overline{(\varphi)}\, d\tau$$
$$+ \xi^2 \left\{\frac{\overline{(\varrho)}^2}{(a)^2} \cos^2 \overline{(\varphi)} + 6(e)\frac{\overline{(\varrho)}}{(a)} \cos \overline{(\varphi)} + \frac{3}{2}(1 + 4(e)^2)\right\} d\tau$$
$$- \xi\eta \left\{2\frac{\overline{(\varrho)}^2}{(a)^2} \cos \overline{(\varphi)} \sin \overline{(\varphi)} + 6(e)\frac{\overline{(\varrho)}}{(a)} \sin \overline{(\varphi)}\right\} d\tau + \eta^2\left\{\frac{\overline{(\varrho)}^2}{(a)^2} \sin^2 \overline{(\varphi)} + \frac{3}{2}(1-(e)^2)\right\} d\tau$$

ex qua integrata emergit

$$\left.\begin{aligned}(n)(\zeta) = {} & (n)(1-b)\tau + (c) + (1-b)\xi\int\left\{2\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)} + 3(e)\right\}(n)d\tau - 2(1-b)\eta\int\frac{\overline{(\varrho)}}{(a)}\sin\overline{(\varphi)}\,(n)d\tau \\ & + \xi^2\int\left\{\frac{\overline{(\varrho)}^2}{(a)^2}\cos^2\overline{(\varphi)} + 6(e)\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)} + \frac{3}{2}(1+4(e)^2)\right\}(n)\,d\tau \\ & - \xi\eta\int\left\{2\frac{\overline{(\varrho)}^2}{(a)^2}\cos\overline{(\varphi)}\sin\overline{(\varphi)} + 6(e)\frac{\overline{(\varrho)}}{(a)}\sin\overline{(\varphi)}\right\}(n)d\tau + \eta^2\int\left\{\frac{\overline{(\varrho)}^2}{(a)^2}\sin^2\overline{(\varphi)} + \frac{3}{2}(1-(e)^2)\right\}(n)\,d\tau\end{aligned}\right\}(26)$$

designante (c) constantem huic integrali addendam. Ex aequatione (25) in seriem evoluta emergit

$$\left.\begin{aligned}(\beta) = {} & \frac{2}{3}b + \frac{1}{3}b^2 - \xi\left\{\frac{\overline{(\varrho)}}{(a)}\cos\overline{(\varphi)} + 2(e)\right\} + \eta\frac{\overline{(\varrho)}}{(a)}\sin\overline{(\varphi)} + \xi^2\left\{\frac{1}{2}\frac{\overline{(\varrho)}^2}{(a)^2}\cos^2\overline{(\varphi)} - (1+(e)^2)\right\} \\ & - \xi\eta\frac{\overline{(\varrho)}^2}{(a)^2}\cos\overline{(\varphi)}\sin\overline{(\varphi)} + \eta^2\left\{\frac{1}{2}\frac{\overline{(\varrho)}^2}{(a)^2}\sin^2\overline{(\varphi)} - (1-(e)^2)\right\}\end{aligned}\right\}\ldots\ldots(27)$$

Integralia, quae haec ipsius (ζ) expressio continet, perfacile obtinentur, postquam $(\bar{\varrho})$, cos $(\overline{\varphi})$ et sin $(\overline{\varphi})$ in series evolutae erunt, id quod infra suscipiemus. Expressio haec ipsius (β) in usum vocanda esset, si expressionem ipsius R in art. 8. datam ad perturbationes logarithmi radii vectoris computandas adhiberemus; quum vero perturbationes has adiumento relationis (12) ex perturbationibus longitudinis computaturi simus, necesse est $f z$, quae in hac relatione continetur, evolvatur. Quae functio adiumento relationis determinanda est huius

$$2\beta + l\left(\frac{d\zeta}{d\tau}\right) - S = f\zeta$$

quae locum habet, quomodocunque functiones $\Lambda\zeta$ et $\Xi\zeta$ determinatae fuerunt. Quem in finem elicimus ex aequatione (15) et ex altera aequatione (14),

$$l\left(\frac{d\zeta}{d\tau}\right) = 2l\bar{\varrho} - 2l\varrho + l\,.\,a^2n\sqrt{1-e^2} - l\,.\,(a)^2(n)\sqrt{1-(e)^2}$$
$$2\beta = 2l\varrho - 2l\bar{\varrho}$$

quibus in aequatione praecedente substitutis, emergit

$$l\,.\,a^2n\sqrt{1-e^2} - l\,.\,(a)^2(n)\sqrt{1-(e)^2} - S = f\zeta$$

Sed quum secundum art. 8. $-S$ denotet perturbationes ipsius lh, habetur, denotante h_0 valorem pure ellipticum ipsius h, haec

$$S = lh_0 - lh$$

sive, propter aequationes

$$h = \frac{\varkappa\,(M+m)}{a^2 n \sqrt{1-e^2}}\;,\; h_0 = \frac{\varkappa\,(M+m)}{a_0^2 n_0 \sqrt{1-e_0^2}}$$

haec

$$S = l\,.\,a^2 n \sqrt{1-e^2} - l\,.\,a_0^2 n_0 \sqrt{1-e_0^2}$$

Substituto hoc valore ipsius S in aequatione pro $f\zeta$, emergit

$$f\zeta = 2\,l a_0 - 2\,l(a) + l n_0 - l(n) + \tfrac{1}{2} l(1-e_0^2) - \tfrac{1}{2} l(1-(e)^2)$$

Quae aequatio monstrat in casu nostro $f\zeta$ constantem esse, itaque $f\zeta = fz$. In praecedentibus obtinuimus aequationes has

$$(a)^3(n)^2 = a_0^3 n_0^2$$

$$n_0 = (n)(1-b)$$

$$1-e_0^2 = (1-(e)^2)\,[1-2(e)\xi-(1-(e)^2)\xi^2-(1-(e)^2)\eta^2]$$

quibus adiuvantibus aequatio praecedens transmutatur in

$$f\zeta = fz = -\tfrac{1}{3} l(1-b) + \tfrac{1}{2} l[1-2(e)\xi-(1-(e)^2)\xi^2-(1-(e)^2)\eta^2]$$

sive in seriem evoluta

(28)...... $$f\zeta = fz = \tfrac{1}{3} b + \tfrac{1}{6} b^2 - (e)\xi - \tfrac{1}{2}(1+(e)^2)\xi^2 - \tfrac{1}{2}(1-(e)^2)\eta^2$$

16.

Indole functionum $\Lambda\zeta$ et $\Xi\zeta$ in praecedentibus determinata, in modum expressionis ipsius T in art. 8. allatae integrandae ulterius nobis inquirendum est. Expressio ista, introducta h, ita se habet

$$T = \left\{2h\frac{\varrho}{r}\cos(v_{\prime}-\lambda) - h + 2\frac{h^3\varrho}{\varkappa(M+m)}[\cos(v_{\prime}-\lambda)-1]\right\}\left(\frac{d\Omega}{dv_{\prime}}\right)$$
$$+\; 2h\frac{\varrho}{r}\sin(v_{\prime}-\lambda)\, r_{\prime}\left(\frac{d\Omega}{dr}\right) - \frac{(n)\,y\,h}{\varkappa\,(M+m)}\cdot\frac{d\,.\,\varrho^2}{d\tau}$$

Quum functio perturbatrix Ω sit functio variabilium $v_{\prime}$, r, p, q, $v_{\prime}'$, r', p', q', expressio praecedens monstrat ipsam T esse functionem variabilium undecim λ, ϱ, h, $v_{\prime}$, r, p, q, $v_{\prime}'$, r', p', q', quarum novem λ, ϱ, h, p, q, $v_{\prime}'$, r', p', q' pro independentibus habentur; id quod in praecedentibus iam supposueram. Quum vero ope expressionis ipsius T et ope expressionum reliquarum, quae in calculum vocabuntur, loco λ et ϱ ipsae ζ et β computandae sint: praestat expressionem ipsius T nec non expressiones reliquas, quibus utemur, ita transformare, ut ζ et β loco λ et ϱ

explicite contineant. Quae transformatio perfacile ope aequationum (13) et (14) perficitur, illae enim quantitates $\overline{\varphi}$ et $\overline{\varrho}$ in functione ipsius ζ exhibent, et hae quantitates λ et ϱ in functione ipsarum $\overline{\varphi}$, $\overline{\varrho}$ et β exprimunt. Quodsi ponimus aequationes (13) in series infinitas secundum sinus et cosinus multiplicium arcus $(n)\zeta$ progredientes evolutas esse, aequationes (13) et (14) suppeditant

$$\lambda = (n)\zeta + (n)yt + \pi + A_1 \sin(n)\zeta + A_2 \sin 2(n)\zeta + A_3 \sin 3(n)\zeta + \text{etc.}$$
$$\varrho = (a)c^{\beta}\{B_0 + B_1 \cos(n)\zeta + B_2 \cos 2(n)\zeta + B_3 \cos 3(n)\zeta + \text{etc.}\}$$

ubi c basin logarithmorum hyperbolicorum, A_1, A_2, A_3, etc. coefficientes aequationis centri et B_0, B_1, B_2, etc. coefficientes evolutionis notae radii vectoris, adiumento excentricitatis (e) computandos, denotant. Quae aequationes mutata τ in t suppeditant

$$v_{,} = (n)z + (n)yt + \pi + A_1 \sin(n)z + A_2 \sin 2(n)z + A_3 \sin 3(n)z + \text{etc.}$$
$$r = (a)c^{w}\{B_0 + B_1 \cos(n)z + B_2 \cos 2(n)z + B_3 \cos 3(n)z + \text{etc.}\}$$

Substitutis his expressionibus, expressio ipsius T quantitates ζ, β, z et w loco quantitatum λ, ϱ, $v_{,}$ et r continet.

Porro quum perturbationes Solis sub forma ea, quam planetarum perturbationibus generaliter attribui, datas esse supposui: loco $v_{,}'$ et r' quantitates z' et w' immediate datae erunt, quamobrem expressio ipsius T etiam functio explicita ipsarum z' et w' reddi debet. Theoria nostra planetarum praebet

$$v_{,}' = \bar{f}' + \pi'$$

ubi $\bar{f}'$ functio ipsius z' est. Sed loco huius aequationis suppono in hac Lunae theoria esse

$$v_{,}' = \bar{f}' + (n)y't + \pi'$$

ubi y' est quantitas ex aliqua ratione ipsi y analoga. Introductione huius quantitatis, quam optimo iure in theoria quoque Lunae omittere potuissemus, effici potest, ut termini in ipsa $(n)'z'$ huius formae $\alpha t \cos g' + \alpha' t \cos 2g'$ + etc., et termini in ipsa w' huius formae $\beta t \sin g' + \beta' t \sin 2g'$ + etc. evanescant, itaque in $(n)'z'$ termini per tempus multiplicati solummodo hi $\delta t \sin g' + \delta' t \sin 2g'$ + etc., et in w' termini solummodo hi $\varepsilon t \cos g'$ + $\varepsilon' t \cos 2g'$ + etc. remaneant, unde computatio perturbationum Lunae secundi ordinis paullulum abbreviatur, et forma earum paullulum simplicior redditur. Nacti igitur sumus problema: datam quantitatem z' quae

ad aequationem $v_{,}' = \bar{f}' + \pi'$ spectet in quantitatem analogam transferre, quae ad aequationem $v_{,}' = \bar{f}' + (n)y't + \pi'$ pertineat. Cuius problematis solutionem, quae iam in formulis art. 15. continetur, infra, ubi de aequationibus nostris in series infinitas evolvendis sermo erit, copiose explicabimus. Quamobrem hoc loco suppono perturbationes Solis aequationibus his

$$v_{,}' = \bar{f}' + (n)y't + \pi'$$
$$lr' = l\bar{r}' + w'$$

datas esse, ubi $\bar{f}'$ et $\bar{r}'$ functiones ipsius z' sunt et ex hac quantitate adiumento aequationum ipsis (13) plane analogarum pendent. Quibus praemissis habemus

$$v_{,}' = (n')z' + (n)y't + \pi' + A_{1}' \sin(n')z' + A_{2}' \sin 2(n')z' + A_{3}' \sin 3(n')z' + \text{etc.}$$
$$r' = (a')c^{w'}\{B_{0}' + B_{1}' \cos(n')z' + B_{2}' \cos 2(n')z' + B_{3}' \cos 3(n')z' + \text{etc.}\}$$

ubi π' longitudinem perigaei Solis denotat, et ubi coefficientes aequationis centri A_{1}', A_{2}', etc. et coefficientes radii vectoris B_{0}', B_{1}', B_{2}', etc. ope excentricitatis terrae computandi sunt. Substitutis his expressionibus, quantitates variabiles $v_{,}'$ et r' in T per z' et w' redditae sunt. Adiumento igitur harum serierum omnium expressio ipsius T modo data in seriem infinitam evolvitur, quae explicata functio variabilium independentium ζ, β, h, p, q, z', w', p', q', atque variabilium z et w ex ζ et β dependentium, facta est, et quae praeterea constantes (a), (n), (e), π, (a'), (n'), (e'), π' continet.

17.

Terminus constans in ipsa h, qui est valor imperturbatus huius quantitatis, est

$$h_0 = \frac{a_0 n_0}{\sqrt{1-e_0^2}}$$

cuius loco quantitatem (h) illis (a), (e), etc. analogam introduci oportet. Sumtis logarithmis hyperbolicis aequationum

$$h_0 = \frac{a_0 n_0}{\sqrt{1-e_0^2}} \text{ et } (h) = \frac{(a)(n)}{\sqrt{1-(e)^2}}$$

subtrahendo invenitur

$$lh_0 - l(h) = la_0 - l(a) + ln_0 - l(n) - \tfrac{1}{2}l(1-e_0^2) + \tfrac{1}{2}l(1-(e)^2)$$

quae eodem modo quo expressio ipsius $f\zeta$ evolvitur in

$$lh_0 = l(h) - \tfrac{1}{3}b - \tfrac{1}{6}b^2 + (e)\xi + \tfrac{1}{2}(1+(e)^2)\xi^2 + \tfrac{1}{2}(1-(e)^2)\eta^2$$

ubi cubi productaque trium dimensionum ipsarum b, ξ et η neglecta sunt. Iam habuimus aequationem hanc

$$lh = lh_0 - S$$

quae, substituto valore ipsius lh_0 modo invento, transit in hanc

$$lh = l(h) - (S+\varepsilon)$$

ubi brevitatis caussa posui

$$\varepsilon = \tfrac{1}{3}b + \tfrac{1}{6}b^2 - (e)\xi - \tfrac{1}{2}(1+(e)^2)\xi^2 - \tfrac{1}{2}(1-(e)^2)\eta^2 \qquad \text{......(29)}$$

quae aequatio insuper monstrat esse

$$\varepsilon = fz$$

Hinc evadit

$$h = (h)\,c^{-(S+\varepsilon)} \qquad \text{......(30)}$$

Substitutis igitur non modo seriebus art. praec. valores ipsarum λ, ϱ, $v_,$, r, $v_,'$, r' suppeditantibus, sed etiam valore praecedenti ipsius h in expressione ipsius T, quantitas haec functio variabilium independentium incognitarum ζ, β, S, p, q, functio variabilium dependentium $v_,$, r, functio variabilium cognitarum $v_,'$, r', p', q' et functio constantium (a), (n), (e), (h), π, (a'), (n'), (e'), π', quae etiam notae supponuntur, explicita reddita est.

Ipsi ζ ex T indagandae aequatio haec

$$T = \frac{d.\dfrac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}}{d\tau}$$

inservit, itaque quantitate $\frac{d\zeta}{d\tau}$ opus est, quae functio ipsarum β et S reddi potest. In art. 13. invenimus aequationem (15) hanc

$$\frac{d\zeta}{d\tau} = \frac{\bar{\varrho}^2 a^2 n \sqrt{1-e^2}}{\varrho^2 (a)^2 (n) \sqrt{1-(e)^2}}$$

quae primum ope aequationum

$$h = \frac{\varkappa(M+m)}{a^2 n \sqrt{1-e^2}}, \quad (h) = \frac{\varkappa(M+m)}{(a)^2 (n) \sqrt{1-(e)^2}}$$

transit in hanc

$$\frac{d\zeta}{d\tau} = \frac{\bar{\varrho}^2 (h)}{\varrho^2 h}$$

sed posterior aequatio (14) suppeditat

$$\frac{\bar{\varrho}}{\varrho} = c^{-\beta}$$

et aequatio (30)

$$\frac{(h)}{h} = c^{S+\varepsilon}$$

itaque

(31)...... $$\frac{d\zeta}{d\tau} = c^{S+\varepsilon-2\beta}$$

quae aequatio etiam ex hac

$$2\beta + l\left(\frac{d\zeta}{d\tau}\right) - S = f\zeta$$

derivari potuisset. Haec enim statim suppeditat, quomodocunque functiones $\Lambda\zeta$ et $\Xi\zeta$ compositae sunt,

$$\frac{d\zeta}{d\tau} = c^{S+f\zeta-2\beta}$$

ex qua in casu nostro, ubi $f\zeta = \varepsilon$ est, aequatio (31) statim emergit.

Hinc igitur factum est, ut aequatio valorem ipsius $\frac{d\zeta}{dt}$ exhibens, scilicet aequatio haec

$$\frac{d\zeta}{dt} = \left(\frac{d\zeta}{d\tau}\right) \int T d\tau$$

functio explicita variabilium et constantium modo enumeratarum evadat, et eodem modo aequationes valores ipsarum $\frac{d\beta}{dt}$ et $\frac{dS}{dt}$ exhibentes functiones explicitae earundem variabilium et constantium redduntur. Quamobrem, et quum aequationes valores ipsarum $\frac{dp}{dt}$ et $\frac{dq}{dt}$ exhibentes, ut infra elucebit, functiones quoque variabilium et constantium earundem reddi possint, aequationes quinque, quae integratione respectu temporis valores veros quantitatum quinque ζ, β, S, p et q suppeditabunt, functiones explicitae earundem variabilium incognitarum, porro functiones variabilium z et w ex

illis dependentium et functiones variabilium cognitarum, z', w', p', et q' factae sunt *), et praeterea solummodo constantes (a), (n), (e), (h), π, (a'), (n'), (e') et π', quarum valores numericos datos esse supponitur, continent.

18.

Quum in aequationibus differentialibus modo descriptis variabiles tali modo illigatae sint, ut aequationes hae directe integrari nequeant, ad plures deinceps approximationes nobis refugiendum est. Quamobrem ab initio valores variabilium indagandarum dari debent, qui a veris earum valoribus non nisi quantitate primi ordinis respectu vis perturbantis aberrant; quibus substitutis, integratione aequationum differentialium valores variabilium obtinentur, qui a veris non nisi quantitate secundi ordinis differunt, et hoc calculo approximatio prima peracta erit. Valores variabilium, quos approximatio prima suppeditabat, quum non nisi quantitatibus secundi ordinis a veris earum valoribus distent, ad approximationem secundam absolvendam in aequationibus nostris differentialibus substituendi sunt, unde post integrationes peractas valores variabilium exibunt, quorum error tertii ordinis est; et eodem modo approximationes subsequentes absolvuntur.

Ad primam igitur approximationem absolvendam valores variabilium a veris earum valoribus quantitate primi ordinis respectu vis perturbantis distantes, ut dixi, cogniti esse debent; haec sola conditio est, et quantitates quaslibet, dummodo huic soli conditioni satisfaciant, substituere nobis licet.

Iam aequatio (26) suppeditat valorem integrum ipsius $(n)\zeta$, qui locum habet, quoties vis perturbans evanescit, qui valor functio ipsarum b, ξ et η est. Sed quum supponam, ipsas b, ξ et η esse quantitates eiusdem ordinis ac vim perturbantem, valor quoque simplicissimus $(n)\tau + (c)$ ipsius $(n)\zeta$, qui nascetur, si b, ξ et η cifrae aequales positae fuerint, a vero eius valore non nisi quantitate primi ordinis differt, itaque conditioni requisitae satisfacit. Hinc sequitur in approximatione prima expressionem simpli-

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*) $\frac{d.\rho^2}{d\tau}$ enim sua sponte functionem solius ρ, itaque functionem ipsarum ζ et β, se praestat.

dissimam $(n)\tau + (o)$ loco $(n)\zeta$ substitui posse. Porro, factis b, ξ et η cifrae aequalibus, aequatio (27) monstrat, fieri quoque $(\beta) = o$, itaque, quum β ipsa sit quantitas eiusdem ordinis ac vis perturbans, in approximatione prima $\beta = o$ ponenda est, et simili modo demonstratur in approximatione prima $S = o$ et $\varepsilon = o$ ponendas esse. Quum loco $(n)z$ et w valores illis valoribus analogi substituendi sint, in approximatione prima $(n)t + (c)$ loco $(n)z$ et cifra ipsa loco w ponitur. Quum vero $(n')z'$, w', p' et q' sint quantitates notae, statim valores earum integros substituere liceret, nisi rei accommodatissimum esset, loco harum quantitatum valores illis valoribus analogos substituere. Denique aequatio (31) suppeditat sub iisdem conditionibus $\frac{d\zeta}{d\tau} = 1$. Congestis his omnibus, in approximatione prima nobis ponendum est:

$$\begin{aligned}
(n)\zeta &= \gamma \\
(n)z &= g \\
(n')z' &= g' \\
\beta &= o \\
w &= o \\
w' &= o \\
S + \varepsilon &= o \\
\frac{d\zeta}{d\tau} &= 1
\end{aligned}$$

ubi

$$\begin{aligned}
\gamma &= (n)\tau + (c) \\
g &= (n)t + (c) \\
g' &= (n')t + (c')
\end{aligned}$$

Quinam valores in approximatione prima et subsequentibus ipsis p, q, p', q', attribuendi sint, infra demonstrabitur. Quibus valoribus omnibus substitutis, aequationes differentiales nostrae functiones evadunt constantium (a), (n), (e) etc. et variabilium independentium τ et t. His aequationibus integratis, constantes arbitrariae additae ita determinandae sunt, ut, deletis terminis a vi perturbanti prolatis, valores pure elliptici variabilium nostrarum redundent. Itaque ad valorem ipsius $(n)\zeta$ tali modo inventum tanquam constans arbitraria addi debet valor ipsius $(n)(\zeta)$, qui per aequatio-

nem (26) datus est, et in quo, postquam $\frac{\overline{(\varrho)}}{(a)} \cos(\bar{\varphi})$, $\frac{\overline{(\varrho)}}{(a)} \sin(\bar{\varphi})$, etc. in series evolutae erunt, etiam $(n)\gamma$ loco $(n)(\zeta)$ ponenda est. Quoties β ex integrata quantitate $R\,dt$ art. 8. elicitur, ad integrale hoc addi debet valor ipsius (β) per aequationem (27) datus, quoties vero loco eius w ex aequatione (12) elicitur, nihil addendum est, quia ipsi fz in hac aequatione contentae iam conditio, quam aequatio (26) exprimit, superstructa, et ope aequationis (28), quae valorem ipsius fz exhibet, expressa est; denique integrali, quod ipsam S suppeditat, tanquam constans arbitraria ε addenda est.

19.

Approximatione prima eo modo quem descripsimus peracta, perturbationes omnes primi ordinis innotescunt; et substituendis iis variabilium valoribus, quos approximatio prima suppeditavit, perturbationes usque ad quantitates tertii ordinis accuratae obtinebuntur. Quum vero in approximatione prima iam primum et maximum ipsarum $(n)\zeta$ et $(n)z$ membrum substitutum sit, in approximatione secunda differentia tantum inter valores ipsarum $(n)\zeta$ et $(n)z$ erutos et maximum earum membrum, quod est resp. γ et g, substituenda est.

In secunda igitur approximatione ad ipsas γ, g et g', quae in aequationibus nostris differentialibus loco $(n)\zeta$, $(n)z$ et $(n)'z'$ substitutae erant, incrementa $(n)\delta\zeta$, $(n)\delta z$ et $(n')\delta z'$ ubique addi debent, quae ita se habent:

$(n)\delta\zeta = -\gamma +$ ei ipsius $(n)\zeta$ valori integro, qui in approximatione prima inventus erat;

$(n)\delta z = -g +$ ei ipsius $(n)z$ valori integro, quem approximatio prima prodiderat;

unde manifestum est, $(n)\delta z$ obtineri mutanda τ in t in valore ipsius $(n)\delta\zeta$;

$(n')\delta z' =$ perturbationibus longitudinis mediae terrae;

in hoc tamen calculo solummodo ii ipsius $(n')\delta z'$ termini, qui per tempus ipsum multiplicati sunt, et termini a Luna ipsa producti in approximatione secunda substituendi sunt.

Quantitatum β, w, w' et $S+\varepsilon$ duplici modo in approximatione secunda ratio haberi potest. Aut per valores earum in approximatione prima computatos functiones exponentiales hae c^{β}, c^{w}, $c^{w'}$ et $c^{-(S+\varepsilon)}$ functiones solius variabilis t reddi et tales in terminis debitis aequationum differentialium substitui, aut β, w, w' et $-(S+\varepsilon)$ pro incrementis ipsarum $l\varrho$, lr, lr' et lh haberi possunt. Modum hunc, qui illo simplicior est, sequar.

Sint igitur $\delta l\varrho$, δlr, $\delta lr'$ et δlh resp. incrementa, quae in approximatione secunda et ulterioribus quantitates $l\varrho$, lr, lr' et lh capiunt, quorum ratio non minus quam ipsorum $(n)\delta\zeta$, etc. ope theorematis Tayloriani habenda est; tum ex praecedentibus facile colligitur in approximatione secunda esse debere:

$\delta l\varrho =$ ei ipsius β valori integro, qui ex approximatione prima elicitus erat;

$\delta lr =$ analogo ipsius w valori;

$\delta lh =$ analogo ipsius $-(S+\varepsilon)$ valori;

$\delta lr' = w'$ quae ex motu terrae nota est.

In w' non minus quam in $(n')\delta z'$ solummodo ad terminos per tempus ipsum multiplicatos et ad terminos a Luna ipsa productos respicere opus est. Denique ad valorem ipsius $\frac{d\zeta}{d\tau}$, qualis in approximatione secunda requiritur, indagandum aequatione finita (31) opus non est, nam differentiato valore ipsius $(n)\zeta$ integro in approximatione prima invento secundum τ, statim valor requisitus ipsius $\frac{d\zeta}{d\tau}$ innotescit.

Substitutis his valoribus nec non valoribus ipsarum δp et δq infra explicandis, aequationes differentiales nostrae usque ad quantitates tertii ordinis accuratae erunt, et functiones explicitae variabilium independentium τ et t se praestabunt, quae integratae variabilium nostrarum terminos secundi ordinis suppeditabunt. Quibus computatis et ad valores quos approximatio prima prodiderat additis, variabiles hae usque ad quantitates tertii ordinis accuratae sunt, et simul valores incrementorum $(n)\delta\zeta$, $(n)\delta z$, $\delta l\varrho$, δlr, etc. innotescunt, quibus ad approximationem tertiam peragendam opus est; et sic porro pro approximationibus subsequentibus, si forte his opus sit.

Computationes vero omnes absolutae faciunt, quando ad approximationem pervenietur, quae ad valores variabilium nostrarum, quales ex approximationibus anterioribus iam innotescunt, terminos limitem fixum determinatumque superantes non addit.

20.

Quantitatibus b, ξ et η adhuc solam conditionem superstruximus, ut quantitates parvulae primi ordinis respectu vis perturbantis sint. Quae quantitates, si indeterminatae manerent, efficerent, ut non modo perturbationes ipsae sed etiam quotientes earum differentiales respectu semiaxis maioris, excentricitatis et longitudinis perigaei Lunae per formulas nostras indagarentur. Quodsi igitur, computatione perturbationum peracta, valores horum elementorum in calculo adhibitos non satis accuratos reperisses, errorem commissum adiumento quotientium illorum corrigere posses. Sed praestat aliam ingredi viam et quantitatibus b, ξ et η determinatos quosdam attribuere valores, quos confestim explicabo.

Si indolem expressionis ipsius T perscrutamur, facili opera comperiemus, valorem ipsius $(n)\zeta$ quem suppeditabit sine respectu constantis arbitrariae ei addendae huius esse formae:

$$\begin{aligned}(n)pt &+ (\tau-t)q\cos g + q'\sin g + q''\sin(-\gamma+2g) + q'''\sin(-2\gamma+3g) + \text{etc.}\\ &\qquad\qquad + q_{,}\sin(2\gamma-g) + q_{,,}\sin(3\gamma-2g) + \text{etc.}\\ &+ (\tau-t)r\cos 2g + r'\sin 2g + r''\sin(-\gamma+3g) + r'''\sin(-2\gamma+4g) + \text{etc.}\\ &\qquad\qquad + r_{,}\sin(\gamma+g) + r_{,,}\sin(3\gamma-g) + \text{etc.}\\ &+ \text{etc.}\\ &+ \Sigma A(\tau-t)\cos(ig+i'g'+D)\\ &+ \Sigma B\sin(\varkappa\gamma+(i-\varkappa)g+i'g+D)\end{aligned}$$

ubi p, q, q', etc. r, r', etc., A et B coefficientes constantes vel numerici sunt, et D arcus huius formae $\mu t+\nu$ est, ubi μ et ν quoque constantes, et ubi inter terminos sub signo summationis contentos ii excludendi sunt, in quibus simul $i'=o$ et $D=o$, quia terminos ad hos valores pertinentes separatim adscripsi. Ex expressione vero (26), quae terminos suppeditat, qui ipsi $(n)\zeta$ tanquam constans arbitraria addendi sunt, facile concluditur, coefficientes

ipsarum ξ, ξ^2 et η^2 in series infinitas secundum sinus, et coefficientes ipsarum η et $\eta\xi$ in series infinitas secundum cosinus multiplicium ipsius γ progredientes evolvi posse; dextrum igitur huius aequationis membrum huius formae est

$$\gamma\text{-}(n)b\tau + \{d(1\text{-}b)\xi + h\xi^2 + k\eta^2\}\sin\gamma + \{d'(1\text{-}b)\xi + h'\xi^2 + k'\eta^2\}\sin 2\gamma + \text{etc.}$$
$$+ \{l(1\text{-}b)\eta + m\eta\xi\}\cos\gamma + \{l'(1\text{-}b)\eta + m'\eta\xi\}\cos 2\gamma + \text{etc.}$$

ubi d, h, k, d', etc. l, m, l', etc. coefficientes constantes ex excentricitate (e) dependentes denotant. Si hanc expressionem ad praecedentem addimus, valorem integrum ipsius $(n)\zeta$ habemus hunc

$$\begin{aligned}(n)\zeta = \gamma\text{-}(n)b\tau + (n)pt &+ \{d(1\text{-}b)\xi + h\xi^2 + k\eta^2\}\sin\gamma + \{d'(1\text{-}b)\xi + h'\xi^2 + k'\eta^2\}\sin 2\gamma + \text{etc.}\\
&+ \{l(1\text{-}b)\eta + m\eta\xi\}\cos\gamma + \{l'(1\text{-}b)\eta + m'\eta\xi\}\cos 2\gamma + \text{etc.}\\
&+ (\tau - t)q\cos g + q'\sin g + q''\sin(-\gamma+2g) + q'''\sin(-2\gamma+3g) + \text{etc.}\\
&\qquad + q_{,}\sin(2\gamma - g) + q_{,,}\sin(3\gamma - 2g) + \text{etc.}\\
&+ (\tau - t)r\cos 2g + r'\sin 2g + r''\sin(-\gamma+3g) + r'''\sin(-2\gamma+4g) + \text{etc.}\\
&\qquad + r_{,}\sin(\gamma + g) + r_{,,}\sin(3\gamma - g) + \text{etc.}\\
&+ \text{etc.}\\
&+ \Sigma A(\tau - t)\cos(ig + i'g' + D)\\
&+ \Sigma B\sin(\varkappa\gamma + (i-\varkappa)g + i'g' + D)\end{aligned}$$

Hinc mutata τ in t invenitur

$$\begin{aligned}(n)z = g + (n)(p\text{-}b)t &+ \{q' + q'' + q''' + \text{etc.} + q_{,} + q_{,,} + \text{etc.} + d(1\text{-}b)\xi + h\xi^2 + k\eta^2\}\sin g\\
&+ \{r' + r'' + r''' + \text{etc.} + r_{,} + r_{,,} + \text{etc.} + d'(1\text{-}b)\xi + h'\xi^2 + k'\eta^2\}\sin 2g\\
&+ \text{etc.}\\
&+ \{l(1\text{-}b)\eta + m\eta\xi\}\cos g + \{l'(1\text{-}b)\eta + m'\eta\xi\}\cos 2g + \text{etc.}\\
&+ \Sigma C\sin(ig + i'g' + D)\end{aligned}$$

Iam quantitates arbitrariae b, ξ et η aptissime ita determinantur, ut fiat

$$\begin{aligned}o &= p - b\\
o &= q' + q'' + q''' + \text{etc.} + q_{,} + q_{,,} + \text{etc.} + d(1\text{-}b)\xi + h\xi^2 + k\eta^2\\
o &= \eta\end{aligned}$$

unde nanciscimur

$$(n)z = g + F\sin 2g + \text{etc.} + \Sigma C\sin(ig + i'g' + D)$$

ubi

$$F = r' + r'' + r''' \text{ etc.} + r_{,} + r_{,,} + \text{etc.} + d'(1-b)\xi + h'\xi^2 + k'\eta^2$$
$$\text{etc.} = \text{etc.}$$

Computatis igitur b, ξ et η per aequationes has

$$b = p$$
$$h\xi^2 + d(1-p)\xi = -(q' + q'' + q''' + \text{etc.} + q_{,} + q_{,,} + \text{etc.})$$
$$\eta = o$$

et substitutis valoribus earum hoc modo erutis in expressione praecedente ipsius F, etc., $(n)z$ praeter terminum $(n)t$ in ipsa g contentum neque terminos huius formae, nec vero terminum formae H sin g, hoc est terminum, qui eiusdem formae esset ac terminus maximus aequationis centri, continet. Hinc factum est, ut computationi numericae perturbationum valores ii motus medii et excentricitatis Lunae superstruendi sint, qui ex observationibus astronomicis immediate proveniunt, ita ut calculo indirecto, quo ii, qui rem ante tractavere, valorem excentricitatis in formulis perturbationum substituendum scrutati sunt, in methodo nostra opus non sit.

Denotat igitur (e) eum excentricitatis Lunae valorem, qui ex observato aequationis centri termino maximo ope indolis eius pure ellipticae eruitur, et (n) (propter terminum $(n)yt$ in longitudine vera $v_{,}$ introductum) observatum motum medium anomaliae mediae Lunae. Valor semiaxis maioris orbitae Lunae (a) in calculo numerico perturbationum adhibendus ex (n) ope aequationis eruitur huius

$$(n)^2 (a)^3 = \varkappa (M + m)$$

et quum in computatione hac, ut infra clarius intelligetur, praecipue quantitate hac $\frac{m'}{M+m} \cdot \frac{(a)^3}{(a')^3}$ opus sit, habemus per aequationem praecedentem et per hanc

$$(n')^2 (a')^3 = \varkappa (M + m')$$

quae ad Solem spectat,

$$\frac{m'}{M+m} \cdot \frac{(a)^3}{(a')^3} = \frac{(n')^2}{(n)^2} \cdot \frac{1}{1 + \frac{M}{m'}}$$

ubi igitur $\frac{M}{m'}$ massam terrae partibus massae Solis expressam repraesentat, et (n') motum medium anomaliae mediae terrae denotat.

Quum perturbationes Lunae ita exhiberi debeant, ut longitudo perigaei ubique in argumentis perturbationum explicite indicata sit, praevidere quidem potuimus, quantitatem η in fine calculi cifrae aequalem necessario inventum iri, sed quum quantitas haec ad problematis in art. 16. enuntiati solutionem spectet, eam in formulis illis praetermittere nolui. Accedit etiam quod eadem quantitas in theoria perturbationum planetarum utilitatem afferre potest; in hac enim theoria perturbationes ipsius $(n)z$ sub forma hac

$$\Sigma C_s \sin (ig + i'g') + \Sigma C_c \cos (ig + i'g')$$

exprimi debent, ut in theoria Iovis atque Saturni amplius explicavi, et tum ξ ita determinanda est, ut terminus $C_s \sin g$, et η ita ut terminus $C_c \cos g$ evanescat. Quo factum erit, ut in hac quoque theoria excentricitatis et longitudinis perihelii valores in calculo numerico perturbationum adhibendi sint, qui observationibus astronomicis immediate reperiuntur.

21.

His de perturbationibus longitudinis radiique vectoris expositis, ad perturbationes quantitatum p et q, e quibus perturbationes latitudinis et reductionis longitudinis pendent, explicandas perventum est.

In praecedentibus inclinationes omnino neglexīmus, habita vero earum ratione, positoque axi coordinatarum x et x' in nodo ascendenti orbitae m cum plano ipsarum xy, habemus

$$\begin{aligned}
x &= r \cos (v-\theta) \\
y &= r \sin (v-\theta) \cos i \\
z &= r \sin (v-\theta) \sin i \\
x' &= r' \sin (\theta-\theta') \sin (v'-\theta') \cos i' + r' \cos (\theta-\theta') \cos (v'-\theta') \\
y' &= r' \cos (\theta-\theta') \sin (v'-\theta') \cos i' - r' \sin (\theta-\theta') \cos (v'-\theta') \\
z' &= r' \sin (v'-\theta') \sin i'
\end{aligned}$$

Substitutis his coordinatarum expressionibus in expressione ipsius Ω in art. 1. data, Ω functio reddita est quantitatum v, r, θ, i ad corpus m, et quantitatum analogarum ad corpus m' spectantium. Quum vero sit $v_{,} = f + \chi$, $v = f + \omega + \theta$, $\chi = \omega + \int \cos i \, d\theta$, habetur

(32)...... $$v = v_{,} + \theta + \omega - \chi = v_{,} + \theta - \int \cos i \, d\theta$$

quacum v eliminari et quantitate $v_{,}$ reddi potest, et eodem modo v quantitate analoga $v'_{,}$ redditur. Quibus factis, Ω functio ipsarum $v_{,}$, r, i, θ, $v'_{,}$, r', i', θ' evadit. Porro adiumento relationum inter i atque θ et p atque q in Sectione prima datarum i atque θ quantitatibus p atque q, et eodem modo i' atque θ' quantitatibus p' atque q' redduntur. Hinc factum est, ut Ω functio evadat quantitatum $v_{,}$, r, p atque q ad corpus m, et quantitatum analogarum ad corpus m' spectantium; id quod in praecedentibus iam supposuimus.

Aequationes e quibus motus corporis m pendet, si ad inclinationes non respicitur, ad aequationes tres, incognitas ζ, β atque S suppeditantes, reduximus, quae aequationes variabiles independentes ζ, β, S, p atque q ad corpus m pertinentes continent, et simul quantitates $v_{,}$ et r, quarum functio Ω est, quantitatibus z et w ex variabilibus ζ et β pendentibus reddere edocuimus. Valoribus perturbatis ipsarum p et q determinandis inserviunt ultimae duae aequationes (1) hae

$$\left.\begin{aligned} dp &= \frac{an}{\sqrt{1-e^2}} \cos^2 i \left(\frac{d\Omega}{dq}\right) dt \\ dq &= -\frac{an}{\sqrt{1-e^2}} \cos^2 i \left(\frac{d\Omega}{dp}\right) dt \end{aligned}\right\} \ldots\ldots(33)$$

quae secundum praecedentia functiones variabilium independentium S, p atque q et variabilium dependentium z et w, quae omnes ad corpus m spectant, sese praestant; quae aequationes quinque omnes praeterea quantitates variabiles z', w', p' atque q', ad corpus m' spectantes, continent. Mutatis quantitatibus ad corpus m pertinentibus in analogas ad m' spectantes quantitates, et vice versa quantitatibus ad hoc corpus spectantibus in quantitates ad illud pertinentes, quinque aequationes illis plane analogas nanciscimur, quae ad motum corporis m' pertinent. In problemate igitur generali trium corporum, quod est problema quo et motus corporis m et motus corporis m' investigandus est, decem aequationes totidem variabiles independentes cohibentes simultanee integrandae sunt, et quum quantitatibus p atque q et reductio longitudinis $v_{,}$ ad planum fundamentale ipsarum xy et latitudo corporis m versus idem planum, nec non quantitatibus p' atque q' et reductio longitudinis $v'_{,}$ ad idem planum et latitudo corporis m' versus idem

planum perfecte determinetur, ad loca corporum m atque m' in spatio integre computanda hae aequationes decem sufficiunt.

In theoria igitur Lunae, in qua propter perparvulam Lunae a Terra distantiam respectu distantiae Terrae a Sole, Solis coordinatae cognitae haberi possunt, quae theoria propterea casus specialis problematis generalis trium corporum dici potest, quinque aequationes totidem variabiles independentes continentes simultanee integrandae sunt, et hae aequationes quinque ad loca Lunae in spatio perfecte computanda sufficiunt, id quod iam in fine Sectionis primae admonuimus. Elementa vero sive variabiles independentes **sex** ad Lunam ipsam pertinentes, quas ab initio aequationes quinque nostrae continebant, in praecedentibus ad variabiles independentes **quinque** reduximus.

22.

Quum quantitas Ω aut per i, θ, i', θ', aut per p, q, p', q' expressa valde implicita, et revera a plano fundamentali independens sit: praestat loco harum quantitatum inclinationem mutuam orbitarum m atque m' nodosque huic respondentes introducere. Quem in finem consideremus triangulum sphaericum ab ambabus orbitis et plano ipsarum xy formatum. Sit Φ huius trianguli latus, quod orbitae m sive Lunae pars, Ψ latus, quod orbitae m' sive Solis pars, et $\theta - \theta'$ latus, quod plani ipsarum xy pars est: tum anguli his lateribus resp. oppositi erunt i', $180° - i$ atque I, denotante I inclinationem mutuam orbitarum m et m'.

Ut quantitates I, Φ atque Ψ in formulis nostris introducantur, pono

$$(34)\ldots\ldots\left\{\begin{array}{l} x = \alpha X + \beta Y\ ,\ x' = \alpha' X' + \beta' Y' \\ y = \alpha_, X + \beta_, Y\ ,\ y' = \alpha'_, X' + \beta'_, Y' \\ z = \alpha_{,,} X + \beta_{,,} Y\ ,\ z' = \alpha'_{,,} X' + \beta'_{,,} Y' \end{array}\right.$$

ubi aequationes conditionales locum habent hae

$$(35)\ldots\ldots\left\{\begin{array}{l} \alpha^2 + \alpha_,^2 + \alpha_{,,}^2 = 1\ ,\ \alpha'^2 + \alpha_,'^2 + \alpha_{,,}'^2 = 1 \\ \beta^2 + \beta_,^2 + \beta_{,,}^2 = 1\ ,\ \beta'^2 + \beta_,'^2 + \beta_{,,}'^2 = 1 \\ \alpha\beta + \alpha_,\beta_, + \alpha_{,,}\beta_{,,} = o\ ,\ \alpha'\beta' + \alpha'_,\beta'_, + \alpha'_{,,}\beta'_{,,} = o \end{array}\right.$$

et ubi statuo esse

$$X = r\cos(v-\theta-\Phi)\ ,\quad X' = r'\cos(v'-\theta'-\Psi)$$
$$Y = r\sin(v-\theta-\Phi)\ ,\quad Y' = r'\sin(v'-\theta'-\Psi)$$

ita ut X atque Y sint coordinatae corporis m in ipsa eius orbita, et X' atque Y' coordinatae corporis m' in ipsa eius orbita tali modo collocatae, ut axis positivus ipsarum X atque X' in nodo ascendenti orbitae m cum orbita m' iaceat.

Comparatis his coordinatarum expressionibus cum earum expressionibus art. praec., invenitur

$$\left.\begin{aligned}
\alpha &= \cos\Phi\\
\beta &= -\sin\Phi\\
\alpha_{,} &= \sin\Phi\cos i\\
\beta_{,} &= \cos\Phi\cos i\\
\alpha_{,,} &= \sin\Phi\sin i\\
\beta_{,,} &= \cos\Phi\sin i
\end{aligned}\right\}\text{......(36)}$$

$$\left.\begin{aligned}
\alpha' &= \cos(\theta-\theta')\cos\Psi+\sin(\theta-\theta')\sin\Psi\cos i'\\
\beta' &= -\cos(\theta-\theta')\sin\Psi+\sin(\theta-\theta')\cos\Psi\cos i'\\
\alpha'_{,} &= -\sin(\theta-\theta')\cos\Psi+\cos(\theta-\theta')\sin\Psi\cos i'\\
\beta'_{,} &= \sin(\theta-\theta')\sin\Psi+\cos(\theta-\theta')\cos\Psi\cos i'\\
&= \sin i'\sin(\theta-\theta').\sin i'\sin\Psi+\cos i'.(\cos(\theta-\theta')\cos\Psi+\sin(\theta-\theta')\sin\Psi\cos i')\\
\alpha'_{,,} &= \sin\Psi\sin i'\\
\beta'_{,,} &= \cos\Psi\sin i'
\end{aligned}\right\}\text{......(37)}$$

ubi posterior ipsius $\beta'_{,}$ expressio, multiplicato priore termino per $\sin^2 i' + \cos^2 i'$, ex priore expressione evasit. Ipsis i' atque θ' ex expressionibus his eliminandis inserviunt relationes inter angulos lateraque trianguli sphaerici supra memorati. Trigonometria sphaerica statim suppeditat

$$\begin{aligned}
\cos\Phi &= \cos(\theta-\theta')\cos\Psi+\sin(\theta-\theta')\sin\Psi\cos i' &&\text{.........}(a)\\
\sin\Phi\cos I &= \cos(\theta-\theta')\sin\Psi-\sin(\theta-\theta')\cos\Psi\cos i' &&\text{.........}(b)\\
\sin\Phi\cos i &= -\sin(\theta-\theta')\cos\Psi+\cos(\theta-\theta')\sin\Psi\cos i' &&\text{.........}(c)\\
\sin\Phi\sin I &= \sin i'\sin(\theta-\theta') &&\text{.........}(d)\\
\sin\Phi\sin i &= \sin i'\sin\Psi &&\text{.........}(e)\\
\cos i' &= \cos I\cos i+\sin I\sin i\cos\Phi &&\text{.........}(f)\\
\cos\Psi\sin i' &= -\sin I\cos i+\cos I\sin i\cos\Phi &&\text{.........}(g)
\end{aligned}$$

quibus relationibus i' atque θ' ex expressionibus (37) statim eliminari possunt, unde evadunt

$$(38) \ldots \left\{ \begin{array}{ll} \alpha' = \cos \Phi & \ldots\ldots \text{ex } (a) \\ \beta' = -\sin \Phi \cos I & \ldots\ldots \text{ex } (b) \\ \alpha'_{,} = \sin \Phi \cos i & \ldots\ldots \text{ex } (c) \\ \beta'_{,} = \sin I \sin i + \cos I \cos i \cos \Phi & \text{ex } (d), (e), (f) \text{ et } (a) \\ \alpha'_{,,} = \sin \Phi \sin i & \ldots\ldots \text{ex } (e) \\ \beta'_{,,} = -\sin I \cos i + \cos I \sin i \cos \Phi & \ldots\ldots \text{ex } (g) \end{array} \right.$$

Iam substitutis aequationibus (34) in formula

$$\Delta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

elicitur propter aequationes conditionales (35)

$$\Delta^2 = X^2 + Y^2 + X'^2 + Y'^2 - 2(\alpha\alpha' + \alpha_{,}\alpha'_{,} + \alpha_{,,}\alpha'_{,,})XX' - 2(\alpha\beta' + \alpha_{,}\beta'_{,} + \alpha_{,,}\beta'_{,,})XY'$$
$$-2(\beta\alpha' + \beta_{,}\alpha'_{,} + \beta_{,,}\alpha'_{,,})X'Y - 2(\beta\beta' + \beta_{,}\beta'_{,} + \beta_{,,}\beta'_{,,})YY'$$

Valoribus vero ipsarum α, β, etc. $\alpha_{,}$, $\beta_{,}$, etc. ex (36) et (38) petendis substitutis, emergunt

$$\begin{aligned} \alpha\alpha' + \alpha_{,}\alpha'_{,} + \alpha_{,,}\alpha'_{,,} &= 1 \\ \alpha\beta' + \alpha_{,}\beta'_{,} + \alpha_{,,}\beta'_{,,} &= 0 \\ \beta\alpha' + \beta_{,}\alpha'_{,} + \beta_{,,}\alpha'_{,,} &= 0 \\ \beta\beta' + \beta_{,}\beta'_{,} + \beta_{,,}\beta'_{,,} &= \cos I \end{aligned}$$

Expressio igitur praecedens ipsius Δ^2, si insuper valores ipsarum X, Y, X' atque Y' substituti erunt, abit in

$$\Delta^2 = r^2 + r'^2 - 2rr'\cos(v-\theta-\Phi)\cos(v'-\theta'-\Psi) - 2rr'\cos I \sin(v-\theta-\Phi)\sin(v'-\theta'-\Psi)$$

quae, positis

$$(39) \ldots \left\{ \begin{array}{l} \varphi = \Phi + \int \cos i \, d\theta = \Phi + \chi - \omega \\ \psi = \Psi + \int \cos i' \, d\theta' = \Psi + \chi' - \omega' \end{array} \right.$$

propter aequationem (32) et eius similem ad corpus m' pertinentem transit in hanc

$$(40) \ldots \quad \Delta^2 = r^2 + r'^2 - 2rr'\cos(v_{,}-\varphi)\cos(v'_{,}-\psi) - 2rr'\cos I \sin(v_{,}-\varphi)\sin(v'_{,}-\psi)$$

Hinc et quum, facili transformatione instituta, sit

$$\Omega = \frac{m'}{M+m}\left\{\frac{1}{\Delta} + \frac{\Delta^2 - r^2 - r'^2}{2r'^3}\right\}$$

manifestum est, Ω functionem ipsarum $v_{,}$, r, $v'_{,}$, r', φ, ψ atque I factam esse.

23.

Quum transformationibus in art. praec. peractis aequationes nostrae ζ, β, S, p atque q suppeditantes, quia ex Ω pendent, functiones ipsarum I, φ, ψ loco ipsarum p, q, p', q' factae sint, necesse est aequationes differentiales investigentur, quibus valores perturbati ipsarum I, φ atque ψ computandi sint. Quem in finem relationibus art. praec. a triangulo sphaerico inter orbitam Lunae, orbitam Solis et planum fundamentale suppeditatis relationes adiungo trigonometricas has

$$\cos I = \cos i \cos i' + \sin i \sin i' \cos(\theta - \theta') \quad \text{.......}(h)$$
$$\sin I \cos \Psi = -\cos i \sin i' + \sin i \cos i' \cos(\theta - \theta') \quad \text{.......}(i)$$
$$\sin I \sin \Psi = \sin i \sin(\theta - \theta') \quad \text{.......}(k)$$
$$\sin I \cos \Phi = \cos i' \sin i - \sin i' \cos i \cos(\theta - \theta') \quad \text{.......}(l)$$
$$\cos \Psi = \cos(\theta - \theta') \cos \Phi - \sin(\theta - \theta') \sin \Phi \cos i \quad \text{.......}(m)$$
$$\cos i' \sin \Psi = \sin(\theta - \theta') \cos \Phi + \cos(\theta - \theta') \sin \Phi \cos i \quad \text{.......}(n)$$
$$\cos I \sin \Psi = \cos(\theta - \theta') \sin \Phi + \sin(\theta - \theta') \cos \Phi \cos i \quad \text{.......}(o)$$
$$\cos i = \cos I \cos i' - \sin I \sin i' \cos \Psi \quad \text{.......}(p)$$

quas idem triangulus subministrat. Differentiata relatione (h), adiumento relationum (d), (i), (k) atque (l) statim prodit

$$dI = \cos \Phi di - \cos \Psi di' + \sin i \sin \Phi d\theta - \sin i' \sin \Psi d\theta'$$

Differentiata relatione (m), ope (n), (o) atque (k) nanciscimur

$$d\Psi = \cos I d\Phi + \cos i' (d\theta - d\theta') - \sin I \sin \Phi di$$

Eodem modo relatio (a) una cum (b), (c) atque (d) suppeditat

$$d\Phi = \cos I d\Psi - \cos i (d\theta - d\theta') + \sin I \sin \Psi di'$$

Eliminata tum $d\Psi$ tum $d\Phi$ ex his aequationibus, nanciscimur

$$d\Phi = (\cos i' \cos I - \cos i) \operatorname{cosec}^2 I (d\theta - d\theta') - \operatorname{cotg} I \sin \Phi di + \operatorname{cosec} I \sin \Psi di'$$
$$d\Psi = (\cos i' - \cos i \cos I) \operatorname{cosec}^2 I (d\theta - d\theta') - \operatorname{cosec} I \sin \Phi di + \operatorname{cotg} I \sin \Psi di'$$

Aequationes vero (39) differentiatae praebent

$$d\varphi = d\Phi + \cos i\, d\theta \quad , \quad d\psi = d\Psi + \cos i'\, d\theta'$$

Substitutis his ipsarum dp, dq, dp' atque dq' valoribus in aequationibus (41), nanciscimur

$$(44)\begin{cases} dI = -\frac{an}{\sqrt{1-e^2}}\left[\left(\frac{d\Omega}{d\varphi}\right)\operatorname{cotg} I + \left(\frac{d\Omega}{d\psi}\right)\operatorname{cosec} I\right]dt - \frac{a'n'}{\sqrt{1-e'^2}}\left[\left(\frac{d\Omega'}{d\psi}\right)\operatorname{cotg} I + \left(\frac{d\Omega'}{d\varphi}\right)\operatorname{cosec} I\right]dt \\ d\varphi = \frac{an}{\sqrt{1-e^2}}\operatorname{cotg} I\left(\frac{d\Omega}{dI}\right)dt + \frac{a'n'}{\sqrt{1-e'^2}}\operatorname{cosec} I\left(\frac{d\Omega'}{dI}\right)dt \\ d\psi = \frac{an}{\sqrt{1-e^2}}\operatorname{cosec} I\left(\frac{d\Omega}{dI}\right)dt + \frac{a'n'}{\sqrt{1-e'^2}}\operatorname{cotg} I\left(\frac{d\Omega'}{dI}\right)dt \end{cases}$$

quae ad problema generale trium corporum spectant. Hoc quidem problema soluturi, nunc novem aequationes totidem variabiles independentes cohibentes habemus, aequationes puta in praecedentibus pro ζ, β, S, ζ', β' et S' evolutas et aequationes (44) pro I, φ et ψ, quae novem aequationes simultanee integrandae sunt. Quibus factis, valores perturbati ipsarum z, w, S, z', w', S', I, φ et ψ innotescunt, qui solam variabilem t continent, quamobrem post substitutos hos valores aequationes (42) valores perturbatos ipsarum p atque q, et aequationes (43) valores perturbatos ipsarum p' atque q' suppeditabunt.

In theoria Lunae, ubi valores perturbati ipsarum p' atque q' noti sunt, valores solummodo ipsarum dp atque dq ex aequationibus (42) petendos in aequationibus (41) substituo, unde

$$(45)\ldots\ldots\begin{cases} dI = -\frac{an}{\sqrt{1-e^2}}\left[\left(\frac{d\Omega}{d\varphi}\right)\operatorname{cotg} I + \left(\frac{d\Omega}{d\psi}\right)\operatorname{cosec} I\right]dt - \sin\psi\frac{dp'}{\cos i'} - \cos\psi\frac{dq'}{\cos i'}, \\ d\varphi = \frac{an}{\sqrt{1-e^2}}\operatorname{cotg} I\left(\frac{d\Omega}{dI}\right)dt - \operatorname{cosec} I\left\{\cos\psi\frac{dp'}{\cos i'} - \sin\psi\frac{dq'}{\cos i'}\right\} \\ d\psi = \frac{an}{\sqrt{1-e^2}}\operatorname{cosec} I\left(\frac{d\Omega}{dI}\right)dt - \operatorname{cotg} I\left\{\cos\psi\frac{dp'}{\cos i'} - \sin\psi\frac{dq'}{\cos i'}\right\} \end{cases}$$

In hac igitur theoria nunc sex habemus aequationes totidem variabiles independentes cohibentes, aequationes puta pro ζ, β et S et aequationes praecedentes pro I, φ atque ψ, quae simultanee integrandae sunt. Quum vero in hac theoria valores ipsarum φ et ψ tali modo coniuncti sint, ut alter ex altero facillima opera eliciatur, revera nunc ut antea quinque aequationes simultanee integrandae sunt. Elicitis valoribus perturbatis ipsarum I, φ et ψ, aequationes (42) integratione valores perturbatos ipsarum p et q suppeditant, quibus latitudo Lunae et reductio longitudinis v, ad planum, ad quod latitudo refertur, computandae sunt.

24.

Transformationibus articulorum praecedentium aequationes, quarum integratione valores perturbati ipsarum ζ, β, S, p et q prodeunt, in functione ipsarum I, φ et ψ loco p, q, p' et q' exhibuimus. Quibus transformationibus conclusiones in huius Sectionis priore parte divulgatae nullo modo perturbantur, quae ab indole ipsarum p, q, p' atque q' plane independentes sunt. Magnum vero calculi compendium per transformationes has adducemus, quia forma ipsius Δ, qualis nunc exhibita est, multo simplicior est quam expressio eiusdem quantitatis in functione ipsarum p, q, p' atque q'. Quantitates igitur I, φ atque ψ pro quantitatibus auxiliaribus habendae sunt, quibus computatio accurata variabilium nostrarum ζ, β, S, p atque q commodissime absolvitur, aut si mavis, quibus evolutio integratioque aequationum differentialium valores variabilium illarum suppeditantium faciliori opera perficitur.

Substituto eo ipsius Δ valore, qui in art. 22. evolvebatur, in omnibus aequationibus differentialibus nostris, factum erit, ut aequationes illae integratione ζ, β et S suppeditantes, et aequationes articuli praecedentis integratione I, φ et ψ suppeditantes functiones explicitae variabilium independentium ζ, β, S, I, φ et ψ sint. Praeterea aequationes illae variabiles notas $v'_{,}$ atque r', et aequationes hae variabiles notas $v'_{,}$, r', p' et q' continent. De integratione aequationum ζ, β et S suppeditantium in praecedentibus disseruimus, hoc vero loco in integrationem aequationum I, φ et ψ suppeditantium nobis inquirendum est.

Valor ipsius Δ sub numero (40) allatus ita quoque exhiberi potest

$$\Delta^2 = r^2 + r'^2 - 2rr'\cos^2\tfrac{1}{2}I\cos(v_{,} - v'_{,} - (\varphi - \psi)) - 2rr'\sin^2\tfrac{1}{2}I\cos(v_{,} + v'_{,} - (\varphi + \psi))$$

unde, substitutis valoribus ipsarum $v_{,}$, r, $v'_{,}$ et r' aequationibus datis his

$$\begin{aligned} v_{,} &= \bar{f} + (n)yt + \pi \\ v'_{,} &= \bar{f}' + (n)y't + \pi' \\ r &= \bar{r}\,c^{w} \\ r' &= \bar{r}'\,c^{w'} \end{aligned}$$

ubi $\bar{f}$ et $\bar{r}$ anomaliam veram et radium vectorem ipsis $\bar{\varphi}$ et $\bar{\phi}$ art. 13. re-

spondentes, et $\bar{f}'$ atque $\bar{r}'$ quantitates analogas ad Solem spectantes denotant, evadit

$$\Delta^2 = \bar{r}^2 c^{2w} + \bar{r}'^2 c^{2w'} - 2\bar{r}\bar{r}' c^{w+w'} \cos^2 \tfrac{1}{2} I \cos[\bar{f} - \bar{f}' + (n)(y-y')t + (\pi - \pi') - (\varphi - \psi)]$$
$$- 2\bar{r}\bar{r}' c^{w+w'} \sin^2 \tfrac{1}{2} I \cos[\bar{f} + \bar{f}' + (n)(y+y')t + (\pi + \pi') - (\varphi + \psi)]$$

Si expressio haec in hac ipsius Ω expressione

$$\Omega = \frac{m'}{M+m}\left\{\frac{1}{\Delta} + \frac{\Delta^2 - \bar{r}^2 c^{2w} - \bar{r}'^2 c^{2w'}}{2\bar{r}'^3 c^{3w'}}\right\}$$

substituta fuerit, erit post evolutionem in seriem, in qua w et w' negligantur et anomaliae mediae g et g' resp. loco $(n)z$ et $(n')z'$ substituuntur, terminus generalis ipsius Ω huius formae

$$X \cos\{ig + i'g' + i''[(n)(y-y')t + (\pi-\pi') - (\varphi-\psi)] + i'''[(n)(y+y')t + (\pi+\pi') - (\varphi+\psi)]\}$$

denotante X functionem aliquam ipsarum (a), (a'), (e), (c), I et numerorum integrorum i, i', i'' atque i'''.

Habita ratione ipsarum w atque w', nec non differentiarum inter g atque $(n)z$ et g' atque $(n')z'$, ope aequationum in praecedentibus erutarum facile demonstratur terminum hunc generalem eandem formam conservare, neque aliud locum habere, si aut Ω aut quotientes differentiales ipsius Ω per valorem ipsius $\frac{an}{\sqrt{1-e^2}}$, qui est $(h)\,c^{-(S+\varepsilon)}$, multiplicatae fuerint.

Quibus praemissis consideremus terminos aequationum (45), qui ex dp' et dq' non pendent. Si in approximatione prima I, φ atque ψ constantes ponimus, erit $\left(\frac{d\Omega}{dI}\right)$ series infinita, cuius terminus generalis huius formae est

$$\frac{dX}{dI} \cos\{ig + i'g' + i''[(n)(y-y')t + (\pi-\pi') - (\varphi-\psi)] + i'''[(n)(y+y')t + (\pi+\pi') - (\varphi+\psi)]\}$$

$\left(\frac{d\Omega}{d\varphi}\right)$ series, cuius terminus generalis huius formae est

$$(i''+i''')\, X \sin\{ig + i'g' + i''[(n)(y-y')t + (\pi-\pi') - (\varphi-\psi)] + i'''[(n)(y+y')t + (\pi+\pi') - (\varphi+\psi)]\}$$

et $\left(\frac{d\Omega}{d\psi}\right)$ series, cuius terminus generalis huius formae est

$$-(i''-i'')\, X \sin\{ig + i'g' + i''[(n)(y-y')t + (\pi-\pi') - (\varphi-\psi)] + i''[(n)(y+y')t + (\pi+\pi') - (\varphi+\psi)]\}$$

Hinc sequitur, valorem specialem indicum hunc $i = i' = i'' = i''' = o$

in $\left(\frac{d\Omega}{dI}\right)$ terminum constantem proferre, in ipsis $\left(\frac{d\Omega}{d\varphi}\right)$ et $\left(\frac{d\Omega}{d\psi}\right)$ vero terminum constantem existere non posse. Expressio igitur ipsius dI in seriem evolvitur huius formae $A \sin(\eta t + \delta) dt$, expressio ipsius $d\varphi$ in seriem huius formae $kdt + B\cos(\eta t + \delta) dt$ et expressio ipsius $d\psi$ in seriem huius formae $hdt + D\cos(\eta t + \delta) dt$, ubi A, B, D, η, δ, k et h constantes sunt, quibus conditio inest, ut pro $\eta = o$ fiat $A = B = D = o$. Approximatio igitur prima suppeditat

$$I = \Sigma A_{,} \cos(\eta t + \delta)$$
$$\varphi = kt + \Sigma B_{,} \sin(\eta t + \delta)$$
$$\psi = ht + \Sigma D_{,} \sin(\eta t + \delta)$$

Substitutis his valoribus in approximatione secunda ope theorematis Tayloriani in aequationibus (45), non modo φ et ψ sed etiam I terminos per tempus ipsum multiplicatos continebunt, et eadem ratione intelligitur, in expressionibus ipsarum ζ atque β in approximatione secunda valores praecedentes ipsarum φ atque ψ terminos per tempus ipsum multiplicatos introducturos esse. Qui termini omnes amoventur adiungendo statim in approximatione prima kt ad φ et ht ad ψ. Sed aliud incommodum grave nascetur factoribus cotg I atque cosec I, qui in expressionibus ipsarum $d\varphi$ atque $d\psi$ continentur, et in approximatione secunda factores $-\frac{\delta I}{\sin^2 I}$ atque $-\frac{\delta I \cos I}{\sin^2 I}$ praebebunt, e quibus propter parvulam orbitae Lunae ad orbitam Solis inclinationem termini permagni prodituri sunt, qui in approximationibus subsequentibus instituendis series divergentes efficient. Quae quum ita sint, necesse est transformationes introducantur, de quibus in articulis sequentibus sermo erit.

25.

Sint

$$\pi + \pi' - (\varphi + \psi) - 2(n)\alpha t = 2N \;,\; \pi - \pi' - (\varphi - \psi) + 2(n)\eta t = 2K \qquad \text{......(46)}$$

ubi quantitates α et η adhuc indeterminatae sunt. Substitutis his valoribus in expressione ipsius Δ in art. praec. data, nanciscimur

$$\Delta^2 = \bar{r}^2 c^{2w} + \bar{r}'^2 c^{2w'} - 2\bar{r}\bar{r}' c^{w+w'} \cos^2 \tfrac{1}{2} I \cos[\bar{f} - \bar{f}' + (n)(y - y' - 2\eta)t + 2K]$$
$$- 2\bar{r}\bar{r}' c^{w+w'} \sin^2 \tfrac{1}{2} I \cos[\bar{f} + \bar{f}' + (n)(y + y' + 2\alpha)t + 2N]$$

et terminum generalem in evoluta quantitate Ω hunc

$$X \cos \{ig + i'g' + i''[2k + (n)(y - y' - 2\eta)t] + i'''[2\nu + (n)(y + y' + 2\alpha)t]\}$$

designante k constantem terminum, qui in valore integro ipsius K, et ν constantem terminum, qui in valore integro ipsius N continetur. Si porro sint

$$P = 2 \sin \tfrac{1}{2} I \sin (N - \nu) \quad , \quad Q = 2 \sin \tfrac{1}{2} I \cos (N - \nu)$$

quantitates P, Q et K loco I, φ et ψ pro variabilibus independentibus habendae sunt. Aequationes praecedentes differentiatae praebent

$$(47)\ldots\ldots \begin{cases} dP = dI \cos \tfrac{1}{2} I \sin (N - \nu) + 2dN \sin \tfrac{1}{2} I \cos (N - \nu) \\ dQ = dI \cos \tfrac{1}{2} I \cos (N - \nu) - 2dN \sin \tfrac{1}{2} I \sin (N - \nu) \end{cases}$$

Aequationes vero (46) suppeditant

$$2dN = - d\varphi - d\psi \quad , \quad 2dK = - d\varphi + d\psi$$

Considerata autem Ω et tanquam functione ipsarum I, φ et ψ, et tanquam functione ipsarum P, Q et K, habetur

$$\begin{aligned} d\Omega &= \left(\frac{d\Omega}{dI}\right) dI + \left(\frac{d\Omega}{d\varphi}\right) d\varphi + \left(\frac{d\Omega}{d\psi}\right) d\psi \\ &= \left(\frac{d\Omega}{dP}\right) dP + \left(\frac{d\Omega}{dQ}\right) dQ + \left(\frac{d\Omega}{dK}\right) dK \end{aligned}$$

unde adiumento aequationum praecedentium nanciscimur

$$(48)\ldots\ldots \begin{cases} \left(\frac{d\Omega}{dI}\right) = \left(\frac{d\Omega}{dP}\right) \cos \tfrac{1}{2} I \sin (N - \nu) + \left(\frac{d\Omega}{dQ}\right) \cos \tfrac{1}{2} I \cos (N - \nu) \\ \left(\frac{d\Omega}{d\varphi}\right) = - \left(\frac{d\Omega}{dP}\right) \sin \tfrac{1}{2} I \cos (N - \nu) + \left(\frac{d\Omega}{dQ}\right) \sin \tfrac{1}{2} I \sin (N - \nu) - \tfrac{1}{2} \left(\frac{d\Omega}{dK}\right) \\ \left(\frac{d\Omega}{d\psi}\right) = - \left(\frac{d\Omega}{dP}\right) \sin \tfrac{1}{2} I \cos (N - \nu) + \left(\frac{d\Omega}{dQ}\right) \sin \tfrac{1}{2} I \sin (N - \nu) + \tfrac{1}{2} \left(\frac{d\Omega}{dK}\right) \end{cases}$$

unde

$$\begin{aligned} \left(\frac{d\Omega}{d\varphi}\right) \operatorname{cotg} I + \left(\frac{d\Omega}{d\psi}\right) \operatorname{cosec} I = &- \left(\frac{d\Omega}{dP}\right) \cos \tfrac{1}{2} I \cos (N - \nu) + \left(\frac{d\Omega}{dQ}\right) \cos \tfrac{1}{2} I \sin (N - \nu) \\ &+ \tfrac{1}{2} \left(\frac{d\Omega}{dK}\right) \operatorname{tg} \tfrac{1}{2} I \end{aligned}$$

$$\begin{aligned} \left(\frac{d\Omega'}{d\psi}\right) \operatorname{cotg} I + \left(\frac{d\Omega'}{d\varphi}\right) \operatorname{cosec} I = &- \left(\frac{d\Omega'}{dP}\right) \cos \tfrac{1}{2} I \cos (N - \nu) + \left(\frac{d\Omega'}{dQ}\right) \cos \tfrac{1}{2} I \sin (N - \nu) \\ &- \tfrac{1}{2} \left(\frac{d\Omega'}{dK}\right) \operatorname{tg} \tfrac{1}{2} I \end{aligned}$$

Differentiatis vero aequationibus (46) respectu temporis, habetur

$$\frac{dN}{dt} = -\tfrac{1}{2}\left[\frac{d\varphi}{dt} + \frac{d\psi}{dt}\right] - (n)\alpha$$
$$\frac{dK}{dt} = -\tfrac{1}{2}\left[\frac{d\varphi}{dt} - \frac{d\psi}{dt}\right] + (n)\eta$$

unde (47) subministrant

$$\frac{dP}{dt} = -(n)\alpha Q + \frac{dI}{dt}\cos\tfrac{1}{2}I\sin(N-\nu) - \left(\frac{d\varphi}{dt} + \frac{d\psi}{dt}\right)\sin\tfrac{1}{2}I\cos(N-\nu)$$
$$\frac{dQ}{dt} = \quad (n)\alpha P + \frac{dI}{dt}\cos\tfrac{1}{2}I\cos(N-\nu) + \left(\frac{d\varphi}{dt} + \frac{d\psi}{dt}\right)\sin\tfrac{1}{2}I\sin(N-\nu)$$

quare, substitutis valoribus ipsarum $\frac{dI}{dt}$, $\frac{d\varphi}{dt}$ et $\frac{d\psi}{dt}$ ex (45) desumendis in his aequationibus, obtinemus ope aequationum (48) has

$$\left.\begin{aligned}
\frac{dP}{dt} &= -(n)\alpha Q - \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos^2\tfrac{1}{2}I + \tfrac{1}{4}P\left(\frac{d\Omega}{dK}\right)\right\} \\
&\qquad + \frac{dp'}{\cos i'dt}\cos\tfrac{1}{2}I\cos[\pi'-\nu+K-(n)(\alpha+\eta)t] - \frac{dq'}{\cos i'dt}\cos\tfrac{1}{2}I\sin[\pi'-\nu+K-(n)(\alpha+\eta)t] \\
\frac{dQ}{dt} &= (n)\alpha P + \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos^2\tfrac{1}{2}I - \tfrac{1}{4}Q\left(\frac{d\Omega}{dK}\right)\right\} \\
&\qquad - \frac{dp'}{\cos i'dt}\cos\tfrac{1}{2}I\sin[\pi'-\nu+K-(n)(\alpha+\eta)t] - \frac{dq'}{\cos i'dt}\cos\tfrac{1}{2}I\cos[\pi'-\nu+K-(n)(\alpha+\eta)t] \\
\frac{dK}{dt} &= (n)\eta + \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\frac{P}{4} + \left(\frac{d\Omega}{dQ}\right)\frac{Q}{4}\right\} \\
&\qquad + \left\{\frac{dp'}{\cos i'dt}\cdot\frac{Q}{4} + \frac{dq'}{\cos i'dt}\cdot\frac{P}{4}\right\}\sec\tfrac{1}{2}I\cos[\pi'-\nu+K-(n)(\alpha+\eta)t] \\
&\qquad + \left\{\frac{dp'}{\cos i'dt}\,\frac{P}{4} - \frac{dq'}{\cos i'dt}\,\frac{Q}{4}\right\}\sec\tfrac{1}{2}I\sin[\pi'-\nu+K-(n)(\alpha+\eta)t]
\end{aligned}\right\}(49)$$

quarum integratione P, Q atque K in theoria Lunae determinandae sunt.

Si vero problema trium corporum generale respicimus, aequationes (44) eidem transformationi subiiciendae sunt, unde prodit

$$\left.\begin{aligned}
\frac{dP}{dt} &= -(n)\alpha Q - \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos^2\tfrac{1}{2}I + \frac{P}{4}\left(\frac{d\Omega}{dK}\right)\right\} - \frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos^2\tfrac{1}{2}I - \frac{P}{4}\left(\frac{d\Omega'}{dK}\right)\right\} \\
\frac{dQ}{dt} &= \quad (n)\alpha P + \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos^2\tfrac{1}{2}I - \frac{Q}{4}\left(\frac{d\Omega}{dK}\right)\right\} + \frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos^2\tfrac{1}{2}I + \frac{Q}{4}\left(\frac{d\Omega'}{dK}\right)\right\} \\
\frac{dK}{dt} &= \quad (n)\eta + \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\frac{P}{4} + \left(\frac{d\Omega}{dQ}\right)\frac{Q}{4}\right\} - \frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\frac{P}{4} + \left(\frac{d\Omega'}{dQ}\right)\frac{Q}{4}\right\}
\end{aligned}\right\}\ldots\ldots(50)$$

quae igitur aequationes ad problema generale trium corporum spectant.

26.

Aequationibus (49) theorema notum demonstrari potest, secundum quod inclinatio mutua orbitarum Lunae et Terrae nodorumque motus lentissima orbitae terrae in spatio transpositione, vi attractiva planetarum producta, non afficitur.

Habetur enim, si quadratum temporis negligitur,

$$p' = t\,b \cos i' \quad , \quad q' = t\,b_{,} \cos i'$$

ubi $b_{,}$ variatio annua obliquitatis eclipticae, et b quantitas est, e qua differentia inter praecessionem lunisolarem et praecessionem generalem potissimum pendet. Hinc

$$\frac{dp'}{\cos i'\,dt} = b \text{ et } \frac{dq'}{\cos i'\,dt} = b_{,}$$

Substitutis his valoribus in aequationibus (49) pro P et Q, nanciscimur post integrationem, in qua I et K pro constantibus haberi licitum est, in ipsa P terminos hos

$$-\frac{b\cos\frac{1}{2}I}{(n)(\alpha+\eta)}\sin[\pi'-v+K-(n)(\alpha+\eta)t]-\frac{b_{,}\cos\frac{1}{2}I}{(n)(\alpha+\eta)}\cos[\pi'-v+K-(n)(\alpha+\eta)t]$$

et in Q terminos hos

$$-\frac{b\cos\frac{1}{2}I}{(n)(\alpha+\eta)}\cos[\pi'-v+K-(n)(\alpha+\eta)t]+\frac{b_{,}\cos\frac{1}{2}I}{(n)(\alpha+\eta)}\sin[\pi'-v+K-(n)(\alpha+\eta)t]$$

Termini igitur in ipsis p' et q' per tempus ipsum multiplicati in inclinatione mutua Lunae orbitae et Terrae, et in longitudine nodorum similes terminos non producunt, sed loco eorum terminos periodicos praecedentes gignunt, quorum periodus revolutioni integrae nodorum horum aequatur. Si in p' et q' ad terminos per temporis quadratum multiplicatos respicitur, termini in ipsis P et Q inveniuntur huius formae

$$t\,h \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \{\pi' - v + K - (n)(\alpha+\eta)t\}$$

ubi h constans. Qui igitur termini eandem habent periodum atque illi, sed eorum coefficiens tempori proportionalis est. Sunt vero termini hi quam minutissimi, itaque negligendi.

27.

Ut aequationes (49) per plures deinceps approximationes integrari possint, oportet valores approximatos ipsarum P, Q et K ab exordio notos esse, quorum substitutione aequationes (49) post integrationes peractas valores accuratiores earundem quantitatum suppeditent. Ad valores tales eruendos aequationibus generalioribus (50) utemur.

Deligamus terminos ipsarum Ω et Ω' ab arcubus $(n)z$ et $(n')z'$ independentes. Iam satis notum est terminos hos solummodo ex distantia mutua $\varDelta$, quae in ipsis Ω et Ω' continetur, nasci posse, quare pono pro hoc calculo

$$\Omega = \frac{m'}{M+m}\,\frac{1}{\varDelta}\ ,\ \Omega' = \frac{m}{M+m'}\,\frac{1}{\varDelta}$$

Ex forma vero ipsius $\varDelta$ in art. 25. inventa manifestum est, terminos hos in ipsa $\frac{1}{\varDelta}$ huius formae esse debere

$$\varepsilon - 2\mu \sin^2 \tfrac{1}{2} I$$

ubi ε ab inclinatione I independens, et μ huius formae est, $\mu_{,} + \mu_{,,} \sin^2 \frac{1}{2} I + \mu_{,,,} \sin^4 \frac{1}{2} I +$ etc. ubi $\mu_{,}$, $\mu_{,,}$, etc. ab I independentes sunt, et $\mu_{,}$ necessario quantitas positiva est. Aequationes

$$P = 2 \sin \tfrac{1}{2} I \sin (N - \nu)$$
$$Q = 2 \sin \tfrac{1}{2} I \cos (N - \nu)$$

suppeditant $P^2 + Q^2 = 4 \sin^2 \frac{1}{2} I$, itaque

$$\Omega = \frac{m'}{M+m} \{\varepsilon - \tfrac{1}{2}\mu (P^2 + Q^2)\}$$
$$\Omega' = \frac{m}{M+m'} \{\varepsilon - \tfrac{1}{2}\mu (P^2 + Q^2)\}$$

Quum in calculo, quem hoc loco peragemus, μ constans censeri possit, aequationes praecedentes subministrant

$$\frac{d\Omega}{dP} = -\frac{m'}{M+m}\mu P\ ;\ \frac{d\Omega}{dQ} = -\frac{m'}{M+m}\mu Q\ ;\ \frac{d\Omega}{dK} = o$$
$$\frac{d\Omega'}{dP} = -\frac{m}{M+m'}\mu P\ ;\ \frac{d\Omega'}{dQ} = -\frac{m}{M+m'}\mu Q\ ;\ \frac{d\Omega'}{dK} = o$$

Posita constante $\varkappa$, quam in art. 2. Sect. I. introduximus, unitati aequali, habetur

$$\frac{an}{\sqrt{1-e^2}} = \frac{M+m}{a^2 n\sqrt{1-e^2}}\;,\quad \frac{a'n'}{\sqrt{1-e'^2}} = \frac{M+m'}{a'^2 n'\sqrt{1-e'^2}}$$

Substitutis his valoribus omnibus in aequationibus (50), nanciscimur

$$(51)\ldots\ldots\left\{\begin{aligned}\frac{dP}{dt} &= -\left\{(n)\alpha - m'\frac{\mu}{w}\cos^2\tfrac{1}{2}I - m\frac{\mu}{w'}\cos^2\tfrac{1}{2}I\right\}Q\\ \frac{dQ}{dt} &= \left\{(n)\alpha - m'\frac{\mu}{w}\cos^2\tfrac{1}{2}I - m\frac{\mu}{w'}\cos^2\tfrac{1}{2}I\right\}P\\ \frac{dK}{dt} &= (n)\eta - m'\frac{\mu}{w}\sin^2\tfrac{1}{2}I + m\frac{\mu}{w'}\sin^2\tfrac{1}{2}I\end{aligned}\right.$$

ubi brevitatis gratia

$$a^2 n\sqrt{1-e^2} = w \text{ et } a'^2 n'\sqrt{1-e'^2} = w'$$

posui. Quum in his aequationibus ipsas w, w' et $\cos^2\frac{1}{2}I$ pro constantibus haberi liceat, primis duabus aequationibus satisfacies, si ponis

$$(52)\ldots\ldots\quad (n)\alpha = \left\{\frac{m'}{w} + \frac{m}{w'}\right\}\mu\cos^2\tfrac{1}{2}(I)$$

$$P = C$$
$$Q = C_{,}$$

ubi (I) valor constans et quasi medius ipsius I, et C atque $C_{,}$ duae constantes sunt. His constantibus determinandis inserviunt aequationes hae

$$C = 2\sin\tfrac{1}{2}I\sin(N-\nu)$$
$$C_{,} = 2\sin\tfrac{1}{2}I\cos(N-\nu)$$

ubi in membro ad dextram valores constantes ipsarum I et N ponendi sunt. Valorem constantem ipsius I modo (I) appellavimus, et valorem constantem ipsius N in art. 25. definivimus esse ν; substitutis his valoribus nanciscimur

$$C = o$$
$$C_{,} = 2\sin^2\tfrac{1}{2}(I)$$

unde pro approximatione prima evadit

$$P = o$$
$$Q = 2\sin\tfrac{1}{2}(I)$$

Substitutis his valoribus in tertia aequatione (51), et posita

$$(n)\eta = \left\{\frac{m'}{w} - \frac{m}{w'}\right\}\cdot\mu\sin^2\tfrac{1}{2}(I)$$

habetur

$$\frac{dK}{dt} = o$$

unde

$$K = k$$

denotante k terminum omnino constantem in valore ipsius K, sicuti iam in art. 25. definivimus.

28.

Ex analysi art. praec. emerserunt valores ipsarum P, Q, K, α et η in approximatiöne prima in aequationibus (50) substituendi; quibus substitutis et integrationibus peractis, termini per tempus ipsum multiplicati oriri non possunt, quia arbitrarias α et η ita determinavimus, ut hi termini evanuerint. Valores vero ipsarum P, Q et K, quos tali modo approximatio prima suppeditabit, veris earum valoribus magis appropinquant, et in approximatione secunda substituendi sunt. Quo factum erit, ut in aequationibus pro $\frac{dP}{dt}$ et $\frac{dK}{dt}$ termini constantes orituri sint, qui post integrationes peractas terminos per tempus ipsum multiplicatos prodant, qui vero termini, additis terminis correctionis $(n)\delta\alpha$ et $(n)\delta\eta$ resp. ad $(n)\alpha$ et $(n)\eta$, et determinatis $\delta\alpha$ et $\delta\eta$ ita ut termini hi constantes in approximatione secunda ex aequationibus pro $\frac{dP}{dt}$ et $\frac{dK}{dt}$ orituri evanescant, tolluntur; et sic porro in approximationibus subsequentibus. Calculis igitur omnino absolutis, P, Q et K terminos per tempus ipsum multiplicatos non continebunt, et α atque η habebunt valores in art. praec. inventos, quibus vero termini e quadrato potestatibusque altioribus vis perturbantis pendentes accesserint. Notandum est terminos, de quibus locuti sumus, solummodo ex aequationibus pro $\frac{dP}{dt}$ et $\frac{dK}{dt}$ tollendos esse, in aequatione enim pro $\frac{dQ}{dt}$ termini tales oriri nequeunt, id quod per calculum praeparatorium art. praec. effecimus.

Quae in praecedentibus de aequationibus (50) disseruimus, nulla fere mutatione facta ad aequationes (49), quae praecipue ad theoriam Lunae spectant, applicari possunt. In his aequationibus terminus $\frac{m}{w}\mu$ in termi-

nis, qui per dp' et dq' multiplicati sunt, continetur, sed quum in hac Lunae theoria denotet m' massam Solis et m massam Lunae per eandem unitatem expressas, terminus $\frac{m}{w'}\mu$ ne minimam quidem vim habet, ita ut loco valorum ipsarum α et η, qui in casu generali problematis trium corporum per aequationes (52) et (53) dati sunt, habeamus in approximatione prima ad perturbationes Lunae obtinendas instituenda

$$(n)\alpha = \frac{m'}{w}\mu\cos^2\tfrac{1}{2}(I)$$
$$(n)\eta = \frac{m'}{w}\mu\sin^2\tfrac{1}{2}(I)$$

Porro forma ea, sub qua ipsas $(n')z'$ et r' in artt. 7. et 16. in theoria Lunae admisimus, efficiet ut in approximatione secunda et subsequentibus termini adsint huius formae $ct + c_{,}t^2 +$ etc. ubi c, $c_{,}$, etc. constantes sunt. Hinc factum erit, ut α et η non omnino constantes sint, sed huius formae

$$\alpha = \alpha_{,} + \alpha_{,,}(n)t + \alpha_{,,,}(n)^2 t^2 + \text{etc.}$$
$$\eta = \eta_{,} + \eta_{,,}(n)t + \eta_{,,,}(n)^2 t^2 + \text{etc.}$$

ubi $\alpha_{,}$, $\alpha_{,,}$, $\alpha_{,,,}$, etc. et $\eta_{,}$, $\eta_{,,}$, $\eta_{,,,}$, etc. verae constantes sunt. Multiplicatis his aequationibus per dt, integratione facta habetur

$$\alpha t = (\alpha_{,} + \tfrac{1}{2}\alpha_{,,}(n)t + \tfrac{1}{3}\alpha_{,,,}(n)^2 t^2 + \text{etc.})\,t$$
$$\eta t = (\eta_{,} + \tfrac{1}{2}\eta_{,,}(n)t + \tfrac{1}{3}\eta_{,,,}(n)^2 t^2 + \text{etc.})\,t$$

qui valores loco αt et ηt in expressione ipsius Δ in art. 25. data substituendi sunt, veluti in art. 7. invenimus expressionem hanc

$$yt = (y_{,} + \tfrac{1}{2}y_{,,}(n)t + \tfrac{1}{3}y_{,,,}(n)^2 t^2 + \text{etc.})\,t$$

loco yt in expressione ipsius Δ quoque substituendam esse. Quae quidem sunt, quibus integratio aequationum (49) ab integratione aequationum (50) differt.

Ad aequationes illas integrandas necesse non est quantitas Ω revera in functione ipsarum P, Q et K exhibeatur; considerata enim Ω et tanquam functione ipsarum P, Q atque K, et tanquam functione ipsarum I, N atque K, habetur

$$d\Omega = \left(\frac{d\Omega}{dP}\right) dP + \left(\frac{d\Omega}{dQ}\right) dQ + \left(\frac{d\Omega}{dK}\right) dK$$
$$= \left(\frac{d\Omega}{dI}\right) dI + \left(\frac{d\Omega}{dN}\right) dN + \left(\frac{d\Omega}{dK}\right) dK$$

unde substitutis valoribus ipsarum dP et dQ ex (47) desumendis, elicitur

$$\left(\frac{d\Omega}{dK}\right) = \left(\frac{d\Omega}{dK}\right)$$
$$\left(\frac{d\Omega}{dI}\right) = \left(\frac{d\Omega}{dP}\right) \cos \tfrac{1}{2} I \sin (N-v) + \left(\frac{d\Omega}{dQ}\right) \cos \tfrac{1}{2} I \cos (N-v)$$
$$\left(\frac{d\Omega}{dN}\right) = 2\left(\frac{d\Omega}{dP}\right) \sin \tfrac{1}{2} I \cos (N-v) - 2\left(\frac{d\Omega}{dQ}\right) \sin \tfrac{1}{2} I \sin (N-v)$$

quae reciproce dant

$$\left.\begin{aligned}\left(\frac{d\Omega}{dP}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{\sin (N-v)}{\cos \frac{1}{2} I} + \left(\frac{d\Omega}{dN}\right) \frac{\cos (N-v)}{2 \sin \frac{1}{2} I} \\ \left(\frac{d\Omega}{dQ}\right) &= \left(\frac{d\Omega}{dI}\right) \frac{\cos (N-v)}{\cos \frac{1}{2} I} - \left(\frac{d\Omega}{dN}\right) \frac{\sin (N-v)}{2 \sin \frac{1}{2} I}\end{aligned}\right\} \ldots\ldots (54)$$

e quibus valores quotientium differentialium, quos aequationes (49) requirunt, computari possunt, manente Δ sub forma ea, quam ei in art. 25. attribuimus.

Quum secundum art. praec. in approximatione prima ponendae sunt

$$P = o$$
$$Q = 2 \sin \tfrac{1}{2} (I)$$

aequationes (54) praebent pro hac approximatione

$$\left(\frac{d\Omega}{dP}\right) = \left(\frac{d\Omega}{dN}\right) \frac{1}{2 \sin \frac{1}{2} (I)} = \left(\frac{d\Omega}{dv}\right) \frac{1}{2 \sin \frac{1}{2} (I)}$$
$$\left(\frac{d\Omega}{dQ}\right) = \left(\frac{d\Omega}{d(I)}\right) \frac{1}{\cos \frac{1}{2} (I)}$$

et quum in eadem approximatione $K = k$ ponendum sit, habetur

$$\left(\frac{d\Omega}{dK}\right) = \left(\frac{d\Omega}{dk}\right)$$

Adiumento harum aequationum aequationes (49) pro approximatione prima ita se habent,

$$(55)\begin{cases}\dfrac{dP}{dt}=-2(n)\alpha\sin\tfrac{1}{2}(I)-\dfrac{(a)(n)}{\sqrt{1-(e)^2}}\left(\dfrac{d\Omega}{d(I)}\right)\cos\tfrac{1}{2}(I)\\ \qquad+\dfrac{dp'}{\cos i'dt}\cos\tfrac{1}{2}(I)\cos[\pi'-v+k-(n)(\alpha+\eta)t]-\dfrac{dq'}{\cos i'dt}\cos\tfrac{1}{2}(I)\sin[\pi'-v+k-(n)(\alpha+\eta)t]\\ \dfrac{dQ}{dt}=\dfrac{(a)(n)}{\sqrt{1-(e)^2}}\left\{\left(\dfrac{d\Omega}{dv}\right)\dfrac{\cos^2\frac{1}{2}(I)}{2\sin\frac{1}{2}(I)}-\tfrac{1}{2}\left(\dfrac{d\Omega}{dK}\right)\sin\tfrac{1}{2}(I)\right\}\\ \qquad-\dfrac{dp'}{\cos i'dt}\cos\tfrac{1}{2}(I)\sin[\pi'-v+k-(n)(\alpha+\eta)t]-\dfrac{dq'}{\cos i'dt}\cos\tfrac{1}{2}(I)\cos[\pi'-v+k-(n)(\alpha+\eta)t]\\ \dfrac{dK}{dt}=(n)\eta+\tfrac{1}{2}\dfrac{(a)(n)}{\sqrt{1-(e)^2}}\left(\dfrac{d\Omega}{d(I)}\right)\operatorname{tg}\tfrac{1}{2}(I)\\ \qquad+\tfrac{1}{2}\dfrac{dp'}{\cos i'dt}\operatorname{tg}\tfrac{1}{2}(I)\cos[\pi'-v+k-(n)(\alpha+\eta)t]-\tfrac{1}{2}\dfrac{dq'}{\cos i'dt}\operatorname{tg}\tfrac{1}{2}(I)\sin[\pi'-v+k-(n)(\alpha+\eta)t]\end{cases}$$

Substitutis igitur in quotientibus differentialibus ipsius Ω in aequationibus (55) non modo (I), v et k loco I, N et K, sed etiam valoribus ipsarum $\bar{f}$, $\bar{f}'$, $\bar{r}$ et $\bar{r}'$ per g et resp. per g' expressis, factisque $w=o$ et $w'=o$, aequationes hae functiones solius variabilis t sunt, quae, determinatis α atque η secundum regulam modo traditam, facili opera ita integrari possunt, ut termini per tempus ipsum multiplicati non adsint. Constantes vero his integralibus addendae sunt ipsi P cifra ipsa, ipsi Q.. $2\sin\frac{1}{2}(I)$ et ipsi K.. k. Tali igitur modo valores accuratiores ipsarum P, Q et K innotescunt, qui approximationi secundae absolvendae inservient.

In hac secunda approximatione adduntur resp. δP, δQ et δK ad valores ipsarum P, Q atque K, et habetur

$\delta P =$ ei ipsius P valori, quem approximatio prima prodiderat;

$dQ = -2\sin\frac{1}{2}(I) +$ analogo ipsius Q valori;

$\delta K = -k +$ analogo ipsius K valori.

In approximatione secunda etiam ipsarum $(n)\delta z$, δlr, δlh, $(n')\delta z'$ et $\delta lr'$ ratio habenda est, et sic porro in approximationibus subsequentibus, si his opus erit.

Iisdem rationibus, quas in his articulis explicavi, substituuntur in aequationibus differentialibus illis, quae integratione valores ipsarum ζ, β et S praebent, in approximatione prima (I), v et k loco I, N, K; in approximatione vero secunda et subsequentibus non modo incrementorum

$(n)\delta\zeta$, δl_ϱ, $(n)\delta z$, etc., quae supra explicavi, sed etiam incrementorum δP, δQ et δK, quae in hoc articulo explicata sunt, ratio habenda est.

29.

Analysis in praecedentibus exposita in fine calculi valores veros ipsarum $(n)z$, w, P, Q et K suppeditabit, restat igitur ut ope harum quantitatum valores veri ipsarum p et q, e quibus latitudo Lunae supra planum fundamentale seu proiectionis et reductio longitudinis ad idem planum pendent, investigentur. Cui computationi inserviunt aequationes (42). Quae aequationes, in quibus Ω pro functione ipsarum I, φ et ψ habita est, ante omnia in alias transformandae sunt, in quibus Ω pro functione ipsarum P, Q et K haberi potest. Haec autem transformatio aequationibus (48) et (46) perficitur, et calculo peracto nanciscimur

$$\begin{aligned}
dp = & \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos\left[\pi - v - K - (n)(\alpha-\eta)t\right]dt \\
& + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin\left[\pi - v - K - (n)(\alpha-\eta)t\right]dt \\
dq = & \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos\left[\pi - v - K - (n)(\alpha-\eta)t\right]dt \\
& - \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin\left[\pi - v - K - (n)(\alpha-\eta)t\right]dt
\end{aligned}$$

et eodem modo, si problema generale trium corporum consideratur, nanciscimur ex (43)

$$\begin{aligned}
dp' = & -\frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos\left[\pi' - v + K - (n)(\alpha+\eta)t\right]dt \\
& - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin\left[\pi' - v + K - (n)(\alpha+\eta)t\right]dt \\
dq' = & -\frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos\left[\pi' - v + K - (n)(\alpha+\eta)t\right]dt \\
& + \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin\left[\pi' - v + K - (n)(\alpha+\eta)t\right]dt
\end{aligned}$$

Quae aequationes, si $\cos i$ et $\cos i'$ excipis, functiones sunt quantitatum $(n)z$, w, S, $(n')z'$, w', S', P, Q atque K, quae secundum praecedentia notae et in functione solius variabilis t exhibitae supponi possunt. Aequationes igitur hae pro functionibus temporis et ipsarum $\cos i$ et $\cos i'$, sive quum

$$\cos i = \sqrt{1-p^2-q^2} \text{ atque } \cos i' = \sqrt{1-p'^2-q'^2}$$

sit, pro functionibus temporis et ipsarum p, q, p' et q' haberi possunt. Quae conditio efficit, ut in integralibus earum termini nascantur, qui per tempus ipsum multiplicati essent, nec admitti possent. Ad hos terminos tollendos transformationem et integrandi methodum assequutus sum, quam statim explicabo. Sint

$$\begin{aligned}
p_{,} &= \sin i \sin(\chi - \omega + (n)\varepsilon t + D) \\
q_{,} &= \sin i \cos(\chi - \omega + (n)\varepsilon t + D) \\
p'_{,} &= \sin i' \sin(\chi' - \omega' + (n)\varepsilon' t + D') \\
q'_{,} &= \sin i' \cos(\chi' - \omega' + (n)\varepsilon' t + D')
\end{aligned}$$

ubi ε, ε', D et D' constantes indeterminatae sunt. Quum vero secundum art. ult. Sect. I. sit

$$\begin{aligned}
p &= \sin i \sin(\chi - \omega) \\
q &= \sin i \cos(\chi - \omega)
\end{aligned}$$

quibus similes sunt hae, quae ad corpus m' spectant

$$\begin{aligned}
p' &= \sin i' \sin(\chi' - \omega') \\
q' &= \sin i' \cos(\chi' - \omega')
\end{aligned}$$

$p_{,}$, $q_{,}$, $p'_{,}$ et $q'_{,}$ ita quoque exprimi possunt

$$\begin{aligned}
p_{,} &= p \cos[(n)\varepsilon t + D] + q \sin[(n)\varepsilon t + D] \\
q_{,} &= -p \sin[(n)\varepsilon t + D] + q \cos[(n)\varepsilon t + D] \\
p'_{,} &= p' \cos[(n)\varepsilon' t + D'] + q' \sin[(n)\varepsilon' t + D'] \\
q'_{,} &= -p' \sin[(n)\varepsilon' t + D'] + q' \cos[(n)\varepsilon' t + D']
\end{aligned}$$

e quibus differentiando inveniuntur hae

$$\begin{aligned}
\frac{dp_{,}}{dt} &= (n)\varepsilon q_{,} + \frac{dp}{dt}\cos[(n)\varepsilon t + D] + \frac{dq}{dt}\sin[(n)\varepsilon t + D] \\
\frac{dq_{,}}{dt} &= -(n)\varepsilon p_{,} - \frac{dp}{dt}\sin[(n)\varepsilon t + D] + \frac{dq}{dt}\cos[(n)\varepsilon t + D] \\
\frac{dp'_{,}}{dt} &= (n)\varepsilon' q'_{,} + \frac{dp'}{dt}\cos[(n)\varepsilon' t + D'] + \frac{dq'}{dt}\sin[(n)\varepsilon' t + D'] \\
\frac{dq'_{,}}{dt} &= -(n)\varepsilon' p'_{,} - \frac{dp'}{dt}\sin[(n)\varepsilon' t + D'] + \frac{dq'}{dt}\cos[(n)\varepsilon' t + D']
\end{aligned}$$

quae aequationes, substitutis valoribus ipsarum $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dp'}{dt}$ et $\frac{dq'}{dt}$ ex praecedentibus aequationibus desumendis, transeunt in has

$$\begin{aligned}
\frac{dp_{,}}{dt} &= (n)\varepsilon q_{,} + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos[\pi-\nu+D-K+(n)(\alpha-\eta-\varepsilon)t] \\
&\quad + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin[\pi-\nu+D-K+(n)(\alpha-\eta-\varepsilon)t] \\
\frac{dq_{,}}{dt} &= -(n)\varepsilon p_{,} + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos[\pi-\nu+D-K+(n)(\alpha-\eta-\varepsilon)t] \\
&\quad - \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin[\pi-\nu+D-K+(n)(\alpha-\eta-\varepsilon)t] \\
\frac{dp'_{,}}{dt} &= (n)\varepsilon' q'_{,} - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos[\pi'-\nu+D'+K-(n)(\alpha+\eta-\varepsilon')t] \\
&\quad - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin[\pi'-\nu+D'+K-(n)(\alpha+\eta-\varepsilon')t] \\
\frac{dq'_{,}}{dt} &= -(n)\varepsilon' p'_{,} - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos[\pi'-\nu+D'+K-(n)(\alpha+\eta-\varepsilon')t] \\
&\quad + \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin[\pi'-\nu+D'+K-(n)(\alpha+\eta-\varepsilon')t]
\end{aligned}$$

Ut aequationes hae quam simplicissimae reddantur, quantitates arbitrariae ε, ε', D et D' ita determinandae sunt, ut anguli, quorum sinus et cosinus in his aequationibus continentur, in quantitates periodicas primi ordinis abeant. Quae conditio subministrat

$$\begin{aligned}
\varepsilon &= \alpha - \eta \\
\varepsilon' &= \alpha + \eta \\
D &= -\pi + \nu + k \\
D' &= -\pi' + \nu - k
\end{aligned}$$

Substitutis his valoribus, aequationes praecedentes abeunt in has

$$\left.\begin{aligned}
\frac{dp_{,}}{dt} &= (n)(\alpha-\eta)q_{,} + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos(K-k) \\
&\quad - \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin(K-k) \\
\frac{dq_{,}}{dt} &= -(n)(\alpha-\eta)p_{,} + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos(K-k) \\
&\quad + \frac{an\cos i}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin(K-k) \\
\frac{dp'_{,}}{dt} &= (n)(\alpha+\eta)q'_{,} - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos(K-k) \\
&\quad - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin(K-k) \\
\frac{dq'_{,}}{dt} &= -(n)(\alpha+\eta)p'_{,} - \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos(K-k) \\
&\quad + \frac{a'n'\cos i'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I - \left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin(K-k)
\end{aligned}\right\}\cdots\cdots(56)$$

30.

Quum

$$\cos i = \sqrt{1-p_{,}^{2}-q_{,}^{2}} \text{ atque } \cos i' = \sqrt{1-p_{,}'^{2}-q_{,}'^{2}}$$

et quum P, Q, K, $(n)z$, etc. iam computatas ideoque in functione temporis expressas esse supponatur, aequationes praecedentes, ut iam dixi, functiones ipsarum $p_{,}$, $q_{,}$, $p'_{,}$, $q'_{,}$ et temporis t sunt. Positis igitur brevitatis caussa

$$(57)\ldots\ldots\left\{\begin{aligned}
ft &= -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I+\left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos(K-k)\\
&\quad+\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I-\left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin(K-k)\\
\varphi t &= -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I-\left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos(K-k)\\
&\quad-\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I+\left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin(K-k)\\
f't &= -\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I-\left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\cos(K-k)\\
&\quad-\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I+\left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\sin(K-k)\\
\varphi' t &= -\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP}\right)\cos\tfrac{1}{2}I+\left(\frac{d\Omega'}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}\cos(K-k)\\
&\quad+\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dQ}\right)\cos\tfrac{1}{2}I-\left(\frac{d\Omega'}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}\sin(K-k)
\end{aligned}\right.$$

erunt ft, φt, $f't$ et $\varphi' t$ functiones notae solius variabilis t. Substitutis his quantitatibus in aequationibus in fine art. praec. inventis, nanciscimur

$$(58)\ldots\ldots\left\{\begin{aligned}
\frac{dp_{,}}{dt} &= (n)(\alpha-\eta)\,q_{,} - ft\,.\cos i\\
\frac{dq_{,}}{dt} &= -(n)(\alpha-\eta)\,p_{,} - \varphi t\,.\cos i\\
\frac{dp'_{,}}{dt} &= (n)(\alpha+\eta)\,q'_{,} + f't\,.\cos i'\\
\frac{dq'_{,}}{dt} &= -(n)(\alpha+\eta)\,p'_{,} + \varphi' t\,.\cos i'
\end{aligned}\right.$$

Ad has aequationes integrandas pono

$$\cos i = u \quad , \quad \cos i' = u'$$

unde

$$u^2 + p_,^2 + q_,^2 = 1 \quad , \quad u'^2 + p_,'^2 + q_,'^2 = 1$$

Quae aequationes differentiatae suppeditant

$$u\frac{du}{dt} + p_,\frac{dp_,}{dt} + q_,\frac{dq_,}{dt} = o \quad , \quad u'\frac{du'}{dt} + p'_,\frac{dp'_,}{dt} + q'_,\frac{dq'_,}{dt} = o$$

Substitutis valoribus ipsarum $\frac{dp_,}{dt}$, $\frac{dq_,}{dt}$, $\frac{dp'_,}{dt}$ et $\frac{dq'_,}{dt}$ ex aequationibus (58) desumendis in his aequationibus, substitutisque u loco cos i et u' loco cos i', nanciscimur

$$\left.\begin{aligned}
\frac{du}{dt} &= \quad p_, f\,t \quad + q_,\varphi\,t \\
\frac{dp_,}{dt} &= \quad (n)\,(\alpha - \eta)\;q_, - u\,f\,t \\
\frac{dq_,}{dt} &= -(n)\,(\alpha - \eta)\;p_, - u\,\varphi\,t \\
\frac{du'}{dt} &= - \quad p'_, f'\,t \quad - q'_,\varphi'\,t \\
\frac{dp'_,}{dt} &= \quad (n)\,(\alpha + \eta)\,q'_, + u' f'\,t \\
\frac{dq'_,}{dt} &= -(n)\,(\alpha + \eta)\,p'_, + u'\varphi'\,t
\end{aligned}\right\} \ldots\ldots(59)$$

Quatuor igitur aequationes (58) per hanc transformationem in sex aequationes differentiales lineares primi ordinis cum coefficientibus variabilibus transformatae sunt, quarum tres priores et tres posteriores simultanee integrandae sunt. Ad has integrationes perficiendas consideremus indolem functionum $f\,t$ et $\varphi\,t$. Ex praecedentibus sequitur $K - k$ esse seriem infinitam, cuius terminus generalis huius formae est A sin $(\beta\,t + B)$, designantibus A, β et B constantes, quae series ita comparata est, ut terminum omnino constantem non habeat. sin $(K - k)$ igitur est series eiusdem formae, et cos $(K - k)$ est series, cuius terminus primus est unitas ipsa et termini reliqui per formam hanc A' cos $(\beta t + B)$ repraesentari possunt. Quantitas

$$\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos\tfrac{1}{2}I + \left(\frac{d\Omega}{dK}\right)\frac{P}{4\cos\frac{1}{2}I}\right\}$$

est series, cuius terminus generalis est huius formae C cos $(\beta t + B)$ et quantitas

$$\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP}\right)\cos\tfrac{1}{2}I-\left(\frac{d\Omega}{dK}\right)\frac{Q}{4\cos\frac{1}{2}I}\right\}$$

series, cuius terminus generalis est huius formae $C'\sin(\beta t+B)$, et in omnibus his seriebus valores singuli ipsarum β et B in quoque argumento coniuncti iidem sunt. Iam quum multiplicatione cosinus per cosinum et sinus per sinum prodeat cosinus et multiplicatione cosinus per sinum prodeat sinus, ex forma serierum allata sequitur, in ft terminum omnino constantem contineri, in φt vero talem terminum contineri non posse. Et eodem modo demonstratur, in $f't$ terminum constantem contineri, in $\varphi' t$ vero talem terminum contiheri non posse. Quibus positis, sit $(n)c$ terminus constans in ft, et $(n)c'$ terminus constans in $f't$. Si igitur primum ad terminos tantum aequationum praecedentium linearium, quorum coefficientes constantes sunt, respicimus, nanciscimur

$$\begin{aligned}
\frac{dp_{,}}{dt} &= \quad (n)(\alpha-\eta)\,q_{,}-(n)\,c\,u\\
\frac{dq_{,}}{dt} &= -(n)(\alpha-\eta)\,p_{,}\\
\frac{du}{dt} &= \quad (n)\,c\,p_{,}\\
\frac{dp'_{,}}{dt} &= \quad (n)(\alpha+\eta)\,q'_{,}+(n)\,c'u'\\
\frac{dq'_{,}}{dt} &= -(n)(\alpha+\eta)\,p'_{,}\\
\frac{du'}{dt} &= -(n)\,c'p'_{,}
\end{aligned}$$

His aequationibus, ut notum est, satisfaciunt valores variabilium hi

$$\begin{aligned}
p_{,} &= Ce^{\beta t}\ ;\ q_{,}=C_{,}e^{\beta t}\ ;\ u=C_{,,}e^{\beta t}\\
p'_{,} &= C'e^{\beta' t}\ ;\ q'_{,}=C'_{,}e^{\beta' t}\ ;\ u'=C'_{,,}e^{\beta' t}
\end{aligned}$$

ubi C, $C_{,}$, $C_{,,}$, C', $C'_{,}$, $C'_{,,}$, β atque β' constantes adhuc indeterminatae et e basis logarithmorum hyperbolicorum est. Substitutis his valoribus variabilium in aequationibus differentialibus praecedentibus, nanciscimur ad ipsas β et β' determinandas aequationes has

$$\beta=(n)\sqrt{-(\alpha-\eta)^2-c^2}\ ;\ \beta'=(n)\sqrt{-(\alpha+\eta)^2-c'^2}$$

qui valores, quum imaginarii sint, manifestant aequationes nostras per sinus et cosinus arcuum realium semper integrari posse. Quare suppono esse

$$p_{,} = C \sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right]$$
$$q_{,} = bC \cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] + C_{,}$$
$$u = b_{,}C \cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] + b_{,,}C_{,}$$

ubi C, $C_{,}$, Θ, b, $b_{,}$ et $b_{,,}$ constantes indeterminatae sunt. Substitutis his valoribus in tribus prioribus aequationibus differentialibus praecedentibus, inveniuntur aequationes hae

$$(n)\sqrt{(\alpha-\eta)^2+c^2}\,C\cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2}-\Theta\right] = (n)\left[(\alpha-\eta)b - cb_{,}\right]C\cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2}-\Theta\right] + (n)\left[\alpha-\eta - cb_{,,}\right]C_{,}$$
$$-(n)b\sqrt{(\alpha-\eta)^2+c^2}\,C\sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2}-\Theta\right] = -(n)(\alpha-\eta)\,C\sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2}-\Theta\right]$$
$$-(n)b_{,}\sqrt{(\alpha-\eta)^2+c^2}\,C\sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2}-\Theta\right] = (n)\;cC\;\sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2}-\Theta\right]$$

quae identicae esse debent. Quarum ultimae duae statim suppeditant

$$b = \frac{\alpha-\eta}{\sqrt{(\alpha-\eta)^2+c^2}}$$
$$b_{,} = -\frac{c}{\sqrt{(\alpha-\eta)^2+c^2}},$$

quae in prima substitutae subministrant

$$b_{,,} = \frac{\alpha-\eta}{c}$$

constantes vero C, $C_{,}$ et Θ indeterminatae remanent. Substitutis his aequationibus in valoribus ipsarum $p_{,}$, $q_{,}$ et u modo assumtis, evadunt

$$p_{,} = C\sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right]$$
$$q_{,} = \frac{\alpha-\eta}{\sqrt{(\alpha-\eta)^2+c^2}}\,C\cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] + C_{,}$$
$$u = -\frac{c}{\sqrt{(\alpha-\eta)^2+c^2}}\,C\cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] + \frac{\alpha-\eta}{c}\,C_{,}$$

quae propter tres constantes arbitrarias C, $C_{,}$ et Θ integralia integra trium priorum aequationum differentialium nostrarum sunt. Sed inter quantitates $p_{,}$, $q_{,}$ et u intercedit aequatio conditionalis haec

$$1 = u^2 + p_{,}^2 + q_{,}^2$$

quare una quaeque illarum constantium e reliquis pendet. Ut aequatio conditionalis inter constantes arbitrarias obtineatur, substituantur valores ipsarum $p_{,}$, $q_{,}$ et u modo inventi in aequatione praecedenti, quo facto, nan-

ciscimur aequationem identicam, e qua aequatio conditionalis inter C et $C_,$ emergit haec

$$1 = C^2 + \frac{(\alpha-\eta)^2 + c^2}{c^2} C_,^2$$

Posita igitur $C = \sin \Gamma$, habemus

$$C_, = \pm \frac{c}{\sqrt{(\alpha-\eta)^2 + c^2}} \cos \Gamma$$

ubi signum algebraicum ex arbitrio eligi potest; electo superiore valores ipsarum $p_,$, $q_,$ et u denique ita se habent

$$(60)\ldots\ldots\begin{cases} p_, = \sin\Gamma \sin\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] \\ q_, = \frac{\alpha-\eta}{\sqrt{(\alpha-\eta)^2+c^2}} \sin\Gamma\cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] + \frac{c}{\sqrt{(\alpha-\eta)^2+c^2}} \cos\Gamma \\ u = -\frac{c}{\sqrt{(\alpha-\eta)^2+c^2}} \sin\Gamma\cos\left[(n)t\sqrt{(\alpha-\eta)^2+c^2} - \Theta\right] + \frac{\alpha-\eta}{\sqrt{(\alpha-\eta)^2+c^2}} \cos\Gamma \end{cases}$$

ubi Γ et Θ constantes sunt e situ plani proiectionis in spatio pendentes. Eodem modo ultimae tres aequationes nostrae differentiales integrantur, et integratae suppeditant

$$(61)\ldots\ldots\begin{cases} p'_, = \sin\Gamma' \sin\left[(n)t\sqrt{(\alpha+\eta)^2+c'^2} - \Theta'\right] \\ q'_, = \frac{\alpha+\eta}{\sqrt{(\alpha+\eta)^2+c'^2}} \sin\Gamma'\cos\left[(n)t\sqrt{(\alpha+\eta)^2+c'^2} - \Theta'\right] - \frac{c'}{\sqrt{(\alpha+\eta)^2+c'^2}} \cos\Gamma' \\ u' = \frac{c'}{\sqrt{(\alpha+\eta)^2+c'^2}} \sin\Gamma'\cos\left[(n)t\sqrt{(\alpha+\eta)^2+c'^2} - \Theta'\right] + \frac{\alpha+\eta}{\sqrt{(\alpha+\eta)^2+c'^2}} \cos\Gamma' \end{cases}$$

ubi Γ' et Θ' etiam constantes sunt e situ plani proiectionis in spatio pendentes. Quibus valoribus praeliminaribus ipsarum $p_,$, $q_,$, u, $p'_,$, $q'_,$ et u' inventis, ad aequationes differentiales rigorosas (59) revertimur. Positis

$$(62)\ldots\ldots\begin{cases} u\,[ft - (n)c] = H \\ u\varphi t = L \\ p_,\,[ft - (n)c] + q_,\varphi t = M \end{cases}$$

priores tres aequationes (59) abeunt in has

$$(63)\ldots\ldots\begin{cases} \frac{dp_,}{dt} = (n)(\alpha-\eta)\,q_, - (n)c\,u - H \\ \frac{dq_,}{dt} = -(n)(\alpha-\eta)\,p_, - L \\ \frac{du}{dt} = (n)c\,p_, + M \end{cases}$$

Quum termini ipsarum $p_{,}$, $q_{,}$ et u, quos modo integratione eruimus, omnium maximi sint, valores approximati ipsarum H, L et M substitutis his ipsarum $p_{,}$, $q_{,}$ et u terminis in expressionibus (62) obtinentur. Quo factum erit, ut H, L et M functiones explicitae temporis sint, et aequationes praecedentes per methodum notam integrari possint. Hac autem integratione valores accuratiores ipsarum $p_{,}$, $q_{,}$ et u innotescent, qui in (62) substituti valores accuratiores ipsarum H, L et M suppeditabunt, quibus in praecedentibus aequationibus substitutis, integratione valores adhuc accuratiores ipsarum $p_{,}$, $q_{,}$ et u elicientur, et sic porro. Integratis aequationibus praecedentibus per methodum notam, integralia sub hac redigi possunt forma

$$p_{,} = C \sin[(n)xt - \Theta] - \int H \cos[(n)xt - (n)x(t)]dt + \int \left\{ \frac{\alpha-\eta}{x} L + \frac{c}{x} M \right\} \sin[(n)xt - (n)x(t)]\, dt$$

$$\begin{aligned} q_{,} = {} & \frac{\alpha-\eta}{x} C \cos[(n)xt - \Theta] + C_{,} + \frac{c}{x} \int \left\{ \frac{\alpha-\eta}{x} M - \frac{c}{x} L \right\} dt \\ & - \frac{\alpha-\eta}{x} \int H \sin[(n)xt - (n)x(t)]dt - \frac{\alpha-\eta}{x} \int \left\{ \frac{\alpha-\eta}{x} L + \frac{c}{x} M \right\} \cos[(n)xt - (n)x(t)]\, dt \end{aligned}$$

$$\begin{aligned} u = {} & -\frac{c}{x} C \cos[(n)xt - \Theta] + \frac{\alpha-\eta}{c} C_{,} + \frac{\alpha-\eta}{x} \int \left\{ \frac{\alpha-\eta}{x} M - \frac{c}{x} L \right\} dt \\ & + \frac{c}{x} \int H \sin[(n)xt - (n)x(t)]dt + \frac{c}{x} \int \left\{ \frac{\alpha-\eta}{x} L + \frac{c}{x} M \right\} \cos[(n)xt - (n)x(t)]\, dt \end{aligned}$$

ubi brevitatis caussa

$$x = \sqrt{(\alpha-\eta)^2 + c^2}$$

feci et ubi in integrationibus (t) constans ponenda, post integrationes vero (t) in t mutanda est.

Si ad rigorosam aequationem conditionalem inter constantes C et $C_{,}$ existentem indagandam hi valores ipsarum $p_{,}$, $q_{,}$ et u in aequatione conditionali hac $1 = p_{,}^2 + q_{,}^2 + u^2$ substituuntur, invenitur primum

$$\begin{aligned} 1 = {} & \left\{ C \sin[(n)xt - \Theta] - \int H \cos[(n)xt - (n)x(t)]dt + \int \left\{ \frac{\alpha-\eta}{x} L + \frac{c}{x} M \right\} \sin[(n)xt - (n)x(t)]dt \right\}^2 \\ & + \left\{ C \cos[(n)xt - \Theta] - \int H \sin[(n)xt - (n)x(t)]dt - \int \left\{ \frac{\alpha-\eta}{x} L + \frac{c}{x} M \right\} \cos[(n)xt - (n)x(t)]dt \right\}^2 \\ & + \left\{ C_{,} \frac{x}{c} + \int \left\{ \frac{\alpha-\eta}{x} M - \frac{c}{x} L \right\} dt \right\}^2 \end{aligned}$$

quae facili computandi ratione transfertur in

$$
\begin{aligned}
1 = \; & \left\{\int H \cos[(n)xt-\Theta]\,dt - \int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\sin[(n)xt-\Theta]\,dt\right\}^2 \\
& + \left\{C - \int H \sin[(n)xt-\Theta]\,dt - \int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\cos[(n)xt-\Theta]\,dt\right\}^2 \\
& + \left\{C_{,}\frac{x}{c} + \int\left\{\frac{\alpha-\eta}{x}M-\frac{c}{x}L\right\}dt\right\}^2
\end{aligned}
$$

Sed facillime perspicitur, integralia ipsa, quae in hac aequatione continentur, terminos constantes habere non posse, quadratis vero horum integralium terminos constantes necessario inesse debere; posita igitur

$$
\lambda = \text{term. const. in} \left(
\begin{aligned}
& \left\{\int H \cos[(n)xt-\Theta]\,dt - \int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\sin[(n)xt-\Theta]\,dt\right\}^2 \\
& + \left\{\int H \sin[(n)xt-\Theta]\,dt + \int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\cos[(n)xt-\Theta]\,dt\right\}^2 \\
& + \left\{\int\left\{\frac{\alpha-\eta}{x}M-\frac{c}{x}L\right\}dt\right\}^2
\end{aligned}
\right)
$$

sive, quae eadem est aequatio,

$$
\lambda = \text{term. const. in} \left(
\begin{aligned}
& \left\{\int H \cos[(n)xt-(n)x(t)]\,dt - \int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\sin[(n)xt-(n)x(t)]\,dt\right\}^2 \\
& + \left\{\int H \sin[(n)xt-(n)x(t)]\,dt - \int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\cos[(n)xt-(n)x(t)]\,dt\right\}^2 \\
& + \left\{\int\left\{\frac{\alpha-\eta}{x}M-\frac{c}{x}L\right\}dt\right\}^2
\end{aligned}
\right)
$$

aequatio praecedens, si non nisi ad terminos constantes respicitur, transit in hanc

$$1 = \lambda + C^2 + C_{,}^2\,\frac{x^2}{c^2}$$

cui valores hi

$$
\begin{aligned}
C &= \sqrt{1-\lambda}\,.\,\sin\Gamma \\
C_{,} &= \frac{c}{x}\sqrt{1-\lambda}\,.\,\cos\Gamma
\end{aligned}
$$

satisfaciunt, quibus constantes arbitrariae nostrae duae C atque $C_{,}$ per constantem arbitrariam unam Γ exprimuntur. Substitutis his constantium valoribus, integralia praecedentia evadunt haec

$$\left.\begin{aligned}
p_{,} &= \sqrt{1-\lambda}\,.\sin\Gamma\sin[(n)xt-\Theta]-\int H\cos[(n)xt-(n)x(t)]\,dt+\int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\sin[(n)xt-(n)x(t)]\,dt\\
q_{,} &= \frac{\alpha-\eta}{x}\sqrt{1-\lambda}\,.\sin\Gamma\cos[(n)xt-\Theta]+\frac{c}{x}\sqrt{1-\lambda}\,.\cos\Gamma+\frac{c}{x}\int\left\{\frac{\alpha-\eta}{x}M-\frac{c}{x}L\right\}dt\\
&\quad-\frac{\alpha-\eta}{x}\int H\sin[(n)xt-(n)x(t)]\,dt-\frac{\alpha-\eta}{x}\int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\cos[(n)xt-(n)x(t)]\,dt\\
u &= -\frac{c}{x}\sqrt{1-\lambda}\,.\sin\Gamma\cos[(n)xt-\Theta]+\frac{\alpha-\eta}{x}\sqrt{1-\lambda}\,.\cos\Gamma+\frac{\alpha-\eta}{x}\int\left\{\frac{\alpha-\eta}{x}M-\frac{c}{x}L\right\}dt\\
&\quad+\frac{c}{x}\int H\sin[(n)xt-(n)x(t)]\,dt+\frac{c}{x}\int\left\{\frac{\alpha-\eta}{x}L+\frac{c}{x}M\right\}\cos[(n)xt-(n)x(t)]\,dt
\end{aligned}\right\}\quad(64)$$

Si expressio ipsius λ supra inventa cum his ipsarum $p_{,}$, $q_{,}$ et u valoribus comparatur, et ponitur

$$\begin{aligned}
p_{,} &= \sqrt{1-\lambda}\,.\sin\Gamma\,[(n)xt-\Theta]+\delta p_{,}\\
q_{,} &= \frac{\alpha-\eta}{x}\sqrt{1-\lambda}\,.\sin\Gamma\cos[(n)xt-\Theta]+\frac{c}{x}\sqrt{1-\lambda}\,.\cos\Gamma+\delta q_{,}\\
u &= -\frac{c}{x}\sqrt{1-\lambda}\,.\sin\Gamma\cos[(n)xt-\Theta]+\frac{\alpha-\eta}{x}\sqrt{1-\lambda}\,.\cos\Gamma+\delta u
\end{aligned}$$

facile invenitur

$$\lambda = \text{term. const. in } \{\delta p_{,}^{2}+\delta q_{,}^{2}+\delta u^{2}\} \qquad \ldots.(64^{*})$$

qua aequatione λ perfacili opera computari potest.

Ex integralibus praecedentibus perspicuum est, terminum per tempus ipsum multiplicatum in valoribus ipsarum $p_{,}$, $q_{,}$ et u oriturum esse, si in quantitate $\frac{\alpha-\eta}{x}M-\frac{c}{x}L$ contineatur terminus constans et in quantitatibus H atque $\frac{\alpha-\eta}{x}L+\frac{c}{x}M$ terminus formae $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left[(n)xt-\Theta\right]$. Indolem vero functionum ft, φt, $p_{,}$, $q_{,}$ et u si quis perscrutatur, facile comperit, nec illi quantitati terminum constantem ullo modo inesse posse, nec in his quantitatibus in approximatione saltem prima terminum per sinum aut per cosinum arcus $(n)xt-\Theta$ multiplicatum contineri. In approximatione tamen secunda et subsequentibus fieri potest, ut termini tales in quantitatibus his nascantur. Ad terminos per tempus ipsum multiplicatos, qui ex his terminis in integralibus nostris nascerentur, tollendos deligamus hos terminos. Suppono igitur in approximatione secunda inventas esse

$$(65)\ldots\ldots\left\{\begin{aligned} H &= (n)A\sin\Gamma\cos\,[(n)xt-\Theta] + (n)B\cos\Gamma + \text{etc.}\\ L &= (n)C\sin\Gamma\sin\,[(n)xt-\Theta] + \text{etc.}\\ M &= (n)D\sin\Gamma\sin\,[(n)xt-\Theta] + \text{etc.}\end{aligned}\right.$$

ubi $(n)B\cos\Gamma$ est terminus constans qui necessario una existere debet, et ubi etc. signum terminos denotat, qui ex angulis aliis pendent. Propter aequationem conditionalem hanc $1 = u^2 + p_{,}^2 + q_{,}^2$ valores ipsarum H, L et M a se invicem independentes non sunt. Inter coefficientes igitur A, B, C, D aequationes conditionales locum habere debent. Ad has inveniendas substituantur valores praecedentes ipsarum H, L et M in aequationibus (63), et tum et aequationes hae et valores ipsarum $p_{,}$, $q_{,}$ et u ex aequationibus (60) desumendi in aequatione conditionali hac

$$o = p_{,}\frac{dp_{,}}{dt} + q_{,}\frac{dq_{,}}{dt} + u\frac{du}{dt}$$

Quo facto obtinetur aequatio haec

$$\begin{aligned} o = &\ \tfrac{1}{2}A\sin^2\Gamma\sin 2\,[(n)xt-\Theta] + B\sin\Gamma\cos\Gamma\sin\,[(n)xt-\Theta]\\ &+ \frac{\alpha-\eta}{2x}C\sin^2\Gamma\sin 2\,[(n)xt-\Theta] + \frac{c}{x}C\sin\Gamma\cos\Gamma\sin\,[(n)xt-\Theta]\\ &+ \frac{c}{2x}D\sin^2\Gamma\sin 2\,[(n)xt-\Theta] - \frac{\alpha-\eta}{x}D\sin\Gamma\cos\Gamma\sin\,[(n)xt-\Theta]\end{aligned}$$

quae identica esse debet, et post comparatos terminos eiusdem formae suppeditat

$$\begin{aligned} o &= xA + (\alpha-\eta)C + cD\\ o &= xB + cC - (\alpha-\eta)D\end{aligned}$$

quae aequationes conditionales requisitae sunt. Quibus positis, ex aequationibus (60) emergunt hae

$$\begin{aligned} \sin\Gamma\sin\,[(n)xt-\Theta] &= p_{,}\\ \sin\Gamma\cos\,[(n)xt-\Theta] &= \frac{x}{\alpha-\eta}q_{,} - \frac{c}{\alpha-\eta}\cos\Gamma\\ \sin\Gamma\cos\,[(n)xt-\Theta] &= -\frac{x}{c}u + \frac{\alpha-\eta}{c}\cos\Gamma\end{aligned}$$

Si ultima harum aequationum per $-D\frac{c}{x}$ multiplicata ad penultimam per $-C\frac{\alpha-\eta}{x}$ multiplicatam additur, nanciscimur propter aequationes conditionales praecedentes

$$A \sin \Gamma \cos [(n)xt - \Theta] = -Cq_{,} + Du - B \cos \Gamma$$

Substituta hac aequatione nec non

$$\sin \Gamma \sin [(n)xt - \Theta] = p_{,}$$

in aequationibus (65), emergunt

$$\begin{aligned} H &= -(n)\,Cq_{,} + (n)\,Du \\ L &= (n)\,Cp_{,} \\ M &= (n)\,Dp_{,} \end{aligned}$$

Substitutis his aequationibus in (63), per se evidens est integralia nostra formam suam non mutare, et solum terminorum illorum effectum in eo consistere, quod ad quantitatem $\alpha - \eta$ terminum correctionis C, et ad quantitatem c terminum correctionis D addant.

Si igitur in approximatione secunda termini ipsarum H, L et M sub (65) allati non tam parvi sunt ut omnino negligi possint, ponatur ubique in integralibus (64)

$$\alpha - \eta + C \text{ loco } \alpha - \eta$$
$$\text{atque} \qquad c + D \text{ loco } c$$

et termini sub (65) allati in valoribus ipsarum H, L et M deleantur. Quibus factis termini per tempus ipsum multiplicati in valoribus ipsarum $p_{,}$, $q_{,}$ et u oriri nequeunt.

Quae quidem methodus, qua termini per tempus ipsum multiplicati ex $p_{,}$ et $q_{,}$ tolluntur, et generaliter methodus haec ipsas $p_{,}$ et $q_{,}$ suppeditans, dum situs plani proiectionis in spatio penitus arbitrarius, sive magnitudo anguli Γ quaecunque est, non modo in problemate trium corporum adhiberi, sed etiam ad problema, ubi numerus corporum se invicem perturbantium quantusvis est, extendi potest.

31.

Constantibus Γ, Γ', Θ et Θ', quas integrationes in expressionibus ipsarum $p_{,}$, $q_{,}$, u, $p'_{,}$, $q'_{,}$ et u' introduxerunt, aequationes conditionales simplicissimae insunt. Ex iis, quae in art. 27. de termino constante ipsarum Ω atque Ω' protuli, sequitur valores approximatos ipsarum c et c' esse hos

32.

Ut significatio constantium Γ et Θ inveniatur, animadverto formam expressionum praecedentium ipsarum $p'_{,}$ et $q'_{,}$, nec non expressionum (60) ipsarum $p_{,}$ et $q_{,}$ indicare, Γ esse debere inclinationem plani alicuius versus planum fundamentale sive ipsarum xy, et Θ arcum quendam in plano illo adhuc ignoto iacentem. Statuta $\Gamma = o$, orbitae ambae ad planum hoc ignotum reducuntur, ita ut in hoc casu planum hoc sit planum fundamentale. Quo facto, habemus

$$p_{,} = o \; ; \; q_{,} = \frac{c}{\sqrt{(\alpha-\eta)^2+c^2}}$$
$$p'_{,} = o \; ; \; q'_{,} = \frac{c'}{\sqrt{(\alpha+\eta)^2+c'^2}}$$

Sed quodquod planum statuis planum fundamentale, semper forma generalis ipsarum $p_{,}$, $q_{,}$, $p'_{,}$ atque $q'_{,}$ sub (67) allata locum habere debet, quare nunc quidem esse debent

$$\sin(i) = \frac{c}{\sqrt{(\alpha-\eta)^2+c^2}} \; ; \; \sin(i') = \frac{c'}{\sqrt{(\alpha+\eta)^2+c'^2}}$$
$$(\chi-\omega) - \pi + v + k = o \; ; \; (\chi'-\omega') - \pi + v - k = 180^\circ$$

ubi (i) atque (i') denotant constantes a vi perturbanti independentes terminos inclinationum orbitarum m atque m' versus planum quod nunc fundamentale est, et ubi in $(\chi-\omega)$ atque in $(\chi'-\omega')$, si posterioribus aequationibus utaris, termini quoque constantes solummodo recipiendi sunt. Substitutis valoribus ipsarum α, η, c atque c', in art. praec. datis in prioribus aequationibus praecedentibus, facile invenitur

$$\sin(i) = \frac{m'w' \sin(I)}{\sqrt{m'^2 w'^2 + m^2 w^2 + 2mm' ww' \cos(I)}}$$
$$\sin(i') = \frac{mw \sin(I)}{\sqrt{m'^2 w'^2 + m^2 w^2 + 2mm' ww' \cos(I)}}$$

unde manifestum est, (i) esse inclinationem orbitae m, et (i') inclinationem orbitae m' versus p l a n u m i n v a r i a b i l e ab ill. Laplace detectum. Nam formulae „Mechanicae coelestis" suppeditant respectu huius plani, quoties motus duorum corporum respectu tertii corporis considerantur, formulas has

$$o = mw \sin i \sin \theta + m'w' \sin i' \sin \theta'$$
$$o = mw \sin i \cos \theta + m'w' \sin i' \cos \theta'$$

in quibus i, θ, i' atque θ' ad planum invariabile spectant. In his quidem formulis termini nonnulli ordinis quadrati vis perturbantis neglecti sunt, quae vero neglectio hoc loco, ubi non nisi termini o[ti] ordinis respectu vis perturbantis in calculum vocantur, nullius momenti est. Quum $\sin i$ et $\sin i'$ semper sint quantitates positivae, et directio motus corporum m et m' eadem supponatur, aequationes praecedentes subministrant has

$$mw \sin i = m'w' \sin i'$$
$$\theta = \theta' + 180^\circ$$

quarum posterior monstrat esse debere

$$I = i + i'$$

Hinc adipiscimur

$$\sin i = \sin I \cos i' - \cos I \sin i'$$
$$\sin^2 I = \sin^2 i + \sin^2 i' + 2 \sin i \sin i' \cos I$$

e qua, eliminato $\sin i'$ ope valoris sui $\frac{mw}{m'w'} \sin i$, evadit

$$\sin i = \frac{m'w' \sin I}{\sqrt{m'^2 w'^2 + m^2 w^2 + 2mm'ww' \cos I}}$$

et eodem modo invenitur

$$\sin i' = \frac{mw \sin I}{\sqrt{m'^2 w'^2 + m^2 w^2 + 2mm'ww' \cos I}}$$

quae formulae, substitutis (i), (i'), (I) resp. loco i, i', I, cum praecedentibus ipsarum $\sin (i)$ et $\sin (i')$ valoribus plane congruunt. Itaque, facta $\Gamma = o$, planum invariabile evasit planum fundamentale, unde concluditur Γ esse inclinationem plani invariabilis versus planum fundamentale sive ipsarum xy.

Quibus positis, consideremus valores ipsarum $p_,$ et $p'_,$ in casu quo Γ cifrae aequalis non est. Habemus, facta $t = o$, valores generales hos

$$\left.\begin{aligned} p_, &= \sin i \sin (\chi - \omega - \pi + \nu + k) \\ p'_, &= \sin i' \sin (\chi' - \omega' - \pi' + \nu - k) \end{aligned}\right\} \ldots\ldots (69)$$

et valorem primi earum, a vi perturbanti independentis, termini hunc

(69*).... $$p_{,} = p'_{,} = \sin \Gamma \sin (-\Theta)$$

qui valores, substitutis in valoribus generalibus illis terminis constantibus et a vi perturbanti independentibus, qui in ipsis i, i', $\chi - \omega$ atque $\chi' - \omega'$ continentur, congruere debent. Aequationes vero (39) et (46), facta $t = o$, suppeditant

$$\chi - \omega - \pi + v + k = -\Phi$$
$$\chi' - \omega' - \pi' + v - k = -\Psi$$

ubi termini maximi constantes et a vi perturbanti independentes valorum ipsarum $\chi - \omega$, $\chi' - \omega'$, Φ atque Ψ solummodo recipiendi sunt. Quibus terminis per litteram respectivam uncis inclusam denotatis, aequationes (69) ope praecedentium abeunt in has

$$p_{,} = \sin (i) \sin [-(\Phi)]$$
$$p'_{,} = \sin (i) \sin [-(\Phi)]$$

quae cum (69*) comparatae subministrant

$$(\Phi) = (\Psi) = \Theta$$

et praeterea valorem ipsarum (i) et (i') praebent, quo tamen non utar. Aequatio haec monstrat, valores perturbatos ipsarum Φ et Ψ eundem terminum constantem a vi perturbanti independentem habere debere, et hunc ipsi Θ aequalem esse. Iam quum Φ et Ψ sint arcus a nodis ascendentibus orbitarum m et m' cum plano ipsarum xy usque ad nodum ascendentem orbitae m cum orbita m' extensi, et quum Θ in plano invariabili iacere debeat: concluditur Θ esse arcum a nodo ascendenti plani invariabilis cum plano fundamentali sive ipsarum xy usque ad nodum ascendentem orbitae m cum orbita m' extensum.

Quum in motu Lunae in quantitatibus α et η massam Lunae optimo iure negligere liceat, denotat in hoc motu Γ inclinationem orbitae Solis versus planum proiectionis, et Θ arcum a nodo ascendenti orbitae Solis cum hoc plano usque ad nodum ascendentem orbitae Lunae cum orbita Solis ductum. Propter planetarum perturbationes orbita Solis immobilis non est, cuius vero motus in formulis (49) iam separatim rationem habuimus, quam-

obrem Γ et Θ ad hoc orbitae Solis planum referendae sunt, quod tempori $t = o$ respondet.

Integrationes ad $p'_{,}$, $q'_{,}$ et u' obtinendas non amplius persequemur, quia integrationibus ad $p_{,}$, $q_{,}$ et u obtinendas explicatis plane similes sunt.

33.

Denotante s sinum latitudinis Lunae supra planum proiectionis, habetur

$$s = \sin i \sin (v - \theta)$$

sive

$$s = \sin i \sin (v_{,} - \chi + \omega)$$

quum vero sit

$$v_{,} = \bar{f} + y(n)t + \pi$$

expressio ipsius s ita quoque exhiberi potest

$$s = \sin i \sin [\bar{f} + \nu + k + (n)(y + \alpha - \eta)t - (\chi - \omega - \pi + \nu + k + (n)(\alpha - \eta)t)]$$

unde secundum aequationes (67)

$$s = q_{,} \sin V - p_{,} \cos V \qquad \text{......(70)}$$

posita brevitatis caussa

$$V = \bar{f} + (n)(y + \alpha - \eta)t + \nu + k$$

computatis igitur $p_{,}$ et $q_{,}$, sicut in praecedentibus explicatum est, sinus latitudinis Lunae ope formulae praecedentis facile evolvitur.

Sit l longitudo Lunae ad planum proiectionis reducta: tum triangulus sphaericus rectangulus ab orbita Lunae, plano proiectionis et plano per Lunae locum ad planum proiectionis perpendiculariter demisso formatus suppeditat

$$\operatorname{tg}(l - \theta) = \cos i \operatorname{tg}(v - \theta)$$

quae aequatio facile transformatur in

$$\operatorname{tg}(l - v) = -\frac{\operatorname{tg}^2 \frac{1}{2} i \sin 2(v - \theta)}{1 + \operatorname{tg}^2 \frac{1}{2} i \cos 2(v - \theta)}$$

Sit

$$V_{,} = \bar{f} + (n)(y + \alpha - \eta - x)t + \nu + k + \Theta$$

unde erit

$$v-\theta = V_{,}-(\chi-\omega+(\alpha-\eta-x)(n)t-\pi+\nu+k+\Theta)$$
$$l-v = l-V_{,}+(\chi-\omega-\theta+(\alpha-\eta-x)(n)t-\pi+\nu+k+\Theta)$$

Substitutis his aequationibus in aequatione praecedenti pro $l-v$, et positis

$$p_{,,} = \sin i \sin[\chi-\omega+(\alpha-\eta-x)(n)t-\pi+\nu+k+\Theta]$$
$$q_{,,} = \sin i \cos[\chi-\omega+(\alpha-\eta-x)(n)t-\pi+\nu+k+\Theta]$$

unde

$$p_{,,} = p_{,} \cos[x(n)t-\Theta]-q_{,}\sin[x(n)t-\Theta]$$
$$q_{,,} = q_{,} \cos[x(n)t-\Theta]+p_{,}\sin[x(n)t-\Theta]$$

emergit, aequatio pro $l-v$ abit in hanc

$$(71)\quad l-V_{,} = \omega-\chi+\theta-(\alpha-\eta-x)(n)t+\pi-\nu-k-\Theta+\text{arc. tg.}\left\{\frac{2p_{,,}q_{,,}\cos 2V_{,}-(q_{,,}^2-p_{,,}^2)\sin 2V_{,}}{\left[1+\sqrt{1-p_{,,}^2-q_{,,}^2}\right]^2+2p_{,,}q_{,,}\sin 2V_{,}+(q_{,,}^2-p_{,,}^2)\cos 2V_{,}}\right\}$$

Ut θ ex hac expressione eliminetur, ad differentiale ipsius $\omega-\chi+\theta$ nobis refugiendum est. Iam habuimus

$$d\chi-d\omega = \cos i\, d\theta$$

unde

$$d(\omega-\chi+\theta) = \frac{1-\cos i}{\cos i}(d\chi-d\omega)$$

Expressiones vero ipsarum $p_{,,}$ et $q_{,,}$ modo datae suppeditant

$$\frac{p_{,,}}{q_{,,}} = \operatorname{tg}[\chi-\omega+(\alpha-\eta-x)(n)t-\pi+\nu+k+\Theta]$$

e qua differentiata elicitur

$$\frac{d\chi}{dt}-\frac{d\omega}{dt} = (n)(x-\alpha+\eta)+\frac{q_{,,}\frac{dp_{,,}}{dt}-p_{,,}\frac{dq_{,,}}{dt}}{\sin^2 i}$$

qua expressione valor ipsius $d(\omega-\chi+\theta)$ modo inventus transit in hunc

$$\frac{d(\omega-\chi+\theta)}{dt} = (n)(x-\alpha+\eta)\frac{p_{,,}^2+q_{,,}^2}{\left[1+\sqrt{1-p_{,,}^2-q_{,,}^2}\right]\sqrt{1-p_{,,}^2-q_{,,}^2}}+\frac{q_{,,}\frac{dp_{,,}}{dt}-p_{,,}\frac{dq_{,,}}{dt}}{\left[1+\sqrt{1-p_{,,}^2-q_{,,}^2}\right]\sqrt{1-p_{,,}^2-q_{,,}^2}}$$

Ad expressionem ipsius $l-V_{,}$ supra datam in seriem evolvendam commodissime differentiale eius, in quo $V_{,}$ pro constanti habita est, adhibetur.

Posita brevitatis caussa $\sqrt{1-p_{,,}^2-q_{,,}^2}=y$ atque

$$z=\frac{2p_{,,}q_{,,}\cos 2V_{,}-(q_{,,}^2-p_{,,}^2)\sin 2V_{,}}{(1+y)^2+2p_{,,}q_{,,}\sin 2V_{,}+(q_{,,}^2-p_{,,}^2)\cos 2V_{,}}$$

aequatio (71) differentiata suppeditat post substitutum ipsius $\frac{d(\omega-\chi+\theta)}{dt}$ valorem modo datum,

$$d(l-V_{,})=\frac{(n)(x-\alpha+\eta)}{\sqrt{1-p_{,,}^2-q_{,,}^2}}dt+\frac{q_{,,}dp_{,,}-p_{,,}dq_{,,}}{y(1+y)}+\frac{dz}{1+z^2}$$

Habita vero ratione aequationis identicae

$$(q_{,,}^2-p_{,,}^2)^2+4p_{,,}^2q_{,,}^2=(q_{,,}^2+p_{,,}^2)^2=(1+y)^2(1-y)^2$$

expressio praecedens ipsius z praebet

$$1+z^2=\frac{4(1+y)^2\left[1-\tfrac{1}{2}(p_{,,}^2+q_{,,}^2)+p_{,,}q_{,,}\sin 2V_{,}+\tfrac{1}{2}(q_{,,}^2-p_{,,}^2)\cos 2V_{,}\right]}{D^2}$$

denotante D denominatorem ipsius z. Differentiata vero expressione ipsius z, dum $V_{,}$ constans tractetur, prodit

$$dz=2\frac{\left\{\begin{array}{l}(p_{,,}dq_{,,}+q_{,,}dp_{,,})y(1+y)^2\cos 2V_{,}-(q_{,,}dq_{,,}-p_{,,}dp_{,,})y(1+y)^2\sin 2V_{,}-2p_{,,}q_{,,}(q_{,,}dq_{,,}-p_{,,}dp_{,,})y\\+(q_{,,}^2-p_{,,}^2)(p_{,,}dq_{,,}+q_{,,}dp_{,,})y+2p_{,,}q_{,,}(p_{,,}dp_{,,}+q_{,,}dq_{,,})(1+y)\cos 2V_{,}-(q_{,,}^2-p_{,,}^2)(q_{,,}dq_{,,}+p_{,,}dp_{,,})(1+y)\sin 2V_{,}\end{array}\right\}}{yD^2}$$

sed identica est

$$\begin{aligned}-2p_{,,}q_{,,}(q_{,,}dq_{,,}-p_{,,}dp_{,,})+(q_{,,}^2-p_{,,}^2)(p_{,,}dq_{,,}+q_{,,}dp_{,,})&=(p_{,,}^2+q_{,,}^2)(q_{,,}dp_{,,}-p_{,,}dq_{,,})\\&=(1+y)(1-y)(q_{,,}dp_{,,}-p_{,,}dq_{,,})\end{aligned}$$

quamobrem

$$dz=2\frac{\left\{\begin{array}{l}(p_{,,}dq_{,,}+q_{,,}dp_{,,})y(1+y)\cos 2V_{,}-(q_{,,}dq_{,,}-p_{,,}dp_{,,})y(1+y)\sin 2V_{,}\\+2p_{,,}q_{,,}(p_{,,}dp_{,,}+q_{,,}dq_{,,})\cos 2V_{,}-(q_{,,}^2-p_{,,}^2)(q_{,,}dq_{,,}+p_{,,}dp_{,,})\sin 2V_{,}+(q_{,,}dp_{,,}-p_{,,}dq_{,,})y(1-y)\end{array}\right\}(1+y)}{yD^2}$$

Hinc evadit

$$\frac{q_{,,}dp_{,,}-p_{,,}dq_{,,}}{y(1+y)}+\frac{dz}{1+z^2}=\frac{\left\{\begin{array}{l}\left[(p_{,,}dq_{,,}+q_{,,}dp_{,,})y(1+y)+2p_{,,}q_{,,}(q_{,,}dq_{,,}+p_{,,}dp_{,,})+(q_{,,}^2-p_{,,}^2)(q_{,,}dp_{,,}-p_{,,}dq_{,,})\right]\cos 2V_{,}\\-\left[(q_{,,}dq_{,,}-p_{,,}dp_{,,})y(1+y)-2p_{,,}q_{,,}(q_{,,}dp_{,,}-p_{,,}dq_{,,})+(q_{,,}^2-p_{,,}^2)(q_{,,}dq_{,,}+p_{,,}dp_{,,})\right]\sin 2V_{,}\\+(q_{,,}dp_{,,}-p_{,,}dq_{,,})\left[y(1-y)+2-(p_{,,}^2+q_{,,}^2)\right]\end{array}\right\}}{2\left[1-\tfrac{1}{2}(p_{,,}^2+q_{,,}^2)+p_{,,}q_{,,}\sin 2V_{,}+\tfrac{1}{2}(q_{,,}^2-p_{,,}^2)\cos 2V_{,}\right]y(1+y)}$$

sed identicae sunt

$$\begin{aligned}2p_{,,}q_{,,}(q_{,,}dq_{,,}+p_{,,}dp_{,,})+(q_{,,}^2-p_{,,}^2)(q_{,,}dp_{,,}-p_{,,}dq_{,,})&=(1+y)(1-y)(q_{,,}dp_{,,}+p_{,,}dq_{,,})\\-2p_{,,}q_{,,}(q_{,,}dp_{,,}-p_{,,}dq_{,,})+(q_{,,}^2-p_{,,}^2)(q_{,,}dq_{,,}+p_{,,}dp_{,,})&=(1+y)(1-y)(q_{,,}dq_{,,}-p_{,,}dp_{,,})\\2-(p_{,,}^2+q_{,,}^2)&=1+y^2\end{aligned}$$

quibus adiuvantibus expressio praecedens abit in

$$\frac{q_{\prime\prime}dp_{\prime\prime}-p_{\prime\prime}dq_{\prime\prime}}{y(1+y)}+\frac{dz}{1+z^2}=\frac{q_{\prime\prime}dp_{\prime\prime}-p_{\prime\prime}dq_{\prime\prime}+(p_{\prime\prime}dp_{\prime\prime}-q_{\prime\prime}dq_{\prime\prime})\sin 2V_{\prime}+(q_{\prime\prime}dp_{\prime\prime}+p_{\prime\prime}dq_{\prime\prime})\cos 2V_{\prime}}{2\left[1-\frac{1}{2}(p_{\prime\prime}^2+q_{\prime\prime}^2)+p_{\prime\prime}q_{\prime\prime}\sin 2V_{\prime}+\frac{1}{2}(q_{\prime\prime}^2-p_{\prime\prime}^2)\cos 2V_{\prime}\right]\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}$$

Sed facili reductione facta, elicitur huius aequationis dextrae partis numerator aequalis huic

$$-2(q_{\prime\prime}\cos V_{\prime}+p_{\prime\prime}\sin V_{\prime})(dq_{\prime\prime}\sin V_{\prime}-dp_{\prime\prime}\cos V_{\prime})$$

nec non denominatoris pars signis [ ] inclusa aequalis huic

$$(1-q_{\prime\prime}\sin V_{\prime}+p_{\prime\prime}\cos V_{\prime})(1+q_{\prime\prime}\sin V_{\prime}-p\cos V_{\prime})$$

hinc denique emergit

$$(72)\ldots\ldots\ \frac{d(l-V_{\prime})}{dt}=\frac{(n)(x-\alpha+\eta)}{\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}-\frac{(q_{\prime\prime}\cos V_{\prime}+p_{\prime\prime}\sin V_{\prime})\left(\frac{dq_{\prime\prime}}{dt}\sin V_{\prime}-\frac{dp_{\prime\prime}}{dt}\cos V_{\prime}\right)}{(1-q_{\prime\prime}\sin V_{\prime}+p_{\prime\prime}\cos V_{\prime})(1+q_{\prime\prime}\sin V_{\prime}-p_{\prime\prime}\cos V_{\prime})\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}$$

De evolutione integrationeque huius expressionis infra agetur, ubi expressiones omnes in series evolvemus, hoc vero loco quaedam de constantibus integralibus addendis annotanda sunt.

Expressio praecedens integrata subministrat

$$l=\Pi+R+V_{\prime}+\int Ft\,.\,dt$$

ubi brevitatis caussa terminos in expressione praecedenti integrandos per Ft reddidi, et ubi $\Pi+R$ constans huic integrationi addita est. Propter terminum in expressione (71) sub signo arc. tg. contentum altera huius constantis pars functio est ipsius $V_{\prime}$ et termini constantis, qui in ipsis $p_{\prime\prime}$ et $q_{\prime\prime}$ continetur, quae quidem constantis integrae $\Pi+R$ pars reductionem ipsius $V_{\prime}$ ad planum proiectionis repraesentat, quae locum haberet, si solummodo ad terminum constantem in ipsis $p_{\prime\prime}$ et $q_{\prime\prime}$ existentem nobis respiciendum esset. Quum vero terminus constans in $p_{\prime\prime}$ et $q_{\prime\prime}$ sit functio ipsius Γ, ea constantis integrae $\Pi+R$ pars, de qua sermo est, functio erit ipsarum Γ atque $V_{\prime}$; quodsi haec pars sit R, habetur

$$\operatorname{tg} R=-\frac{\operatorname{tg}^2\frac{1}{2}\Gamma\sin 2V_{\prime}}{1+\operatorname{tg}^2\frac{1}{2}\Gamma\cos 2V_{\prime}}$$

altera autem pars, scilicet Π, vera constans est.

Formulae praecedentes ad Lunam sive generalius ad corpus m spectant, et eodem modo obtinetur pro corpore m'

$$l'=\Pi'+R'+V_{\prime}'+\int F't\,.\,dt$$

Si vero inclinationem mutuam orbitarum m et m' cifrae aequalem facimus, necessario Ft atque $F't$ evanescunt, unde in hoc casu emergunt

$$l = \Pi + R + V_{,}$$
$$l' = \Pi' + R' + V'_{,}$$

quum vero R sit functio ipsarum Γ atque $V_{,}$, et R' functio analoga ipsarum Γ atque $V'_{,}$, habetur eo temporis momento, quo $V_{,} = V'_{,}$ est, etiam $R = R'$, et quum in casu quem nunc tractamus hoc temporis momento necessario esse debeat $l = l'$, elicitur ex aequationibus praecedentibus

$$\Pi' = \Pi$$

Quum $V_{,}$ atque $V'_{,}$ denotent argumenta latitudinis, quae locum haberent, si corpora m et m' in plano invariabili se moverent, Π est longitudo nodi ascendentis plani invariabilis cum plano proiectionis.

34.

Si formulas intueris, quibus longitudinem reductam, latitudinem, logarithmum radii vectoris eorumque perturbationes in hac commentatione determinavimus, facile videbis constantes π et π', quas in art. 13. in valoribus ipsarum $v_{,}$ atque $v'_{,}$ introduximus, ex formulis omnibus evanuisse, neque formulas has constantes (φ) atque (ψ) continere. Constantes vero in his formulis existentes, quae quidem non nisi observationibus determinari possunt, sunt (n), (n'), (a), (a'), (e), (e'), (h), (h'), (c), (c'), ν, k, (I), Γ, Θ et Π, quarum vero (a), (a'), (h) atque (h') e reliquis ita pendent, ut sit

$$(a) = \left(\frac{\varkappa(M+m)}{(n)^2}\right)^{\frac{1}{3}}$$
$$(a') = \left(\frac{\varkappa(M+m')}{(n)'^2}\right)^{\frac{1}{3}}$$
$$(h) = \frac{(a)(n)}{\sqrt{1-(e)^2}}$$
$$(h') = \frac{(a)'(n)'}{\sqrt{1-(e)'^2}}$$

Restant igitur, si massas non adnumeras, observationibus determinandae duodecim constantes independentes, id quod secus se habere non potest. Significatio harum constantium, quam repeto, est haec:

(n) et (n') ... motus anomaliarum mediarum observati;

(c) et (c') ... anomaliae mediae tempori $t = o$ respondentes;

(e) et (e') ... excentricitates, quae ex observato maximo coefficiente aequationis centri ope formulae pure ellipticae computantur;

(I) ... inclinatio mutua media;

$v + k$ ... arcus orbitae m inter locum perigaei corporis m et nodum ascendentem huius orbitae cum plano invariabili interceptus, et tempori $t = o$ respondens;

$v - k$ ... arcus orbitae m' inter locum perigaei corporis m' et nodum descendentem huius orbitae cum plano invariabili interceptus, et tempori $t = o$ respondens;

Γ ... inclinatio plani invariabilis versus planum proiectionis;

Θ ... arcus plani invariabilis inter nodum ascendentem huius plani cum plano proiectionis et nodum ascendentem orbitae m tempori $t = o$ respondentem, sive, quod idem punctum est, nodum descendentem orbitae m' tempori $t = o$ respondentem cum plano invariabili interceptus;

Π ... longitudo nodi ascendentis plani invariabilis cum plano proiectionis.

Quae significationes in problemate generali, ubi utriusque corporis motus incognitus est, locum habent, in theoria vero Lunae, ubi motus Solis notus est, quia perturbationes, quas Luna Solis cursui affert, perparvulae sunt, paullulum mutantur. In hoc problemate (n'), (c'), (e'), $v - k + \Theta$, Γ et Π ipsae datae et (n), (c), (e), $v + k$, (I) et Θ ex motu Lunae observationibus eliciendae sunt, et quidem denotat

Π ... longitudinem nodi ascendentis eclipticae tempori $t = o$ respondentis cum plano proiectionis, cuius situs ex arbitrio eligi potest;

Γ... inclinationem huius eclipticae versus planum proiectionis;
$v - k + \Theta$... arcum huius eclipticae a nodo eius ascendenti cum plano proiectionis usque ad locum perigaei Solis extensum.

Formulae in hac Sectione evolutae monstrant, expressiones analyticas perturbationum latitudinis et reductionis longitudinis ad planum proiectionis simplicissimas evadere, si planum hoc eligatur planum invariabile, pro quo in theoria Lunae planum eclipticae habendum est. Quum vero in astronomia practica hodierna longitudinibus latitudinibusque Lunae non utamur, sed e contrario computationes omnes ope ascensionum rectarum declinationumque Lunae absolvantur, idque optimo iure rationeque certissima fiat: praestat absque ambagibus locum Lunae ad aequatorem relatum per theoriam quoque suppeditari, sicuti loca planetarum ad aequatorem relata computare iam edocuimus.

Esto nobis igitur planum aequatoris tempori $t = o$ respondens planum proiectionis. Hinc factum est, ut sit $\Pi = o$;

Γ... obliquitas eclipticae tempori $t = o$ respondens;
$v - k + \Theta$... longitudo perigaei Solis tempori eidem respondens;
Θ... longitudo nodi ascendentis orbitae Lunae cum ecliptica eidem tempori respondens;

manentibus significationibus constantium reliquarum iisdem ut antea.

Significationes quantitatum y, α et η cognosci oportet. Facile reperitur V esse argumentum latitudinis Lunae erga eclipticam, itaque secundum notationem vulgarem esse debere

$$(n)(y + \alpha - \eta) = \text{motui perigaei Lunae} - \text{motui nodi eius.}$$

Si in expressione ipsius s sub (70) data substituitur primus ipsarum p, et q, terminus ex (60) petendus nec non valor ipsius V, invenitur coefficiens ipsius t in maximi termini argumento $=$

$$(n)(y + \alpha - \eta) - (n)\sqrt{(\alpha - \eta)^2 + c^2}$$

quae quantitas motui perigaei Lunae aequari debet. Facta vero $m = o$, formulae art. 31. praebent

$$\sqrt{(\alpha - \eta)^2 + c^2} = \alpha + \eta$$

habemus igitur quam proxime

$(n)(y-2\eta)$ = motui perigaei Lunae progressivo;
$(n)(\alpha+\eta)$ = motui nodorum retrogrado.

Loco radii vectoris Lunae semper parallaxis eius horizontalis, vel potius logarithmus sinus parallaxeos in calculis astronomicis adhibetur; quum vero

$$l \,.\, \sin\,(\text{par. hor.}) = lD - lr$$

ubi D radium telluris designat, et

$$lr = l(r) + w \text{ atque } l(r) = l(a) + l \,.\, \frac{1-(e)^2}{1+(e)\cos f}$$

sit, habetur

$$l \,.\, \sin\,(\text{par. hor.}) = l\,\frac{D}{(a)} - l \,.\, \frac{1-(e)^2}{1+(e)\cos f} - w$$

ubi loco $l \,.\, \frac{1-(e)^2}{1+(e)\cos f}$ nota series infinita haec

$$\tfrac{1}{2}(e)^2 + \tfrac{1}{32}(e)^4 + \text{etc.} - [(e) - \tfrac{3}{8}(e)^3 - \text{etc.}]\cos(n)z - [\tfrac{3}{4}(e)^2 - \text{etc.}]\cos 2(n)z - \text{etc.}$$

substituenda est. Ex huius vero articuli initio invenimus

$$\frac{D}{(a)} = \left(\frac{D^3(n)^2}{\varkappa(M+m)}\right)^{\frac{1}{3}}$$

ubi $\varkappa$ per longitudinem penduli exprimi potest. Vis enim attractiva Terrae in eius superficie et in punctis, quorum sinus latitudinis geographicae $= \sqrt{\tfrac{1}{3}}$, habetur quam proxime $= \varkappa\frac{M}{D^2}$, si radius D ad haec puncta refertur. Quamobrem, denotante P longitudinem penduli simplicis, quod in his Terrae superficiei punctis intra temporis t intervallum oscillationem integram absolveret, si vis centrifugalis non existeret, et π rationem semiperipheriae circuli ad radium, per principia Mechanicae evadit

$$t = \pi\sqrt{\frac{P}{\varkappa\frac{M}{D^2}}}$$

Hinc sequitur

$$\varkappa = \frac{\pi^2 PD^2}{Mt^2}$$

et quum praeterea habeatur $(n) = \frac{2\pi}{T}$, ubi T tempus revolutionis anomalisticae Lunae designat, expressio praecedens ipsius $\frac{D}{(a)}$ transit in

$$\frac{D}{(a)} = \left(\frac{M}{M+m} \cdot \frac{D}{P} \cdot \frac{4t^2}{T^2}\right)^{\frac{1}{3}}$$

SECTIO III.

GENERALIS AEQVATIONVM IN PRAECEDENTIBVS EXHIBITARVM EVOLVTIO.

1.

In theoria planetarum, qualis in theoria mea Iovis et Saturni exposita est, non quantitatem β, sed loco eius et in prima et in secunda approximatione statim w computavi, et formulam perturbationes secundi ordinis respectu massarum ipsius $(n)\zeta$ suppeditantem ita comparavi, ut β non contineret, quare hac quantitate opus non erat. Quantitatem vero $(n)\zeta$ et in prima et in secunda approximatione computavi, et valore huius quantitatis eo, qui in prima approximatione erutus erat, opus fuit ad perturbationes secundi ordinis ipsius $n(\zeta)$ computandas, quia formula perturbationes secundi ordinis huius quantitatis suppeditans functio est perturbationum primi ordinis ipsius $(n)\zeta$. Qua quantitate inventa, mutanda τ in t ipsam $(n)z$ elicui.

Hoc vero loco aliam ingrediar viam, qua neque β neque ζ ipsa computabitur. Formulas evolvam quae immediate z atque w suppeditant, et in his formulis perturbationes ordinum inferiorum ipsarum ζ et β eliminatae et perturbationibus eiusmodi ipsarum z et w redditae erunt. Tali modo formulis constructis, computatio perturbationum multo brevior reddita est.

quoniam termini in ζ per $(\tau - t)$ multiplicati, qui in valore ipsius z quidem evanescunt, quorum vero numerus in ζ praesertim in approximatione secunda et altioribus admodum magnus est, omnino non adsunt.

Ad hoc propositum assequendum quantitatem T, quae, integratione priore respectu ipsius τ instituta, respectu ipsius t integranda esset, primum respectu ipsius t integrabo, et deinde, mutata τ in t, alteram integrationem respectu ipsius t peragam. Qui integrandi ordo etiam adhibendus est, si formulas nostras ad cometarum perturbationes, quae ope quadraturarum mechanicarum, quas dicunt, eliciuntur, investigandas adhibere velis.

Quum in motu Lunae non modo termini secundi ordinis respectu vis perturbantis, sed etiam termini nonnulli tertii ordinis vim habeant, formulas valorem ipsius z et ipsius w suppeditantes usque ad quantitates quarti ordinis omnibus partibus expletas evolvam.

2.

Resumamus formulam

$$T = \frac{d . \frac{\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)}}{d\tau}$$

quae, indicata differentiatione instituta, fit

$$T = \frac{\left(\frac{d^2\zeta}{d\tau\, dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)} - \frac{\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)}{\left(\frac{d\zeta}{d\tau}\right)^2}$$

Quum secundum art. 18. Sect. II. $\frac{d\zeta}{d\tau}$ sit quantitas cuius terminus a vi perturbante independens unitas ipsa est, ponamus

$$\left(\frac{d\zeta}{d\tau}\right) = 1 + \left(\frac{d\delta\zeta}{d\tau}\right)$$

ubi igitur $\left(\frac{d\delta\zeta}{d\tau}\right)$ terminos solummodo continet a vi perturbante prolatos. Substituta hac expressione in praecedente expressione ipsius T, habetur, si evolutio usque ad quantitates quarti ordinis perfecta erit,

$$T=\left(\frac{d^2\zeta}{d\tau dt}\right)-\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)+\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau dt}\right)-\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) \qquad \text{.......}(1)$$
$$+\,2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)$$

ubi notandum est, quantitates $\left(\frac{d^2\zeta}{d\tau^2}\right)$, $\left(\frac{d\zeta}{dt}\right)$ et $\left(\frac{d^2\zeta}{d\tau dt}\right)$ esse ipsas quantitates primi ordinis, quas cum differentialibus respectivis ipsius $\delta\zeta$ permutare nobis liceat. Expressio praecedens suppeditat

$$\frac{d^2\zeta}{d\tau dt}=T+\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)+\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)-\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau dt}\right)-2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)$$

3.

Quum sit $f\zeta=\varepsilon$ (secundum art. 15. Sect. II.), habetur

$$\beta=\tfrac{1}{2}(S+\varepsilon)-\tfrac{1}{2}\,l\left(\frac{d\zeta}{d\tau}\right)$$

sive usque ad quantitates tertii ordinis, quae in hac quantitate sufficiunt ad quantitates tertii ordinis in T retinendas,

$$\delta l\varrho=\beta=\tfrac{1}{2}(S+\varepsilon)-\tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{4}\left(\frac{d\delta\zeta}{d\tau}\right)^2$$

atque

$$\delta l\varrho^2=\tfrac{1}{4}(S+\varepsilon)^2+\tfrac{1}{4}\left(\frac{d\delta\zeta}{d\tau}\right)^2-\tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon)$$

Habetur porro

$$\begin{aligned}\delta lh &= -(S+\varepsilon)\\ \delta lr &= w\\ \delta lr' &= w'\end{aligned}$$

Valores incrementorum reliquarum variabilium designantur praefixa littera δ, ita ut sit

$$\begin{aligned}(n)\zeta &= \gamma+(n)\delta\zeta\\ \zeta &= \tau+\frac{(c)}{(n)}+\delta\zeta\\ (n)z &= g+(n)\delta z\\ (n')z' &= g'+(n')\delta z'\\ P &= (P)+\delta P\\ Q &= (Q)+\delta Q\\ K &= k+\delta K\end{aligned}$$

Valores ipsarum (P), (Q), k in Sectione secunda explicati sunt. Denique aequatio

$$h = (h)\, c^{-(S+\varepsilon)}$$

praebet usque ad quantitates tertii ordinis, quae sufficiunt ad quantitates tertii ordinis in ipsa ζ obtinendas, hanc

(2)........ $$h = (h)\left[1 - (S+\varepsilon) + \tfrac{1}{2}(S+\varepsilon)^2\right]$$

4.

Quibus positis, revertamur ad expressionem ipsius T, qualis in art. 16. Sect. II. data est, scilicet

$$T = \left\{2h\,\frac{\varrho}{r}\cos(v_{,}-\lambda) - h + 2\,\frac{h^3\varrho}{\varkappa(M+m)}\left[\cos(v_{,}-\lambda)-1\right]\right\}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$+\, 2\,h\,\frac{\varrho}{r}\sin(v_{,}-\lambda)\,r\left(\frac{d\Omega}{dr}\right) - \frac{(n)\,y\,h}{\varkappa(M+m)}\cdot\frac{d\,.\,\varrho^2}{d\tau}$$

Posita

$$\mathring{T} = \left\{2h\,\frac{\varrho}{r}\cos(v_{,}-\lambda) - h + 2\,\frac{h^3\,\varrho}{\varkappa(M+m)}\left[\cos(v_{,}-\lambda)-1\right]\right.\left(\frac{d\Omega}{dv_{,}}\right)$$
$$+\,2h\frac{\varrho}{r}\sin(v_{,}-\lambda)\,r\left(\frac{d\Omega}{dr}\right)$$

habetur

(3)........ $$T = \mathring{T} - \frac{(n)\,y\,h}{\varkappa(M+m)}\cdot\frac{d\,.\,\varrho^2}{d\tau}$$

Quum ϱ sit functio ipsarum ζ et β, habemus ope theorematis Tayloriani usque ad quantitates tertii ordinis, quae sufficiunt, quia y est quantitas ordinis primi

$$\varrho^2 = (\varrho)^2 + \frac{d.(\varrho)^2}{d\tau}\delta\zeta + (\varrho)^2(S+\varepsilon) - (\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right) + (\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)^2 + \tfrac{1}{2}\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta^2$$
$$+\,\frac{d.(\varrho)^2}{d\tau}\delta\zeta(S+\varepsilon) - \frac{d.(\varrho)^2}{d\tau}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right) + \tfrac{1}{2}(\varrho)^2(S+\varepsilon)^2 - (\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon)$$

ubi (ϱ) valorem ipsius ϱ ope elementorum constantium (a), (e), (c) et (n) computandum denotat.

Differentiata aequatio praecedens suppeditat

$$\frac{d.\varrho^2}{d\tau}=\frac{d.(\varrho)^2}{d\tau}+\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta+\frac{d.(\varrho)^2}{d\tau}(S+\varepsilon)-(\varrho)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d^3.(\varrho)^2}{d\tau^3}\delta\zeta^2+\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta(S+\varepsilon)$$
$$-\frac{d.(\varrho)^2}{d\tau}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}(S+\varepsilon)^2+2(\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)-(\varrho)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)(S+\varepsilon).$$

Substituto hoc ipsius $\frac{d.\varrho^2}{d\tau}$ valore, nec non valore ipsius h per (2) dato, aequatio (3) transit in hanc

$$\mathring{T}=T+\frac{(n)y(h)}{\varkappa(M+m)}\left\{\frac{d.(\varrho)^2}{d\tau}+\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta-(\varrho)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d^3.(\varrho)^2}{d\tau^3}\delta\zeta^2-\frac{d.(\varrho)^2}{d\tau}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+2(\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\right\}$$

Si in hac aequatione valor ipsius T ex (1) petendus substitutus fuerit, differentiatione invenientur aequationes usque ad quantitates tertii et resp. secundi ordinis accuratae hae

$$\left.\begin{aligned}
\mathring{T}&=\left(\frac{d^2\zeta}{d\tau dt}\right)-\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)-\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)+\frac{(n)y(h)}{\varkappa(M+m)}\left\{\frac{d.(\varrho)^2}{d\tau}+\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta-(\varrho)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\right\}\\
\frac{d\mathring{T}}{d\tau}&=\left(\frac{d^3\zeta}{d\tau^2 dt}\right)-2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)-\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^3\zeta}{d\tau^2 dt}\right)-\left(\frac{d^3\delta\zeta}{d\tau^3}\right)\left(\frac{d\zeta}{dt}\right)\\
&\quad+\frac{(n)y(h)}{\varkappa(M+m)}\left\{\frac{d^2.(\varrho)^2}{d\tau^2}+\frac{d^2.(\varrho)^2}{d\tau^2}\left(\frac{d\delta\zeta}{d\tau}\right)+\frac{d^3.(\varrho)^2}{d\tau^3}\delta\zeta-\frac{d.(\varrho)^2}{d\tau}\left(\frac{d^2\delta\zeta}{d\tau^2}\right)-(\varrho)^2\left(\frac{d^3\delta\zeta}{d\tau^3}\right)\right\}\\
\frac{d^2\mathring{T}}{d\tau^2}&=\left(\frac{d^4\zeta}{d\tau^3 dt}\right)+\frac{(n)y(h)}{\varkappa(M+m)}\cdot\frac{d^3.(\varrho)^2}{d\tau^3}
\end{aligned}\right\}\cdots\cdots(4)$$

quibus infra utemur. Porro, quum sit

$$(T)=(\mathring{T})-\frac{(n)y(h)}{\varkappa(M+m)}\cdot\frac{d.(\varrho)^2}{d\tau}$$

denotantibus (T) et $(\mathring{T})$ resp. eos ipsarum T et $\mathring{T}$ valores ex elementis (a), (e), (h), etc. pendentes, in quibus β, $(S+\varepsilon)$, w, etc. omissae et γ, g atque g' resp. loco $(n)\zeta$, $(n)z$ atque $(n)z'$ positae sunt, aequatio praecedens inter T et $\mathring{T}$ inventa suppeditat

$$T=\mathring{T}-(\mathring{T})+(T)-\frac{(n)y(h)}{\varkappa(M+m)}\left\{\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta-(\varrho)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d^3.(\varrho)^2}{d\tau^3}\delta\zeta^2-\frac{d.(\varrho)^2}{d\tau}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+2(\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\right\}$$

et si hic ipsius T valor in ultima art. 2. aequatione substitutus erit, emergit

$$\frac{d^2\zeta}{d\tau dt}=\mathring{T}-(\mathring{T})+(T)+\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)-\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau dt}\right)+\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)-2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)$$
$$-\frac{(n)y(h)}{\varkappa(M+m)}\left\{\frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta-(\varrho)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d^3.(\varrho)^2}{d\tau^3}\delta\zeta^2-\frac{d.(\varrho)^2}{d\tau}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+2(\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\right.$$

$$\mathring{T}-(\mathring{T})=\frac{d(\mathring{T})}{d\tau}\delta\zeta-\tfrac{1}{2}(\mathring{T})\left(\frac{d\delta\zeta}{d\tau}\right)-\tfrac{1}{2}\frac{d(S)}{dt}\left(\frac{d\delta\zeta}{d\tau}\right)-\tfrac{1}{2}(\mathring{T})(S+\varepsilon)+\tfrac{1}{2}\frac{d(S)}{dt}(S+\varepsilon)-2U(S+\varepsilon)$$
$$+\frac{d(\mathring{T})}{dg}(n)\delta z+r\frac{d(\mathring{T})}{dr}w+\frac{d(\mathring{T})}{dg'}(n')\delta z'+r'\frac{d(\mathring{T})}{dr'}w'+\frac{d(\mathring{T})}{dP}\delta P+\frac{d(\mathring{T})}{dQ}\delta Q+\frac{d(\mathring{T})}{dK}\delta K$$
$$+\tfrac{1}{2}\frac{d^2(\mathring{T})}{d\tau^2}\delta\zeta^2+\tfrac{3}{8}\frac{d(S)}{dt}\left(\frac{d\delta\zeta}{d\tau}\right)^2-\tfrac{1}{2}\frac{d(\mathring{T})}{d\tau}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)-\tfrac{1}{2}\frac{d(\mathring{T})}{d\tau}\delta\zeta(S+\varepsilon)-2\frac{dU}{d\tau}\delta\zeta(S+\varepsilon)+\tfrac{3}{8}(\mathring{T})\left(\frac{d\delta\zeta}{d\tau}\right)^2$$
$$+\tfrac{1}{4}(\mathring{T})\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon)+\tfrac{1}{4}\frac{d(S)}{dt}\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon)+U\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon)+\tfrac{1}{8}(\mathring{T})(S+\varepsilon)^2-\tfrac{3}{8}\frac{d(S)}{dt}(S+\varepsilon)^2$$
$$+3U(S+\varepsilon)^2+\frac{d^2(\mathring{T})}{d\tau.dg}\delta\zeta.(n)\delta z+r\frac{d^2(\mathring{T})}{d\tau.dr}\delta\zeta.w+\frac{d^2(\mathring{T})}{d\tau.dg'}\delta\zeta.(n')\delta z'+r'\frac{d^2(\mathring{T})}{d\tau.dr'}\delta\zeta.w'$$
$$+\frac{d^2(\mathring{T})}{d\tau.dP}\delta\zeta.\delta P+\frac{d^2(\mathring{T})}{d\tau.dQ}\delta\zeta.\delta Q+\frac{d^2(\mathring{T})}{d\tau.dK}\delta\zeta.\delta K$$
$$-\tfrac{1}{2}\left[\frac{d(\mathring{T})}{dg}(n)\delta z+r\frac{d(\mathring{T})}{dr}w+\frac{d(\mathring{T})}{dg'}(n')\delta z'+r'\frac{d(\mathring{T})}{dr'}w'+\frac{d(\mathring{T})}{dP}\delta P+\frac{d(\mathring{T})}{dQ}\delta Q+\frac{d(\mathring{T})}{dK}\delta K\right]\left(\frac{d\delta\zeta}{d\tau}\right)$$
$$-\tfrac{1}{2}\left[\frac{d^2(S)}{dg.dt}(n)\delta z+r\frac{d^2(S)}{dr.dt}w+\frac{d^2(S)}{dg'.dt}(n')\delta z'+r'\frac{d^2(S)}{dr'.dt}w'+\frac{d^2(S)}{dP.dt}\delta P+\frac{d^2(S)}{dQ.dt}\delta Q+\frac{d^2(S)}{dK.dt}\delta K\right]\left(\frac{d\delta\zeta}{d\tau}\right)$$
$$-\tfrac{1}{2}\left[\frac{d(\mathring{T})}{dg}(n)\delta z+r\frac{d(\mathring{T})}{dr}w+\frac{d(\mathring{T})}{dg'}(n')\delta z'+r'\frac{d(\mathring{T})}{dr'}w'+\frac{d(\mathring{T})}{dP}\delta P+\frac{d(\mathring{T})}{dQ}\delta Q+\frac{d(\mathring{T})}{dK}\delta K\right](S+\varepsilon)$$
$$+\tfrac{1}{2}\left[\frac{d^2(S)}{dg.dt}(n)\delta z+r\frac{d^2(S)}{dr.dt}w+\frac{d^2(S)}{dg'.dt}(n')\delta z'+r'\frac{d^2(S)}{dr'.dt}w'+\frac{d^2(S)}{dP.dt}\delta P+\frac{d^2(S)}{dQ.dt}\delta Q+\frac{d^2(S)}{dK.dt}\delta K\right](S+\varepsilon)$$
$$-2\left[\frac{dU}{dg}(n)\delta z+r\frac{dU}{dr}w+\frac{dU}{dg'}(n')\delta z'+r'\frac{dU}{dr'}w'+\frac{dU}{dP}\delta P+\frac{dU}{dQ}\delta Q+\frac{dU}{dK}\delta K\right](S+\varepsilon)$$
$$+\tfrac{1}{2}\frac{d^2(\mathring{T})}{dg^2}(n)^2\delta z^2+r\frac{d^2(\mathring{T})}{dg.dr}(n)\delta z.w+\text{etc.}$$

quae aequatio usque ad quantitates tertii ordinis reciproce suppeditat

$$(\mathring{T})=\mathring{T}-\frac{d(\mathring{T})}{d\tau}\delta\zeta+\tfrac{1}{2}(\mathring{T})\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{2}\frac{d(S)}{dt}\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{2}(\mathring{T})(S+\varepsilon)-\tfrac{1}{2}\frac{d(S)}{dt}(S+\varepsilon)+2U(S+\varepsilon)$$
$$-\frac{d(\mathring{T})}{dg}(n)\delta z-r\frac{d(\mathring{T})}{dr}w-\frac{d(\mathring{T})}{dg'}(n')\delta z'-r'\frac{d(\mathring{T})}{dr'}w'-\frac{d(\mathring{T})}{dP}\delta P-\frac{d(\mathring{T})}{dQ}\delta Q-\frac{d(\mathring{T})}{dK}\delta K$$

e qua differentiata elicitur haec

$$\frac{d(\mathring{T})}{d\tau}=\frac{d\mathring{T}}{d\tau}-\frac{d^2(\mathring{T})}{d\tau^2}\delta\zeta-\tfrac{1}{2}\frac{d(\mathring{T})}{d\tau}\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{2}(\mathring{T})\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d(S)}{dt}\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{2}\frac{d(\mathring{T})}{d\tau}(S+\varepsilon)+2\frac{dU}{d\tau}(S+\varepsilon)$$
$$-\frac{d^2(\mathring{T})}{d\tau.dg}(n)\delta z-r\frac{d^2(\mathring{T})}{d\tau.dr}w-\frac{d^2(\mathring{T})}{d\tau.dg'}(n')\delta z'-r'\frac{d^2(\mathring{T})}{d\tau.dr'}w'-\frac{d^2(\mathring{T})}{d\tau.dP}\delta P-\frac{d^2(\mathring{T})}{d\tau.dQ}\delta Q-\frac{d^2(\mathring{T})}{d\tau.dK}\delta K$$

et usque ad quantitates secundi ordinis hae

Porro, quum sit

$$\frac{dS}{dt} = h\left(\frac{d\Omega}{dv_r}\right)$$

erit $\frac{dS}{dt}$ functio variabilium lh, z, lr, z', lr', P, Q atque K, unde habetur ope theorematis Tayloriani

$$\frac{d(S)}{dt} = \frac{dS}{dt} + \frac{d(S)}{dt}(S+\varepsilon) - \frac{d^2(S)}{dg\,.\,dt}(n)\delta z - r\frac{d^2(S)}{dr\,.\,dt}w - \frac{d^2(S)}{dg'.\,dt}(n')\delta z'$$
$$- r'\frac{d^2(S)}{dr'.\,dt}w' - \frac{d^2(S)}{dP.\,dt}\delta P - \frac{d^2(S)}{dQ.\,dt}\delta Q - \frac{d^2(S)}{dK.\,dt}\delta K$$

et respective

$$\frac{d(S)}{dt} = \frac{dS}{dt}$$
$$\frac{d^2(S)}{dg\,.\,dt} = \frac{d^2(S)}{dg\,.\,dt}$$
$$\text{etc.} = \text{etc}$$

Substitutis his ipsarum $(\mathring{T})$, $\frac{d(\mathring{T})}{d\tau}$, $\frac{d^2(\mathring{T})}{d\tau^2}$, $\frac{d(S)}{dt}$, $\frac{d^2(S)}{dg\,.\,dt}$, etc. valoribus in dextro membro expressionis praecedentis pro $\mathring{T} - (\mathring{T})$, nanciscimur, si termini quarti ordinis et ordinum altiorum ubique negliguntur,

$$\mathring{T} - (\mathring{T}) = \frac{d\mathring{T}}{d\tau}\delta\zeta - \tfrac{1}{2}\mathring{T}\left(\frac{d\delta\zeta}{d\tau}\right) - \tfrac{1}{2}\left(\frac{dS}{dt}\right)\left(\frac{d\delta\zeta}{d\tau}\right) - \tfrac{1}{2}\mathring{T}(S+\varepsilon) + \tfrac{1}{2}\frac{dS}{dt}(S+\varepsilon) - 2U(S+\varepsilon)$$
$$+ \frac{d(\mathring{T})}{dg}(n)\delta z + r\frac{d(\mathring{T})}{dr}w + \frac{d(\mathring{T})}{dg'}(n')dz' + r'\frac{d(\mathring{T})}{dr'}w' + \frac{d(\mathring{T})}{dP}\delta P + \frac{d(\mathring{T})}{dQ}\delta Q + \frac{d(\mathring{T})}{dK}\delta K$$
$$- \tfrac{1}{2}\frac{d^2\mathring{T}}{d\tau^2}\delta\zeta^2 - \tfrac{1}{2}\frac{d\mathring{T}}{d\tau}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right) + \tfrac{1}{2}\mathring{T}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \tfrac{1}{2}\frac{dS}{dt}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \tfrac{1}{4}\mathring{T}\left(\frac{d\delta\zeta}{d\tau}\right)^2 + \tfrac{1}{4}\frac{dS}{dt}\left(\frac{d\delta\zeta}{d\tau}\right)^2$$
$$- \tfrac{1}{4}\mathring{T}\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon) - \tfrac{1}{4}\frac{dS}{dt}\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon) + \tfrac{1}{2}\frac{d\mathring{T}}{d\tau}\delta\zeta(S+\varepsilon) - \tfrac{1}{4}\mathring{T}(S+\varepsilon)^2 + \tfrac{3}{4}\frac{dS}{dt}(S+\varepsilon)^2$$
$$+ 2U(S+\varepsilon)^2 - 2\frac{dU}{dg}(S+\varepsilon)(n)\delta z - 2r\frac{dU}{dr}(S+\varepsilon)w - 2\frac{dU}{dg'}(S+\varepsilon)(n')\delta z' - 2r'\frac{dU}{dr'}(S+\varepsilon)w'$$
$$- 2\frac{dU}{dP}(S+\varepsilon)\delta P - 2\frac{dU}{dQ}(S+\varepsilon)\delta Q - 2\frac{dU}{dK}(S+\varepsilon)\delta K + \tfrac{1}{2}\frac{d^2(\mathring{T})}{dg^2}(n)^2\delta z^2 + r\frac{d^2(\mathring{T})}{dg\,.\,dr}(n)\delta z\,.\,w + \text{etc.}$$

Ex hac expressione quantitates $\mathring{T}$, $\frac{d\mathring{T}}{d\tau}$ atque $\frac{d^2\mathring{T}}{d\tau^2}$ ope aequationum (4) in art. 4. datarum eliminandae sunt; quo facto emergit

$$\begin{aligned}
\dot{\mathring{T}} - (\mathring{T}) + (T) = \; & \frac{dX}{dt} + \delta\zeta\left(\frac{d^3\zeta}{d\tau^2 dt}\right) - \tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right) - \tfrac{1}{2}(S+\varepsilon)\left(\frac{d^2\zeta}{d\tau dt}\right) - \tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dS}{dt} + \tfrac{1}{2}(S+\varepsilon)\frac{dS}{dt} \\
& - \tfrac{3}{2}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d^2\zeta}{d\tau dt}\right) - \tfrac{3}{2}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^3\zeta}{d\tau^2 dt}\right) - \delta\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right)\left(\frac{d\zeta}{dt}\right) - \tfrac{1}{2}\delta\zeta^2\left(\frac{d^4\zeta}{d\tau^3 dt}\right) + \tfrac{3}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau dt}\right) \\
& + \tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) + \tfrac{1}{4}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right) + \tfrac{1}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^2\frac{dS}{dt} + \tfrac{1}{2}(S+\varepsilon)\delta\zeta\left(\frac{d^3\zeta}{d\tau^2 dt}\right) \\
& + \tfrac{1}{2}(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) + \tfrac{1}{2}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\frac{dS}{dt} - \tfrac{1}{8}(S+\varepsilon)^2\left(\frac{d^2\zeta}{d\tau dt}\right) - \tfrac{1}{4}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dS}{dt} + \tfrac{3}{8}(S+\varepsilon)^2\frac{dS}{dt} \\
& + \frac{(n)y(h)}{\varkappa(M+m)}\left\{\begin{aligned}
& \frac{d^2.(\varrho)^2}{d\tau^2}\delta\zeta - \tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\left(\frac{d\delta\zeta}{d\tau}\right) - \tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}(S+\varepsilon) + \tfrac{1}{2}\frac{d^3.(\varrho)^2}{d\tau^3}\delta\zeta^2 - \tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \tfrac{1}{8}\frac{d.(\varrho)^2}{d\tau}\left(\frac{d\delta\zeta}{d\tau}\right)^2 \\
& - \tfrac{1}{4}\frac{d(\varrho)^2}{d\tau}\left(\frac{d\delta\zeta}{d\tau}\right)(S+\varepsilon) - \tfrac{1}{8}\frac{d.(\varrho)^2}{d\tau}(S+\varepsilon)^2 - (\varrho)^2\delta\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right) + \tfrac{1}{2}(\varrho)^2\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \tfrac{1}{2}(\varrho)^2(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)
\end{aligned}\right\}
\end{aligned}$$

ubi brevitatis caussa posui

$$X = \int \left\{\begin{aligned}
& (T) - 2U(S+\varepsilon) + \frac{d(\mathring{T})}{dg}(n)\delta\zeta + r\frac{d(\mathring{T})}{dr}w + \frac{d(\mathring{T})}{dg'}(n')\delta z' + r'\frac{d(\mathring{T})}{dr'}w' + \frac{d(\mathring{T})}{dP}\delta P + \frac{d(\mathring{T})}{dQ}\delta Q + \frac{d(\mathring{T})}{dK}\delta K \\
& + 2U(S+\varepsilon)^2 - 2\frac{dU}{dg}(S+\varepsilon)(n)\delta z - 2r\frac{dU}{dr}(S+\varepsilon)w - 2\frac{dU'}{dg'}(S+\varepsilon)(n')\delta z' - 2r'\frac{dU}{dr'}(S+\varepsilon)w' - 2\frac{dU}{dP}(S+\varepsilon)\delta P \\
& \qquad - 2\frac{dU}{dQ}(S+\varepsilon)\delta Q - 2\frac{dU}{dK}(S+\varepsilon)\delta K \\
& + \tfrac{1}{2}\frac{d^2(\mathring{T})}{dg^2}(n)^2\delta z^2 + r\frac{d^2(\mathring{T})}{dg.dr}(n)\delta z.w + \frac{d^2(\mathring{T})}{dg.dg'}(n)\delta z.(n')\delta z' + r'\frac{d^2(\mathring{T})}{dg.dr'}(n)\delta z.w' + \frac{d^2(\mathring{T})}{dg.dP}(n)\delta z.\delta P \\
& \qquad + \frac{d^2(\mathring{T})}{dg.dQ}(n)\delta z.\delta Q + \frac{d^2(\mathring{T})}{dg.dK}(n)\delta z.\delta K \\
& + \tfrac{1}{2}\left\{r^2\frac{d^2(\mathring{T})}{dr^2} + r\frac{d(\mathring{T})}{dr}\right\}w^2 + r\frac{d^2(\mathring{T})}{dr.dg'}w.(n')\delta z' + rr'\frac{d^2(\mathring{T})}{dr.dr'}w.w' + r\frac{d^2(\mathring{T})}{dr.dP}w.\delta P + r\frac{d^2(\mathring{T})}{dr.dQ}w.\delta Q \\
& \qquad + r\frac{d^2(\mathring{T})}{dr.dK}w.\delta K \\
& + \tfrac{1}{2}\frac{d^2(\mathring{T})}{dg'^2}(n')^2\delta z'^2 + r'\frac{d^2(\mathring{T})}{dg'.dr'}(n')\delta z'.w' + \frac{d^2(\mathring{T})}{dg'.dP}(n')\delta z'.\delta P + \frac{d^2(\mathring{T})}{dg'.dQ}(n')\delta z'.\delta Q + \frac{d^2(\mathring{T})}{dg'.dK}(n')\delta z'.\delta K \\
& + \tfrac{1}{2}\left\{r'^2\frac{d^2(\mathring{T})}{dr'^2} + r'\frac{d(\mathring{T})}{dr'}\right\}w'^2 + r'\frac{d^2(\mathring{T})}{dr'.dP}w'.\delta P + r'\frac{d^2(\mathring{T})}{dr'.dQ}w'.\delta Q + r'\frac{d^2(\mathring{T})}{dr'.dK}w'.\delta K \\
& + \tfrac{1}{2}\frac{d^2(\mathring{T})}{dP^2}\delta P^2 + \frac{d^2(\mathring{T})}{dP.dQ}\delta P.\delta Q + \frac{d^2(\mathring{T})}{dP.dK}\delta P.\delta K \\
& + \tfrac{1}{2}\frac{d^2(\mathring{T})}{dQ}\delta Q^2 + \frac{d^2(\mathring{T})}{dQ.dK}\delta Q.\delta K \\
& + \tfrac{1}{2}\frac{d^2(\mathring{T})}{dK^2}\delta K^2
\end{aligned}\right\} dt$$

In hac ipsius X expressione terminos tertii ordinis omnes, ut praesto sint, adscripsi, quamquam maxima eorum pars in motu Lunae nullam vim habet. Praeter terminos enim tertii ordinis, qui in terminis secundi ordinis expressionis praecedentis implicite continentur, termini tertii ordinis, qui ex producto quadratisque ipsarum $(n)\delta z$ et w pendent, fere sunt, qui vim habent. Receptis his terminis nec non maximis reliquorum terminorum, in motu Lunae poni potest

$$X = \int \left\{ \begin{array}{l} (T) - 2U(S+\varepsilon) + \frac{d(\mathring{T})}{dg}(n)\delta z + r\frac{d(\mathring{T})}{dr}w + \frac{d(\mathring{T})}{dg'}(n')\delta z' + r'\frac{d(\mathring{T})}{dr'}w' + \frac{d(\mathring{T})}{dP}\delta P + \frac{d(\mathring{T})}{dQ}\delta Q \\ \qquad\qquad + \frac{d(\mathring{T})}{dK}\delta K \\ + 2U(S+\varepsilon)^2 - 2\frac{dU}{dg}(S+\varepsilon)(n)\delta z - 2r\frac{dU}{dr}(S+\varepsilon)w + \frac{1}{2}\frac{d^2(\mathring{T})}{dg^2}(n)^2 dz^2 + r\frac{d^2(\mathring{T})}{dg\,.\,dr}(n)\delta z\,.\,w \\ + \frac{d^2(\mathring{T})}{dg\,.\,dP}(n)\delta z\,.\,\delta P + \frac{d^2(\mathring{T})}{dg\,.\,dQ}(n)\delta z\,.\,\delta Q + \frac{1}{2}\left\{ r^2\frac{d^2(\mathring{T})}{dr^2} + r\frac{d(\mathring{T})}{dr} \right\} w^2 + r\frac{d^2(\mathring{T})}{dr\,.\,dP}w\,.\,\delta P \\ + \frac{d^2(\mathring{T})}{dr\,.\,dQ}w\,.\,\delta Q \end{array} \right\} dt$$

quae expressio iam terminos nonnullos continet, qui omitti potuissent. Substituto praecedente ipsius $\mathring{T} - (\mathring{T}) + (T)$ valore in aequatione ultima art. 4., emergit denique

$$\begin{array}{l} \frac{d^2\zeta}{d\tau\,dt} = \frac{dX}{dt} + \delta\zeta\left(\frac{d^3\zeta}{d\tau^2 dt}\right) + \left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) + \frac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau\,dt}\right) - \frac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dS}{dt} - \frac{1}{2}(S+\varepsilon)\left(\frac{d^2\zeta}{d\tau\,dt}\right) + \frac{1}{2}(S+\varepsilon)\frac{dS}{dt} \\ \quad - \frac{3}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau\,dt}\right) - \frac{3}{4}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^3\zeta}{d\tau^2 dt}\right) - \frac{3}{4}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d^2\zeta}{d\tau\,dt}\right) - \frac{3}{4}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) - \frac{1}{2}\delta\zeta^2\left(\frac{d^4\zeta}{d\tau^3 dt}\right) \\ \quad - d\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right)\left(\frac{d\zeta}{dt}\right) + \frac{1}{4}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau\,dt}\right) + \frac{1}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^2\frac{dS}{dt} + \frac{1}{2}(S+\varepsilon)\delta\zeta\left(\frac{d^3\zeta}{d\tau^2 dt}\right) + \frac{1}{2}(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) \\ \quad + \frac{1}{2}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\frac{dS}{dt} - \frac{1}{8}(S+\varepsilon)^2\left(\frac{d^2\zeta}{d\tau\,dt}\right) - \frac{1}{4}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dS}{dt} + \frac{3}{8}(S+\varepsilon)^2\frac{dS}{dt} \\ \quad + \frac{(n)y(h)}{\varkappa(M+m)}\left\{ \begin{array}{l} (\varrho)^2\left\{ \left(\frac{d^2\delta\zeta}{d\tau^2}\right) - \frac{3}{4}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right) - \delta\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right) + \frac{1}{2}(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right) \right\} \\ - \frac{1}{2}\frac{d\,.\,(\varrho)^2}{d\tau}\left\{ \left(\frac{d\delta\zeta}{d\tau}\right) + (S+\varepsilon) - \delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right) - \frac{1}{4}\left(\frac{d\delta\zeta}{d\tau}\right)^2 + \frac{1}{2}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right) + \frac{1}{4}(S+\varepsilon)^2 \right\} \end{array} \right\} \end{array}$$

6.

Secundum ea, quae in Sectione secunda tradidimus, expressio modo evoluta primum respectu ipsius τ integranda est, quo $\frac{d\zeta}{dt}$ obtineatur, et con-

stans arbitraria huic integrali addita, quae functio ipsius t erit, ita determinanda est, ut fiat

$$\overline{\left(\frac{d\zeta}{dt}\right)} = -\frac{(n)\,y}{\left(\frac{d.\Delta z}{dz}\right)}$$

Quibus factis, expressio respectu ipsius t integratur, cui integrali tamquam constans arbitraria functio ipsius τ in art. 15. Sect. II. definita et (ζ) nominata est addenda. Tali modo ipsa ζ inventa, mutata τ in t ipsa z prodibit. In sequentibus vero aliam ingressuri viam, ordinem integrationum invertemus, quo factum erit, ut formulam simpliciorem nanciscamur. Cuius calculi fundamenta nunc exponamus.

Quantitas $\frac{d^2\zeta}{d\tau dt}$ generaliter sumta functio ipsarum t et τ est, quam per $\varphi(\tau, t)$ denotabo. Tum habetur

(5)........ $$\frac{d^2\zeta}{d\tau} = \varphi(\tau, t)\, dt$$

e qua integrata elicitur

$$\frac{d\zeta}{d\tau} = \int\varphi(\tau, t)\, dt + F\tau$$

ubi $F\tau$ functio adhuc ignota ipsius τ est, quae huic integrali tamquam constans arbitraria addenda erat. Hinc, mutata τ in t, nanciscimur

$$\overline{\left(\frac{d\zeta}{d\tau}\right)} = \overline{\left[\int\varphi(\tau, t)\, dt\right]} + Ft$$

Aequatio vero (10) art. 9. Sect. II. est haec

(6)........ $$\frac{dz}{dt} = \overline{\left(\frac{d\zeta}{d\tau}\right)} - \frac{(n)\,y}{\left(\frac{d.\Delta z}{dz}\right)}$$

quae, substituto valore ipsius $\overline{\left(\frac{d\zeta}{d\tau}\right)}$ modo eruto, suppeditat

(7)........ $$\frac{dz}{dt} = \overline{\left[\int\varphi(\tau, t)\, dt\right]} + Ft - \frac{(n)\,y}{\left(\frac{d.\Delta z}{dz}\right)}$$

Qua aequatione per dt multiplicata et integrata, ipsam z nanciscimur, cui integrali vera constans addenda est. Quibus absolutis, ad indolem ipsius

Ft investigandam $\frac{dz}{dt}$ methodo illa computabimus. Aequatio (5) etiam suppeditat

$$\frac{d^2\zeta}{dt} = \varphi(\tau,\ t)\,d\tau$$

itaque

$$\frac{d\zeta}{dt} = \int\varphi(\tau,\ t)\,d\tau + \psi t \qquad \text{........(8)}$$

ubi ψt denotat functionem illam, quae tamquam constans arbitraria integrali huic addenda erat. Mutata τ in t, obtinemus ex praecedenti integrali

$$\overline{\left(\frac{d\zeta}{dt}\right)} = \overline{[\int\varphi(\tau,\ t)\,d\tau]} + \psi t$$

Conditio igitur in huius articuli initio memorata subministrat

$$-\frac{(n)\,y}{\left(\frac{d\,.\,\varDelta z}{dz}\right)} = \overline{[\int\varphi(\tau,\ t)\,d\tau]} + \psi t$$

Valore ipsius ψt ex hac aequatione elicito et in (8) substituto, emergit

$$d\zeta = dt\int\varphi(\tau,\ t)\,d\tau - dt\,\overline{[\int\varphi(\tau,\ t)\,d\tau]} - \frac{(n)\,y}{\left(\frac{d\,.\,\varDelta z}{dz}\right)}\,dt$$

unde

$$\zeta = (\zeta) + \int dt\int\varphi(\tau,\ t)\,d\tau - \int dt\,\overline{[\int\varphi(\tau,\ t)\,d\tau]} - \int\frac{(n)\,y}{\left(\frac{d\,.\,\varDelta z}{dz}\right)}\,dt$$

Hinc, mutata τ in t, evadit

$$z = (z) + \overline{[\int dt\int\varphi(\tau,\ t)\,d\tau]} - \int dt\,\overline{[\int\varphi(\tau,\ t)\,d\tau]} - \int\frac{(n)\,y}{\left(\frac{d\,.\,\varDelta z}{dz}\right)}\,dt$$

atque hinc

$$\frac{dz}{dt} = \frac{d\,(z)}{dt} + \frac{d\,.\,\overline{[\int dt\int\varphi(\tau,\ t)\,d\tau]}}{dt} - \overline{[\int\varphi(\tau,\ t)\,d\tau]} - \frac{(n)\,y}{\left(\frac{d\,.\,\varDelta z}{dz}\right)}$$

Si hic ipsius $\frac{dz}{dt}$ valor cum valore eiusdem quantitatis sub (7) dato comparatus erit, invenitur

$$Ft = \frac{d\,(z)}{dt} + \frac{d\,.\,\overline{[\int dt\int\varphi(\tau,\ t)\,d\tau]}}{dt} - \overline{[\int\varphi(\tau,\ t)\,d\tau]} - \overline{[\int\varphi(\tau,\ t)\,dt]}$$

Sed manifestum est, esse

$$\frac{d.\left[\overline{\int dt\int \varphi(\tau,t)\,d\tau}\right]}{dt} = \left[\overline{\int \varphi(\tau,t)\,d\tau}\right] + \overline{\left(\frac{d.\int dt\int \varphi(\tau,t)\,d\tau}{d\tau}\right)}$$

atque

$$\overline{\left(\frac{d.\int dt\int \varphi(\tau,t)\,d\tau}{d\tau}\right)} = \overline{\left(\int dt\,\frac{d.\int \varphi(\tau,t)\,d\tau}{d\tau}\right)} = \overline{\left[\int \varphi(\tau,t)\,dt\right]}$$

quibus expressionibus in valore praecedenti ipsius Ft substitutis, emergit

$$Ft = \frac{d(z)}{dt}$$

unde

$$F\tau = \frac{d(\zeta)}{d\tau}$$

Hinc colligitur, ad ipsam z immediate ex expressione ipsius $\frac{d^2\zeta}{d\tau\,dt}$ in art. praec. data eliciendam, expressionem hanc primum respectu ipsius t integrandam esse, cui integrali tamquam constans arbitraria $\frac{d(\zeta)}{d\tau}$ addatur, deinde, mutata τ in t, valorem ipsius $\overline{\left(\frac{d\zeta}{d\tau}\right)}$ hoc modo inventum in aequatione (6) substituendum esse, e qua denique ipsam z, integratione respectu ipsius t iterum peracta, elicitum iri, cui vera constans addatur, quae nihil aliud est quam constans (c) in art. 15. Sectionis secundae introducta, quae anomaliam mediam temporis epochae respondentem denotat.

Ex eodem vero articulo habetur

$$\frac{d(\zeta)}{d\tau} = 1 - b + A(1-b)\xi + B\xi^2$$

(secundum art. 20. Sect. II. enim in motu Lunae statuitur $\eta = o$), ubi brevitatis caussa posui

$$A = 2\,\frac{(\bar\varrho)}{(a)}\cos(\bar\varphi) + 3(e)$$
$$B = \frac{(\bar\varrho)^2}{(a)^2}\cos^2(\bar\varphi) + 6(e)\,\frac{(\bar\varrho)}{(a)}\cos(\bar\varphi) + \tfrac{3}{2}\left[1 + 4(e)^2\right]$$

ubi $(\bar\varrho)$ et $(\bar\varphi)$ functiones ipsius (ζ) sunt, quae ipsa functio ipsius ξ censenda est. Habemus igitur, si tertia et altiores ipsius ξ potestates negliguntur,

$$A = A_{,} + \frac{dA_{,}}{d(\zeta)}\,\delta(\zeta)$$
$$B = B_{,}$$

ubi $A_{,}$ et $B_{,}$ functiones ipsarum (ϱ) et (φ) sunt, quae ipsae non minus quam quantitates reliquae, quibus formulae in hac Sectione evolutae compositae sunt, ope ipsius γ et elementorum (a), (e) et (c) computandae sunt. Quantitas $\delta(\zeta)$ vero hac datur formula

$$\delta(\zeta) = \xi\int\left\{2\frac{(\varrho)}{(a)}\cos(\varphi) + 3(e)\right\}d\tau = \xi\frac{(\varrho)^2 + (a)(\varrho)[1-(e)^2]}{(n)(a)^2\sqrt{1-(e)^2}}\sin(\varphi)\,\text{*)}$$

et $\frac{dA_{,}}{d(\zeta)}$ hac

$$\frac{dA_{,}}{d(\zeta)} = -2\xi\frac{(n)\sin(\varphi)}{\sqrt{1-(e)^2}}$$

unde nanciscimur

$$\frac{dA_{,}}{d(\zeta)}\delta\zeta + B_{,} = A_{,,} = 3\frac{(\varrho)^2}{(a)^2}\cos^2(\varphi) + 12(e)\frac{(\varrho)}{(a)}\cos(\varphi) + 2\frac{(\varrho)}{(a)}\cos^2(\varphi) - \tfrac{5}{2} + 10(e)^2$$

cui accedit

$$A_{,} = 2\frac{(\varrho)}{(a)}\cos(\varphi) + 3(e)$$

Quibus positis erit denique

$$\frac{d(\zeta)}{d\tau} = 1 - b + A_{,}(1-b)\xi + A_{,,}\xi^2$$

7.

Ad integrationem primam expressionis pro $\frac{d^2\zeta}{d\tau dt}$ instituendam animadverto esse

$$\delta\zeta\left(\frac{d^3\zeta}{d\tau^2 dt}\right) + \left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right) = \frac{d.\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)}{dt}$$

$$\tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right) = \tfrac{1}{4}\frac{d.\left(\frac{d\delta\zeta}{d\tau}\right)^2}{dt}$$

$$-\tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dS}{dt} - \tfrac{1}{2}(S+\varepsilon)\left(\frac{d^2\zeta}{d\tau dt}\right) = -\tfrac{1}{2}\frac{d.(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)}{dt}$$

$$\tfrac{1}{2}(S+\varepsilon)\frac{dS}{dt} = \tfrac{1}{4}\frac{d.(S+\varepsilon)^2}{dt}$$

$$-\tfrac{3}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau dt}\right) = -\tfrac{1}{8}\frac{d.\left(\frac{d\delta\zeta}{d\tau}\right)^3}{dt}$$

*) Vide Unt. über die gegens. Störungen des Jupiters und Saturns pag. 8.

$$-\tfrac{3}{2}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^3\zeta}{d\tau^2 dt}\right)-\tfrac{3}{2}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)-\tfrac{3}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)=-\tfrac{3}{2}\,\frac{d.\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)}{dt}$$

$$-\tfrac{1}{2}\delta\zeta^2\left(\frac{d^4\zeta}{d\tau^3 dt}\right)-\delta\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right)\left(\frac{d\zeta}{dt}\right)=-\tfrac{1}{2}\,\frac{d.\delta\zeta^2\left(\frac{d^3\delta\zeta}{d\tau^3}\right)}{dt}$$

$$\tfrac{1}{4}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau dt}\right)+\tfrac{1}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^2\frac{dS}{dt}=\tfrac{1}{8}\,\frac{d.(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)^2}{dt}$$

$$\tfrac{1}{2}(S+\varepsilon)\delta\zeta\left(\frac{d^3\zeta}{d\tau^2 dt}\right)+\tfrac{1}{2}(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\left(\frac{d\zeta}{dt}\right)+\tfrac{1}{2}\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\frac{dS}{dt}=\tfrac{1}{2}\,\frac{d.(S+\varepsilon)\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)}{dt}$$

$$-\tfrac{1}{8}(S+\varepsilon)^2\left(\frac{d^2\zeta}{d\tau dt}\right)-\tfrac{1}{4}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dS}{dt}=-\tfrac{1}{8}\,\frac{d.(S+\varepsilon)^2\left(\frac{d\delta\zeta}{d\tau}\right)}{dt}$$

$$\tfrac{3}{8}(S+\varepsilon)^2\frac{dS}{dt}=\tfrac{1}{8}\,\frac{d.(S+\varepsilon)^3}{dt}$$

His positis nec non substituta

$$y = y_{,} + (n)y_{,,}t$$

ubi secundum art. 7. Sect. II. $y_{,}$ et $y_{,,}$ non minus quam (n) verae constantes sunt, expressio pro $\frac{d^2\zeta}{d\tau dt}$ in art. 5. evoluta suppeditat usque ad quantitates quarti ordinis accuratam expressionem hanc

$$\begin{aligned}\frac{d\zeta}{d\tau}=1+W+\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)+\tfrac{1}{4}\left(\frac{d\delta\zeta}{d\tau}\right)^2-\tfrac{1}{2}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{4}(S+\varepsilon)^2-\tfrac{1}{8}\left(\frac{d\delta\zeta}{d\tau}\right)^3\\ -\tfrac{3}{2}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)-\tfrac{1}{2}\delta\zeta^2\left(\frac{d^3\delta\zeta}{d\tau^3}\right)+\tfrac{3}{8}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)^2+\tfrac{1}{2}(S+\varepsilon)\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\\ -\tfrac{1}{8}(S+\varepsilon)^2\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{8}(S+\varepsilon)^3\end{aligned}$$

ubi posui

$$\begin{aligned}W=-b+A_{,}(1-b)\xi+A_{,,}\xi^2+X\\ +\frac{(n)(h)y_{,}}{\varkappa(M+m)}\left\{\begin{array}{l}(\varrho)^2\int\left\{\left(\frac{d^2\delta\zeta}{d\tau^2}\right)-\tfrac{3}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)-\delta\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right)+\tfrac{1}{2}(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)\right\}dt\\ -\tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\int\left\{\left(\frac{d\delta\zeta}{d\tau}\right)+(S+\varepsilon)-\delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right)-\tfrac{1}{4}\left(\frac{d\delta\zeta}{d\tau}\right)^2+\tfrac{1}{2}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{4}(S+\varepsilon)^2\right\}dt\end{array}\right\}\\ +\frac{(n)^2(h)y_{,,}}{\varkappa(M+m)}\left\{(\varrho)^2\int\left(\frac{d^2\delta\zeta}{d\tau^2}\right)t\,dt-\tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\int\left\{\left(\frac{d\delta\zeta}{d\tau}\right)+(S+\varepsilon)\right\}t\,dt\right\}\end{aligned}$$

Antequam integrationem subsequentem aggredimur, propositum est aequationes praecedentes ita transformare, ut in aequatione finali ζ et β non adsint. Quem in finem aequatio praecedens ipsam $\frac{d\zeta}{d\tau}$ exhibens ipsa praebet usque ad quantitates tertii ordinis

$$\frac{d\delta\zeta}{d\tau} = W + \delta\zeta\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \tfrac{1}{4}\left(\frac{d\delta\zeta}{d\tau}\right)^2 - \tfrac{1}{2}(S+\varepsilon)\left(\frac{d\delta\zeta}{d\tau}\right) + \tfrac{1}{4}(S+\varepsilon)^2$$

e qua differentiata elicitur haec

$$\frac{d^2\delta\zeta}{d\tau^2} = \frac{dW}{d\tau} + \tfrac{3}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \delta\zeta\left(\frac{d^3\delta\zeta}{d\tau^3}\right) - \tfrac{1}{2}(S+\varepsilon)\left(\frac{d^2\delta\zeta}{d\tau^2}\right)$$

atque usque ad quantitates secundi ordinis accuratae hae

$$\frac{d\delta\zeta}{d\tau} = W$$

$$\frac{d^2\delta\zeta}{d\tau^2} = \frac{dW}{d\tau}$$

$$\frac{d^3\delta\zeta}{d\tau^3} = \frac{d^2W}{d\tau^2}$$

Porro habetur

$$\frac{(h)}{\varkappa(M+m)} = \frac{1}{(a)^2(n)\sqrt{1-(e)^2}}$$

$$\frac{d\delta\zeta}{d\tau} = (S+\varepsilon) - 2\beta + \tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)^2$$

unde usque ad quartum ordinem

$$\left(\frac{d\delta\zeta}{d\tau}\right)^2 = (S+\varepsilon)^2 - 4\beta(S+\varepsilon) + 4\beta^2 + (S+\varepsilon)^3 - 6\beta(S+\varepsilon)^2 + 12\beta^2(S+\varepsilon) - 8\beta^3$$

$$\left(\frac{d\delta\zeta}{d\tau}\right)^3 = (S+\varepsilon)^3 - 6\beta(S+\varepsilon)^2 + 12\beta^2(S+\varepsilon) - 8\beta^3$$

et usque ad tertium ordinem

$$\left(\frac{d\delta\zeta}{d\tau}\right) = (S+\varepsilon) - 2\beta + \tfrac{1}{2}(S+\varepsilon)^2 - 2\beta(S+\varepsilon) + 2\beta^2$$

Quibus expressionibus adiuvantibus, expressio praecedens ipsius $\frac{d\zeta}{d\tau}$ facile transformatur in hanc

$$\frac{d\zeta}{d\tau} = 1 + W + \delta\zeta\left\{\frac{dW}{d\tau} + \tfrac{1}{2}\delta\zeta\frac{d^2W}{d\tau^2}\right\} + \beta^2\{1 - \beta + (S+\varepsilon)\} \quad \text{........(9)}$$

et expressio ipsius W in hanc

$$(10)\ldots\ldots\ W = -b + A_{,}(1+b)\xi + A_{,,}\xi^2 + X$$
$$+ \frac{y_{,}}{\sqrt{1-(e)^2}}\left\{\frac{(\varrho)^2}{(a)^2}\int\frac{dW}{d\tau}dt - \tfrac{1}{2}\frac{d.(\varrho)^2}{(a)^2 d\tau}\int\{W+(S+\varepsilon)+\tfrac{1}{2}(S+\varepsilon)^2\}dt\right\}$$
$$+ \frac{y_{,,}}{\sqrt{1-(e)^2}}\left\{\frac{(\varrho)^2}{(a)^2}\int\frac{dW}{d\tau}tdt - \tfrac{1}{2}\frac{d.(\varrho)^2}{(a)^2 d\tau}\int\{W+(S+\varepsilon)\}tdt\right\}$$

Illa, mutata τ in t, suppeditat

$$\overline{\left(\frac{d\zeta}{d\tau}\right)} = 1 + \overline{W} + \delta z\left\{\overline{\left(\frac{dW}{d\tau}\right)} + \tfrac{1}{2}\delta z\overline{\left(\frac{d^2W}{d\tau^2}\right)}\right\} + w^2\{1-w+(S+\varepsilon)\}.$$

ubi ζ et β nec non quotientes earum differentiales ubique eliminati sunt.

Ut ex hac aequatione $\frac{dz}{dt}$ derivari possit, necesse est $\frac{1}{\left(\frac{d.\varDelta z}{dz}\right)}$ derivetur. Quum secundum art. 13. Sect. II. sit

$$\varDelta z = \bar{f} + \pi$$

erit

$$\frac{1}{\left(\frac{d.\varDelta z}{dz}\right)} = \frac{\bar{r}^2}{(a)^2(n)\sqrt{1-(e)^2}}$$

et quum $\bar{r}$ sit functio solius variabilis z, habetur

$$\frac{1}{\left(\frac{d.\varDelta z}{dz}\right)} = \frac{1}{(n)\sqrt{1-(e)^2}}\left\{\frac{(r)^2}{(a)^2} + \frac{d.(r)^2}{(a)^2 dg}(n)\delta z + \tfrac{1}{2}\frac{d^2.(r)^2}{(a)^2 dg^2}(n)^2\delta z^2\right\}$$

itaque aequatio (6) transit in hanc

$$\frac{dz}{dt} = \overline{\left(\frac{d\zeta}{d\tau}\right)} - \frac{y_{,}}{\sqrt{1-(e)^2}}\left\{\frac{(r)^2}{(a)^2} + \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta z + \tfrac{1}{2}\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2}(n)^2\delta z^2\right\}$$
$$-\frac{y_{,,}(n)t}{\sqrt{1-(e)^2}}\left\{\frac{(r)^2}{(a)^2} + \frac{d.\frac{(a)^2}{(r)^2}}{dg}(n)\delta z\right\} - \frac{y_{,,,}(n)^2t^2}{\sqrt{1-(e)^2}}\frac{(r)^2}{(a)^2}$$

Substituto valore ipsius $\overline{\left(\frac{d\zeta}{d\tau}\right)}$ modo eruto in hac aequatione, nanciscimur valorem ipsius $\frac{dz}{dt}$, qui per $(n)dt$ multiplicatus et integratus suppeditat

$$(n)z = (c) + (n)t + (n)\int \left\{ \begin{array}{l} \overline{W} + (n)\delta z \left\{ \left(\overline{\frac{dW}{d\gamma}}\right) - \frac{y_{\prime}}{\sqrt{1-(e)^2}} \frac{d\,.\frac{(r)^2}{(a)^2}}{dg} - \frac{y_{\prime\prime}}{\sqrt{1-(e)^2}} \frac{d\,.\frac{(r)^2}{(a)^2}}{dg}(n)t \right\} \\ + \tfrac{1}{2}(n)^2 \delta z^2 \left\{ \left(\overline{\frac{d^2 W}{d\gamma^2}}\right) - \frac{y_{\prime}}{\sqrt{1-(e)^2}} \frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2} \right\} + w^2 \{1 - w + (S+\varepsilon)\} \end{array} \right\} dt$$
$$- \frac{y_{\prime}}{\sqrt{1-(e)^2}}(n)\int \frac{(r)^2}{(a)^2} dt - \frac{y_{\prime\prime}}{\sqrt{1-(e)^2}}(n)\int \frac{(r)^2}{(a)^2}(n)\,t dt - \frac{y_{\prime\prime\prime}}{\sqrt{1-(e)^2}} \int \frac{(r)^2}{(a)^2}(n)^2 t^2 dt \;\ldots\ldots (11)$$

ubi (c) est constans huic integrali addita, ita ut termini ipsius $(n)z$ e vi perturbante independentes sint $(c) + (n)t$ sive g. In hac formula etiam quotientes differentiales ipsius W respectu τ in quotientes differentiales respectu ipsius γ converti, id quod in calculo numerico commodissimum est.

Haec antem formula terminos omnes usque ad quartum ordinem respectu vis perturbantis continet, et in hac concinna forma perfacile in usum vocatur. Ceterum, si excipiuntur quae ad ipsam y spectant, haec formula fere est, cuius in fine art. 30. Commentationis meae de perturbationibus Iovis atque Saturni mentionem feci.

Superest ut quantitas W amplius explicetur. Si in valore huius quantitatis sub (10) dato substituitur expressio ipsius $\mathring{X}$ ex art. 5. sumenda, nec non valor ipsius (T) per $(\mathring{T})$ expressus, obtinetur

$$W = -b + A_{\prime}(1-b)\xi + A_{\prime\prime}\xi^2 + Z$$

ubi

$$Z = (n)\int \left\{ \begin{array}{l} \frac{1}{(n)}(\mathring{T}) - \frac{y_{\prime}}{\sqrt{1-(e)^2}} \frac{d\,.\frac{(\rho)^2}{(a)^2}}{d\gamma} - \frac{2}{(n)} U(S+\varepsilon) + \frac{1}{(n)} \frac{d(\mathring{T})}{dg}(n)\delta z + \frac{1}{(n)} r \frac{d(\mathring{T})}{dr} w + \frac{1}{(n)} \frac{d(\mathring{T})}{dg'}(n')\delta z' \\ + \frac{1}{(n)} r' \frac{d(\mathring{T})}{dr'} w' + \frac{1}{(n)} \frac{d(\mathring{T})}{dP}\delta P + \frac{1}{(n)} \frac{d(\mathring{T})}{dQ}\delta Q + \frac{1}{(n)} \frac{d(\mathring{T})}{dK}\delta K - \frac{y_{\prime\prime}}{\sqrt{1-(e)^2}} \frac{d\,.\frac{(\rho)^2}{(a)^2}}{d\gamma}(n)t \\ + \frac{y_{\prime}}{\sqrt{1-(c)^2}} \left\{ \frac{(\rho)^2}{(a)^2} \frac{dW}{d\gamma} - \tfrac{1}{2} \frac{d\,.\frac{(\rho)^2}{(a)^2}}{d\gamma} \left[W + (S+\varepsilon) + \tfrac{1}{2}(S+\varepsilon)^2 \right] \right\} \\ + 2\frac{1}{(n)} U(S+\varepsilon)^2 - 2\frac{1}{(n)} \frac{dU}{dg}(S+\varepsilon)(n)\delta z - 2\frac{1}{(n)} r \frac{dU}{dr}(S+\varepsilon) w + \tfrac{1}{2}\frac{1}{(n)} \frac{d^2(\mathring{T})}{dg^2}(n)^2 \delta z^2 \\ + \frac{1}{(n)} r \frac{d^2(\mathring{T})}{dg\,.\,dr}(n)\delta z\,.\,w + \frac{1}{(n)} \frac{d^2(\mathring{T})}{dg\,.\,dP}(n)\delta z\,.\,\delta P + \frac{1}{(n)} \frac{d^2(\mathring{T})}{dg\,.\,dQ}(n)\delta z\,.\,\delta Q \\ + \tfrac{1}{2}\frac{1}{(n)} \left\{ r^2 \frac{d^2(\mathring{T})}{dr^2} + r\frac{d(\mathring{T})}{dr} \right\} w^2 + \frac{1}{(n)} r \frac{d^2(\mathring{T})}{dr\,.\,dP} w\,.\,\delta P + \frac{1}{(n)} r \frac{d^2(\mathring{T})}{dr\,.\,dQ} w\,.\,\delta Q + \text{etc.} - \frac{y_{\prime\prime\prime}}{\sqrt{1-(e)^2}} \frac{d\,.\frac{(\rho)^2}{(a)^2}}{d\gamma}(n)^2 t^2 \\ + \frac{y_{\prime\prime}}{\sqrt{1-(e)^2}} \left\{ \frac{(\rho)^2}{(a)^2} \frac{dW}{d\gamma}(n)t - \tfrac{1}{2} \frac{d\,.\frac{(\rho)^2}{(a)^2}}{d\gamma} \left[W + (S+\varepsilon) \right](n)t \right\} \end{array} \right\} dt$$

Post evolutionem in seriem termini omnes ipsius Z qui e τ pendent insigni gaudent proprietate, magnum calculi compendium afferente; quae quidem proprietas infra explicabitur.

8.

Revertamur ad perturbationes logarithmi radii vectoris, adiumento formulae (12) in art. 11. Sect. II. datae computandas. Quum sit

$$\Xi z = l\bar{r}$$

erit

$$\Xi_{,}z \text{ sive } \frac{d.\Xi z}{dz} = \frac{(a)(e)\sin\bar{f}}{\bar{r}\sqrt{1-(e)^2}}$$

atque

$$\frac{\Xi_{,}z}{\Delta_{,}z} = \frac{(e)\bar{r}\sin\bar{f}}{(a)(1-(e)^2)} = \tfrac{1}{2}\,\frac{\frac{d.\bar{r}^2}{dz}}{(a)^2(n)\sqrt{1-(e)^2}}$$

Quum porro $fz = \varepsilon$, unde $f_{,}z = o$, atque

$$\frac{1}{\left(\frac{d\zeta}{d\tau}\right)} = 1 - \left(\frac{d\delta\zeta}{d\tau}\right) + \left(\frac{d\delta\zeta}{d\tau}\right)^2 \mp \text{etc.}$$

formula illa transit in hanc

$$\text{(12)......} \qquad w = C + \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\int \frac{\frac{d.\bar{r}^2}{dz}}{(a)^2\sqrt{1-(e)^2}}\, y\, dt - \tfrac{1}{2}\int \bar{Y}\, dt$$

ubi C constans est et

$$Y = \left(\frac{d^2\zeta}{d\tau^2}\right) - \left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\delta\zeta}{d\tau^2}\right) + \left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\delta\zeta}{d\tau^2}\right)$$

quae formula usque ad quantitates quarti ordinis accurata est.

9.

Ad quantitatem Y evolvendam differentietur aequatio (9) art. 7. respectu ipsius τ, unde emergit

$$\frac{d^2\zeta}{d\tau^2} = \frac{dW}{d\tau} + \left(\frac{d\delta\zeta}{d\tau}\right)\frac{dW}{d\tau} + \delta\zeta\frac{d^2W}{d\tau^2} + \delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\frac{d^2W}{d\tau^2}$$
$$+ \tfrac{1}{2}\delta\zeta^2\frac{d^3W}{d\tau^3} + 2\beta\frac{d\beta}{d\tau} - 3\beta^2\frac{d\beta}{d\tau} + 2\,(S+\varepsilon)\beta\frac{d\beta}{d\tau}$$

hinc, reiectis terminis quarti ordinis et ordinum altiorum, nanciscimur producta haec

$$\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau^2}\right)=\left(\frac{d\delta\zeta}{d\tau}\right)\frac{dW}{d\tau}+\left(\frac{d\delta\zeta}{d\tau}\right)^2\frac{dW}{d\tau}+\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)\frac{d^2W}{d\tau^2}+2\left(\frac{d\delta\zeta}{d\tau}\right)\beta\frac{d\beta}{d\tau}$$

$$\left(\frac{d\delta\zeta}{d\tau}\right)^2\left(\frac{d^2\zeta}{d\tau^2}\right)=\left(\frac{d\delta\zeta}{d\tau}\right)^2\frac{dW}{d\tau}$$

Adiumento harum expressionum formula art. praec. ipsam Y exhibens transit in hanc

$$Y=\frac{dW}{d\tau}+\delta\zeta\frac{d^2W}{d\tau^2}+\tfrac{1}{2}\delta\zeta^2\frac{d^3W}{d\tau^3}+2\beta\frac{d\beta}{d\tau}-3\beta^2\frac{d\beta}{d\tau}+2(S+\varepsilon)\beta\frac{d\beta}{d\tau}-2\left(\frac{d\delta\zeta}{d\tau}\right)\beta\frac{d\beta}{d\tau}$$

Sed aequatio haec

$$\beta=\tfrac{1}{2}(S+\varepsilon)-\tfrac{1}{2}l\left(\frac{d\zeta}{d\tau}\right)$$

suppeditat usque ad terminos tertii ordinis

$$\frac{d\beta}{d\tau}=-\tfrac{1}{2}\left(\frac{d^2\zeta}{d\tau^2}\right)+\tfrac{1}{2}\left(\frac{d\delta\zeta}{d\tau}\right)\left(\frac{d^2\zeta}{d\tau^2}\right)$$

quae, substituto valore praecedenti ipsius $\frac{d^2\zeta}{d\tau^2}$, neglectisque terminis tertii ordinis, subministrat

$$\frac{d\beta}{d\tau}=-\tfrac{1}{2}\frac{dW}{d\tau}-\tfrac{1}{2}\delta\zeta\frac{d^2W}{d\tau^2}+\tfrac{1}{2}\beta\frac{dW}{d\tau}$$

habemus porro usque ad terminos secundi ordinis

$$\left(\frac{d\delta\zeta}{d\tau}\right)=(S+\varepsilon)-2\beta$$

Eliminatis adminiculo harum aequationum quantitatibus $\frac{d\beta}{d\tau}$ et $\left(\frac{d\delta\zeta}{d\tau}\right)$ ex praecedenti expressione ipsius Y, emergit

$$Y=\frac{dW}{d\tau}+\delta\zeta\left[\frac{d^2W}{d\tau^2}-\beta\frac{d^2W}{d\tau^2}+\tfrac{1}{2}\delta\zeta\frac{d^3W}{d\tau^3}\right]-\beta[1-\tfrac{1}{2}\beta]\frac{dW}{d\tau}$$

Denique $\frac{d.\bar{r}^2}{dz}$ evoluta evadit

$$\frac{d.\bar{r}^2}{dz}=(n)\frac{d.(r)^2}{dg}+(n)\frac{d^2.(r)^2}{dg^2}(n)\delta z+\tfrac{1}{2}(n)\frac{d^3(r)^2}{dg^3}(n)^2\delta z^2$$

unde

$$\frac{\frac{d.\bar{r}^2}{dz}}{(a)^2\sqrt{1-(e)^2}}\,y = \frac{y_{,}}{\sqrt{1-(e)^2}}(n)\left\{\frac{d.\frac{(r)^2}{(a)^2}}{dg}+\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2}(n)\delta z+\frac{1}{2}\,\frac{d^3.\frac{(r)^2}{(a)^2}}{dg^3}(n)^2\delta z^2\right\}$$

$$+\frac{y_{,,}}{\sqrt{1-(e)^2}}(n)\left\{\frac{d.\frac{(r)^2}{(a)^2}}{dg}+\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2}(n)\,\delta z\right\}(n)t$$

$$+\frac{y_{,,,}}{\sqrt{1-(e)^2}}(n)\frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)^2t^2$$

Substitutis his expressionibus in aequatione (12) art. 8., elicitur

$$(13)\; w = C+\tfrac{1}{2}\varepsilon-\tfrac{1}{2}(n)\int\left\{\begin{array}{l}\left(\overline{\frac{dW}{d\gamma}}\right)+(n)\delta z\left\{\left(\overline{\frac{d^2W}{d\gamma^2}}\right)-w\left(\overline{\frac{d^2W}{d\gamma^2}}\right)-\frac{y_{,}}{\sqrt{1-(e)^2}}\,\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2}-\frac{y_{,,}}{\sqrt{1-(e)^2}}\,\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2}(n)t\right\}\\ +\tfrac{1}{2}(n)^2\delta z^2\left\{\left(\overline{\frac{d^3W}{d\gamma^3}}\right)-\frac{y_{,}}{\sqrt{1-(e)^2}}\,\frac{d^3.\frac{(r)^2}{(a)^2}}{dg^3}\right\}-w\left[1-\tfrac{1}{2}w\right]\left(\overline{\frac{dW}{d\gamma}}\right)\end{array}\right\}dt$$

$$+\tfrac{1}{2}\frac{y_{,}}{\sqrt{1-(e)^2}}(n)\int\frac{d.\frac{(r)^2}{(a)^2}}{dg}dt+\tfrac{1}{2}\frac{y_{,,}}{\sqrt{1-(e)^2}}(n)\int\frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)t\,dt+\tfrac{1}{2}\frac{y_{,,,}}{\sqrt{1-(e)^2}}(n)\int\frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)^2t^2dt$$

quae formula terminos omnes usque ad quartum ordinem continet. Termini tamen tertii ordinis in logarithmo sinus parallaxeos horizontalis Lunae nullam fere vim habent, et solummodo integritatis caussa et ut monstrarem quomodo hi termini se haberent, terminos tertii ordinis qui expressioni logarithmi radii vectoris insunt omnes evolvi.

10.

Restat nobis ut constans C definiatur, quae secundum art. 11. Sect. II. adiumento aequationis huius

$$w = \tfrac{1}{2}(S+\varepsilon)-\tfrac{1}{2}\,l\left(\overline{\frac{d\zeta}{d\tau}}\right)$$

determinanda est. Termini sub signo integrationis in expressione ipsius w in art. praec. data existentes post evolutionem in seriem terminos constantes continere nequeunt; quodsi igitur ad terminos constantes tantum respicimus, aequatio haec suppeditat

$$w = C+\tfrac{1}{2}\varepsilon$$

Quantitas quoque S constantem terminum non continet, quare aequatio praecedens sub eadem restrictione praebet

$$w = \tfrac{1}{2}\varepsilon - \tfrac{1}{2}\, l\, \overline{\left(\frac{d\zeta}{d\tau}\right)}$$

His duobus valoribus termini constantis comparatis, emergit

$$C = \text{term. const. in} \left\{ -\tfrac{1}{2}\, l\, \overline{\left(\frac{d\zeta}{dt}\right)} \right\}$$

quae ope aequationis huius

$$\overline{\left(\frac{d\zeta}{d\tau}\right)} = \frac{dz}{dt} + \frac{y}{\sqrt{1-(e)^2}} \left\{ \frac{(r)^2}{(a)^2} + \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta z \right\}$$

ubi terminos tertii ordinis, qui nullam vim habent, omisi, abit in hanc

$$C = \text{term. const. in} \left\{ -\tfrac{1}{2}\, l \left[\frac{dz}{dt} + \frac{y_{\prime}}{\sqrt{1-(e)^2}} \frac{(r)^2}{(a)^2} + \frac{y_{\prime}}{\sqrt{1-(e)^2}} . \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta z \right] \right\}$$

Posito vero

$$\frac{dz}{dt} = 1 + \delta \frac{dz}{dt}$$

habetur usque ad quantitates tertii ordinis

$$l\left[1 + \delta\frac{dz}{dt} + \frac{y_{\prime}}{\sqrt{1-(e)^2}}\frac{(r)^2}{(a)^2} + \frac{y_{\prime}}{\sqrt{1-(e)^2}} . \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta z\right] =$$

$$\delta\frac{dz}{dt} + \frac{y_{\prime}}{\sqrt{1-(e)^2}}\frac{(r)^2}{(a)^2} - \tfrac{1}{2}\left(\delta\frac{dz}{dt}\right)^2 - \frac{y_{\prime}}{\sqrt{1-(e)^2}}\left[\frac{(r)^2}{(a)^2}\delta\frac{dz}{dt} - \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta z\right] - \frac{\tfrac{1}{2}y_{\prime}^2}{1-(e)^2}\frac{(r)^4}{(a)^4}$$

itaque quum $\delta\,\frac{dz}{dt}$ terminum constantem non habeat, et terminus constans in $\frac{(r)^2}{(a)^2}\delta\,\frac{dz}{dt} - \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta z$, quia per $(e)^4$ multiplicatus est, minutissimus sit, elicitur

$$C = \text{term. const. in} \left\{ -\tfrac{1}{2}\,\frac{y_{\prime}}{\sqrt{1-(e)^2}}\frac{(r)^2}{(a)^2} + \tfrac{1}{4}\left(\delta\frac{dz}{dt}\right)^2 + \tfrac{1}{4}\,\frac{y_{\prime}^2}{1-(e)^2}\frac{(r)^4}{(a)^4} \right\}$$

qua aequatione constans haec perfacile computari potest.

11.

Formulae in praecedentibus evolutae monstrant ad $(n)z$ et w usque ad terminos quarti ordinis computandas, terminos et primi et secundi ordinis ipsius $S+\varepsilon$ requiri. Qui quidem termini facili opera adiumento huius aequationis

$$S+\varepsilon = 2w + l\,\overline{\left(\frac{d\zeta}{dt}\right)}$$

computantur. Substituta enim evoluta quantitate $l\,\overline{\left(\frac{d\zeta}{d\tau}\right)}$ in art. praec. data, elicitur formula

$$(14)\ldots\ldots\quad S+\varepsilon = 2w + \delta\frac{dz}{dt} + \frac{y_{,}}{\sqrt{1-(e)^2}}\cdot\frac{(r)^2}{(a)^2} - \tfrac{1}{2}\left(\delta\frac{dz}{dt}\right)^2 - \frac{y_{,}}{\sqrt{1-(e)^2}}\left[\frac{(r)^2}{(a)^2}\delta\frac{dz}{dt} - \frac{d.(r)^2}{(a)^2 dg}(n)\delta z\right]$$
$$+ \frac{y_{,,}(n)t}{\sqrt{1-(e)^2}}\cdot\frac{(r)^2}{(a)^2} - \frac{1}{2}\,\frac{y_{,}^2}{1-(e)^2}\cdot\frac{(r)^4}{(a)^4}$$

usque ad terminos tertii ordinis accurata.

Si quantitas S non modo per hanc formulam sed etiam calculo directo computatur, computatio perturbationum numerica confirmari potest, et calculi errores, si forte adsint, detegi et corrigi possunt. Directo vero modo ope theorematis Tayloriani invenitur usque ad quantitates tertii ordinis

$$(15)\ldots\ldots\quad S+\varepsilon = \varepsilon + \int\frac{d(S)}{dt}dt - \int\frac{d(S)}{dt}(S+\varepsilon)dt + \int\frac{d^2(S)}{dt.dg}(n)\delta z.dt + \int r\frac{d^2(S)}{dt.dr}w.dt$$
$$+\int\frac{d^2(S)}{dt.dg'}(n')\delta z'.dt + \int r'\frac{d^2(S)}{dt.dr'}w'.dt + \int\frac{d^2(S)}{dt.dP}\delta P.dt + \int\frac{d^2(S)}{dt.dQ}\delta Q.dt$$
$$+\int\frac{d^2(S)}{dt.dK}\delta K.dt$$

cui formulae praeter ε constans non est addenda.

12.

In quantitatibus P, Q et K termini expliciti tertii ordinis nullam vim habent, quamobrem ad hos non respiciam. Itaque, habitis aequationibus (49) Sect. II., quae P, Q et K determinant, pro functionibus ipsarum lh, $(n)z$, lr, $(n')z'$, lr', P, Q et K, erit

$$\frac{dP}{dt}=\frac{d(P)}{dt}-h\frac{d^2(P)}{dt\,.\,dh}(S+\varepsilon)+\frac{d^2(P)}{dt\,.\,dg}(n)\delta z+r\frac{d^2(P)}{dt\,.\,dr}w+\frac{d^2(P)}{dt\,.\,dg'}(n')\delta z' \quad ......(16)$$
$$+r'\frac{d^2(P)}{dt\,.\,dr'}w'+\frac{d^2(P)}{dt\,.\,dP}\delta P+\frac{d^2(P)}{dt\,.\,dQ}\delta Q+\frac{d^2(P)}{dt\,.\,dK}\delta K$$

ubi in quotientibus differentialibus ad dextram valores elementorum constantes in praecedentibus definiti ubique substituendi sunt; et similes aequationes nanciscimur pro $\frac{dQ}{dt}$ et $\frac{dK}{dt}$. Quum in iis aequationum (49) Sect. II. terminis, qui per dp' et dq' multiplicati sunt, perturbationes ipsarum P, Q et K negligere liceat, prima harum aequationum transit in hanc

$$\frac{dP}{dt}=-(n)\alpha Q-\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ}\right)\cos^2\tfrac{1}{2}I+\tfrac{1}{4}P\left(\frac{d\Omega}{dK}\right)\right\} \quad ......(17)$$
$$+\frac{dp'}{\cos i'dt}\cos\tfrac{1}{2}(I)\cos[\pi'-\nu+k-(n)(\alpha+\eta)t]-\frac{dq'}{\cos i'dt}\cos\tfrac{1}{2}(I)\sin[\pi'-\nu+k-(n)(\alpha+\eta)t]$$

et eodem modo reliquae duae aequationes (49) abbreviantur. Quum secundum art. 27. Sect. II. valores constantes ipsarum I, N, P, Q et K substituendi sint hi

$$\begin{aligned} I &= (I)\\ N &= \nu\\ P &= o\\ Q &= 2\sin\tfrac{1}{2}(I)\end{aligned}$$

quumque sit

$$\cos^2\tfrac{1}{2}I=1-\tfrac{1}{4}P^2-\tfrac{1}{4}Q^2$$

sequitur ponendas esse

$$\begin{aligned} \frac{d.\cos^2\frac{1}{2}I}{dP} &= o\\ \frac{d.\cos^2\frac{1}{2}I}{dQ} &= -\sin\tfrac{1}{2}(I)\\ \left(\frac{d\Omega}{dP}\right) &= \left(\frac{d\Omega}{dN}\right)\frac{1}{2\sin\frac{1}{2}(I)}\\ \left(\frac{d\Omega}{dQ}\right) &= \left(\frac{d\Omega}{dI}\right)\frac{1}{\cos\frac{1}{2}(I)}\end{aligned}$$

Habitis vero valoribus generalibus ipsarum $\left(\frac{d\Omega}{dP}\right)$ atque $\left(\frac{d\Omega}{dQ}\right)$ et pro functionibus ipsarum P atque Q, et pro functionibus ipsarum I atque N, invenitur

$$d\left(\frac{d\Omega}{dP}\right)=\left(\frac{d^2\Omega}{dP^2}\right)dP+\left(\frac{d^2\Omega}{dP.dQ}\right)dQ=\left(\frac{d.\left(\frac{d\Omega}{dP}\right)}{dI}\right)dI-\left(\frac{d.\left(\frac{d\Omega}{dP}\right)}{dN}\right)dN$$

$$d\left(\frac{d\Omega}{dQ}\right)=\left(\frac{d^2\Omega}{dP.dQ}\right)dP+\left(\frac{d^2\Omega}{dQ^2}\right)dQ=\left(\frac{d.\left(\frac{d\Omega}{dQ}\right)}{dI}\right)dI+\left(\frac{d.\left(\frac{d\Omega}{dQ}\right)}{dN}\right)dN$$

quae adiumento aequationum

$$dI = dP\,\frac{\sin(N-v)}{\cos\frac{1}{2}I}+dQ\,\frac{\cos(N-v)}{\cos\frac{1}{2}I}$$

$$dN = dP\,\frac{\cos(N-v)}{2\sin\frac{1}{2}I}-dQ\,\frac{\sin(N-v)}{2\sin\frac{1}{2}I}$$

et positis $N=v$ et $I=(I)$, suppeditant

$$\left(\frac{d^2\Omega}{dP^2}\right)=\left(\frac{d\left(\frac{d\Omega}{dP}\right)}{dN}\right)\frac{1}{2\sin\frac{1}{2}(I)}$$

$$\left(\frac{d^2\Omega}{dP.dQ}\right)=\left(\frac{d.\left(\frac{d\Omega}{dP}\right)}{dI}\right)\frac{1}{\cos\frac{1}{2}(I)}=\left(\frac{d.\left(\frac{d\Omega}{dQ}\right)}{dN}\right)\frac{1}{2\sin\frac{1}{2}(I)}$$

$$\left(\frac{d^2\Omega}{dQ^2}\right)=\left(\frac{d\left(\frac{d\Omega}{dQ}\right)}{dI}\right)\frac{1}{\cos\frac{1}{2}(I)}$$

Ex aequationibus vero differentiatis his

$$\left(\frac{d\Omega}{dP}\right)=\left(\frac{d\Omega}{dI}\right)\frac{\sin(N-v)}{\cos\frac{1}{2}I}+\left(\frac{d\Omega}{dN}\right)\frac{\cos(N-v)}{2\sin\frac{1}{2}I}$$

$$\left(\frac{d\Omega}{dQ}\right)=\left(\frac{d\Omega}{dI}\right)\frac{\cos(N-v)}{\cos\frac{1}{2}I}-\left(\frac{d\Omega}{dN}\right)\frac{\sin(N-v)}{2\sin\frac{1}{2}I}$$

elicitur, positis post differentiationes $N=v$ et $I=(I)$,

$$\left(\frac{d\left(\frac{d\Omega}{dP}\right)}{dI}\right)=\left(\frac{d^2\Omega}{dI.dN}\right)\frac{1}{2\sin\frac{1}{2}(I)}-\left(\frac{d\Omega}{dN}\right)\frac{\cos\frac{1}{2}(I)}{4\sin^2\frac{1}{2}(I)}$$

$$\left(\frac{d\left(\frac{d\Omega}{dP}\right)}{dN}\right)=\left(\frac{d^2\Omega}{dN^2}\right)\frac{1}{2\sin\frac{1}{2}(I)}+\left(\frac{d\Omega}{dI}\right)\frac{1}{\cos\frac{1}{2}(I)}$$

$$\left(\frac{d\left(\frac{d\Omega}{dQ}\right)}{dI}\right)=\left(\frac{d^2\Omega}{dI^2}\right)\frac{1}{\cos\frac{1}{2}(I)}+\left(\frac{d\Omega}{dI}\right)\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}$$

$$\left(\frac{d\left(\frac{d\Omega}{dQ}\right)}{dN}\right)=\left(\frac{d^2\Omega}{dI.dN}\right)\frac{1}{\cos\frac{1}{2}(I)}-\left(\frac{d\Omega}{dN}\right)\frac{1}{2\sin\frac{1}{2}(I)}$$

itaque

$$\left(\frac{d^2\Omega}{dP^2}\right)=\left(\frac{d^2\Omega}{dN^2}\right)\frac{1}{4\sin^2\frac{1}{2}(I)}+\left(\frac{d\Omega}{dI}\right)\frac{1}{2\sin\frac{1}{2}(I)\cos\frac{1}{2}(I)}$$

$$\left(\frac{d^2\Omega}{dP.dQ}\right)=\left(\frac{d^2\Omega}{dI.dN}\right)\frac{1}{2\sin\frac{1}{2}(I)\cos\frac{1}{2}(I)}-\left(\frac{d\Omega}{dN}\right)\frac{1}{4\sin^2\frac{1}{2}(I)}$$

$$\left(\frac{d^2\Omega}{dQ^2}\right)=\left(\frac{d^2\Omega}{dI^2}\right)\frac{1}{\cos^2\frac{1}{2}(I)}+\left(\frac{d\Omega}{dI}\right)\frac{\sin\frac{1}{2}(I)}{2\cos^3\frac{1}{2}(I)}$$

Iam posita

$$A=-\frac{(a)}{\sqrt{1-(e)^2}}\left(\frac{d\Omega}{d(I)}\right)\cos\tfrac{1}{2}(I)$$

ubi valores constantes elementorum ubique substituendi sunt, aequatio (17) differentiata praebet

$$h\frac{d^2(P)}{dt.dh}=(n)A;\qquad \frac{d^2(P)}{dt.dg}=(n)\frac{dA}{dg};\ r\frac{d^2(P)}{dt.dr}=(n)r\frac{dA}{dr};$$

$$\frac{d^2(P)}{dt.dg'}=(n)\frac{dA}{dg'};\ r'\frac{d^2(P)}{dt.dr'}=(n)r'\frac{dA}{dr'};\ \frac{d^2(P)}{dt.dK}=(n)\frac{dA}{dk}$$

Positis porro

$$D=-\frac{(a)}{\sqrt{1-(e)^2}}\left\{\left(\frac{d^2\Omega}{d(I).dv}\right)\frac{\cos\frac{1}{2}(I)}{2\sin\frac{1}{2}(I)}-\left(\frac{d\Omega}{dv}\right)\frac{\cos^2\frac{1}{2}(I)}{4\sin^2\frac{1}{2}(I)}+\tfrac{1}{4}\left(\frac{d\Omega}{dk}\right)\right\}$$

$$E=-\frac{(a)}{\sqrt{1-(e)^2}}\left\{\left(\frac{d^2\Omega}{d(I)^2}\right)-\left(\frac{d\Omega}{d(I)}\right)\frac{\sin\frac{1}{2}(I)}{2\cos\frac{1}{2}(I)}\right\}$$

ubi itidem valores constantes elementorum ubique substituendi sunt, erit

$$\frac{d^2(P)}{dt.dP}=(n)D;\ \frac{d^2(P)}{dt.dQ}=-(n)\alpha+(n)E$$

Positis insuper

$$B=\frac{(a)}{\sqrt{1-(e)^2}}\left\{\left(\frac{d\Omega}{dv}\right)\frac{\cos^2\frac{1}{2}(I)}{2\sin\frac{1}{2}(I)}-\left(\frac{d\Omega}{dk}\right)\frac{\sin\frac{1}{2}(I)}{2}\right\}$$

$$F=\frac{(a)}{\sqrt{1-(e)^2}}\left\{\left(\frac{d^2\Omega}{dv^2}\right)\frac{\cos^2\frac{1}{2}(I)}{4\sin^2\frac{1}{2}(I)}+\left(\frac{d\Omega}{d(I)}\right)\frac{\cos\frac{1}{2}(I)}{2\sin\frac{1}{2}(I)}-\tfrac{1}{4}\left(\frac{d^2\Omega}{dv.dk}\right)\right\}$$

$$G=\frac{(a)}{\sqrt{1-(e)^2}}\left\{\left(\frac{d^2\Omega}{d(I).dv}\right)\frac{\cos\frac{1}{2}(I)}{2\sin\frac{1}{2}(I)}-\left(\frac{d\Omega}{dv}\right)\frac{1+\sin^2\frac{1}{2}(I)}{4\sin^2\frac{1}{2}(I)}-\left(\frac{d^2\Omega}{d(I).dk}\right)\frac{\sin\frac{1}{2}(I)}{2\cos\frac{1}{2}(I)}-\tfrac{1}{4}\left(\frac{d\Omega}{dk}\right)\right\}$$

ubi quoque valores constantes elementorum ubique substituendi sunt, aequationes ipsi (17) analogae pro $\frac{dQ}{dt}$ et $\frac{dK}{dt}$ suppeditant

$$h\frac{d^2(Q)}{dt.dh}=(n)B;\ \frac{d^2(Q)}{dt.dg}=(n)\frac{dB}{dg};\ r\frac{d^2(Q)}{dt.dr}=(n)r\frac{dB}{dr};\ \frac{d^2(Q)}{dt.dg'}=(n)\frac{dB}{dg'};$$

$$r'\frac{d^2(Q)}{dt.dr'}=(n)r'\frac{dB}{dr'};\ \frac{d^2(Q)}{dt.dP}=(n)\alpha+(n)F;\ \frac{d^2(Q)}{dt.dQ}=(n)G;\ \frac{d^2(Q)}{dt.dK}=(n)\frac{dB}{dk};$$

$$h\frac{d^2(K)}{dt.dh}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)A;\ \frac{d^2(K)}{dt.dg}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)\frac{dA}{dg};$$

$$r\frac{d^2(K)}{dt.dr}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)r\frac{dA}{dr};\ \frac{d^2(K)}{dt.dg'}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)\frac{dA}{dg'};$$

$$r'\frac{d^2(K)}{dt.dr'}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)r'\frac{dA}{dr'};\ \frac{d^2(K)}{dt.dP}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)D+\frac{1}{4\cos^2\frac{1}{2}(I)}(n)B;$$

$$\frac{d^2(K)}{dt.dQ}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)E-\frac{1+\sin^2\frac{1}{2}(I)}{4\cos^4\frac{1}{2}(I)}(n)A;\ \frac{d^2(K)}{dt.dK}=-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)\frac{dA}{dk}$$

Substitutis his valoribus nec non valoribus ipsarum α et η his

$$\alpha=\alpha_{,}+(n)\alpha_{,,}t$$
$$\eta=\eta_{,}+(n)\eta_{,,}t$$

in aequatione (16) et in similibus pro $\frac{dQ}{dt}$ et $\frac{dK}{dt}$, inveniuntur post exactas integrationes hae

$$P=-2\sin\tfrac{1}{2}(I)\,\alpha_{,}(n)t-\sin\tfrac{1}{2}(I)\,\alpha_{,,}(n)^2t^2+(n)\int A\,dt$$
$$+(n)\int\left(\frac{dp'}{(n)\cos i'dt}\right)\cos\tfrac{1}{2}(I)\cos[\pi'-v+k-(n)(\alpha+\eta)t]dt-(n)\int\left(\frac{dq'}{(n)\cos i'dt}\right)\cos\tfrac{1}{2}(I)\sin[\pi'-v+k-(n)(\alpha+\eta)t]dt$$
$$+(n)\int\left\{\begin{matrix}-A(S+\varepsilon)+\frac{dA}{dg}(n)\delta z+r\frac{dA}{dr}w+\frac{dA}{dg'}(n')\delta z'+r'\frac{dA}{dr'}w'\\+D\delta P-\alpha_{,}\delta Q+E\delta Q+\frac{dA}{dk}\delta K\end{matrix}\right\}dt$$

$$Q=2\sin\tfrac{1}{2}(I)+(n)\int B\,dt$$
$$-(n)\int\left(\frac{dp'}{(n)\cos i'dt}\right)\cos\tfrac{1}{2}(I)\sin[\pi'-v+k-(n)(\alpha+\eta)t]dt-(n)\int\left(\frac{dq'}{(n)\cos i'dt}\right)\cos\tfrac{1}{2}(I)\cos[\pi'-v+k-(n)(\alpha+\eta)t]dt$$
$$+(n)\int\left\{\begin{matrix}-B(S+\varepsilon)+\frac{dB}{dg}(n)\delta z+r\frac{dB}{dr}w+\frac{dB}{dg'}(n')\delta z'+r'\frac{dB}{dr'}w'\\+\alpha_{,}\delta P+F\delta P+G\delta Q+\frac{dB}{dk}\delta K\end{matrix}\right\}dt$$

$$K=k+\eta_{,}(n)t+\tfrac{1}{2}\eta_{,,}(n)^2t^2-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)\int A\,dt$$
$$+(n)\int\left(\frac{dp'}{(n)\cos i'dt}\right)\frac{tg\frac{1}{2}(I)}{2}\cos[\pi'-v+k-(n)(\alpha+\eta)t]dt-(n)\int\left(\frac{dq'}{(n)\cos i'dt}\right)\frac{tg\frac{1}{2}(I)}{2}\sin[\pi'-v+k-(n)(\alpha+\eta)t]dt$$
$$-\frac{\sin\frac{1}{2}(I)}{2\cos^2\frac{1}{2}(I)}(n)\int\left\{\begin{matrix}-A(S+\varepsilon)+\frac{dA}{dg}(n)\delta z+r\frac{dA}{dr}w+\frac{dA}{dg'}(n')\delta z'+r'\frac{dA}{dr'}w'\\+D\delta P+E\delta Q+\frac{dA}{dk}\delta K\end{matrix}\right\}dt$$
$$+(n)\int\left\{\frac{1}{4\cos^2\frac{1}{2}(I)}B\delta P-\frac{1+\sin^2\frac{1}{2}(I)}{4\cos^4\frac{1}{2}(I)}A\delta Q\right\}dt$$

ubi $\alpha_{,}$ et $\alpha_{,,}$ ita determinandae sunt ut in P, et $\eta_{,}$ atque $\eta_{,,}$ ita ut in K termini resp. tempori ipsi et temporis quadrato proportionales evanescant. Formula praecedens valorem ipsius K exhibens monstrat, maximam valoris huius quantitatis partem facillima opera ex valore ipsius P derivari posse.

Functiones quoque ad $p_{,}$ et $q_{,}$ computandas requisitae, in art. 30. Sect. II. definitae et ft atque φt nominatae adiumento quantitatum, quae in aequationibus praecedentibus continentur, evolvi possunt. Comparatis enim aequationibus (57) Sect. II. cum (49) Sect. II., facili computandi ratione invenitur

$$ft = (n)\sec\tfrac{1}{2}(I)\left\{A - A(S+\varepsilon) + \frac{dA}{dg}(n)\delta z + r\frac{dA}{dr}w + \frac{dA}{dg'}(n')\delta z' + r'\frac{dA}{dr'}w' + D\delta P + E\delta Q + \frac{dA}{dk}dK\right\}$$
$$+ (n)\frac{\sin\frac{1}{2}(I)}{2\cos^3\frac{1}{2}(I)}A\delta Q + (n)\sec\tfrac{1}{2}(I)\,B\delta K$$

$$\varphi t = -(n)\sec\tfrac{1}{2}(I)\left\{B - B(S+\varepsilon) + \frac{dB}{dg}(n)\delta z + r\frac{dB}{dr}w + \frac{dB}{dg'}(n')\delta z' + r'\frac{dB}{dr'}w' + F\delta P + G\delta Q + \frac{dB}{dk}\delta K\right\}$$
$$- (n)\frac{\sin\frac{1}{2}(I)}{2\cos^3\frac{1}{2}(I)}B\delta Q + (n)\sec\tfrac{1}{2}(I)\,A\delta K$$

unde evidens est, maximam ipsarum ft et φt partem ex computatione ipsarum P et Q iam notam esse.

SECTIO IV.

EXPOSITIO CALCVLI, QVO APPROXIMATIO PRIMA AD VALORES VEROS PERTVRBATIONVM LVNAE OBTINENDOS INSTITVENDA ABSOLVITVR.

1.

Hucusque formulas omnes ita comparavimus, ut non modo ad motum Lunae, sed etiam ad motum planetarum investigandum adhiberi possint. Factis enim $y = \alpha = \eta = o$, formulae prodeunt eaedem, quas ad planetarum perturbationes computandas alioquin dedi. Evolutio quidem generalis in Sectione praecedenti instituta ab evolutione generali in Theoria mea Iovis atque Saturni data differt, sed non est dubium, quin haec illi in theoria quoque planetarum praeferenda sit, quoniam terminorum per $t - \tau$ multiplicatorum, quorum numerus non est parvus, eliminatio calculi compendium non contemnendum adducit. Quantitates porro auxiliares in hac Lunae theoria introductae cum quantitatibus auxiliaribus, quibus in theoria planetarum usus sum, plane conveniunt; quantitates enim in Astr. Nachr. No. 244. per P, Q et $\varphi_{,} - \varphi$ denotatae, postquam arcus ibidem Θ designatus, ut praescripsi, subtractus est, et quantitates hoc loco P, Q et K denotatae, factis $y = \alpha = \eta = o$, eaedem sunt; porro quantitates ibidem p et q nominatae sub iisdem conditionibus cum quantitatibus hoc

loco $p_{,}$ et $q_{,}$ denotatis identicae sunt, denique quantitates in Astr. Nachr. No. 295. l atque l' denotatae cum quantitatibus hoc loco $p_{,,}$ atque $q_{,,}$ denotatis congruunt.

Sicut in praecedentibus formulas tali modo exhibuimus, ut, factis $y = \alpha = \eta = o$, ad motum planetarum investigandum statim adhiberi possint, ita in subsequentibus evolutiones speciales instituendas perficere nobis liceret, quia quantitatis Ω evolutio, qua potissimum opus erit, in Lunae theoria eodem modo atque in planetarum theoria peragi posset; sed quum distantia Lunae a Terra quadringenties fere distantia Solis a Terra minor sit, maximam utilitatem commodumque haud spernendum nobis parabimus, si in theoria Lunae quantitatem perturbatricem Ω in seriem infinitam secundum potestates ascendentes ipsius $\frac{r}{r'}$ progredientem evolvimus. Qua ratione inductus evolutionem ipsius Ω hoc modo instituam, et haec huius Lunae theoriae sola conditio est, quae ad planetarum theoriam applicari nequit.

2.

Revertamur ad ipsarum $(n)z$ et w expressiones quae in Sectionis III. artt. 7. et 9. sub (11) et (13) inventae sunt. Si in his expressionibus termini secundi atque tertii ordinis omittuntur, habetur

$$(n)z = g + (n)\int \overline{W}\,dt - \frac{y_{,}}{\sqrt{1-(e)^2}}(n)\int \frac{(r)^2}{(a)^2}\,dt$$

$$w = C + \tfrac{1}{2}\varepsilon - \tfrac{1}{2}(n)\int \left(\frac{d\overline{W}}{d\gamma}\right) dt + \tfrac{1}{2}\frac{y_{,}}{\sqrt{1-(e)^2}}(n)\int \frac{d.\frac{(r)^2}{(a)^2}}{dg}\,dt$$

ubi secundum art. 7. Sect. III.

$$W = -b + A_{,}(1+b)\xi + A_{,,}\xi^2 + \int (\mathring{T})\,dt - \frac{y_{,}}{\sqrt{1-(e)^2}}(n)\int \frac{d.\frac{(\varrho)^2}{(a)^2}}{d\gamma}\,dt$$

secundum art. 10. Sect. III.

$$C = \text{term. const. in}\left\{-\tfrac{1}{2}\frac{y_{,}}{\sqrt{1-(e)^2}}\frac{(r)^2}{(a)^2}\right\}$$

et secundum art. 17. Sect. II.

$$\tfrac{1}{2}\varepsilon = \tfrac{1}{6}b + \tfrac{1}{12}b^2 - \tfrac{1}{2}(e)\xi - \tfrac{1}{4}[1+(e)^2]\xi^2$$

et ubi constantes omnes integralibus addendae iam appositae sunt. Termini per ξ^2 multiplicati proprie quidem ad quantitates secundi ordinis referendi sunt, sed quum nihil obstet, quo minus eorum ratio statim habeatur, eos in hac approximatione prima omittere nolui.

Formulae praecedentes monstrant, ad perturbationes primi ordinis obtinendas ipsam $(\mathring{T})$, et quum $(\mathring{T})$ sit functio ipsius Ω, ante omnia hanc quantitatem in seriem infinitam evolvendam esse. Quem in finem animadverto ex praecedentibus haberi

$$\begin{aligned} v_{,} &= \bar{f} + (n)yt + \pi \\ lr &= l\bar{r} + w \\ v_{,}' &= \bar{f}' + (n)y't + \pi' \\ lr' &= l\bar{r}' + w' \end{aligned}$$

quae in approximatione prima abeant in has

$$\begin{aligned} v_{,} &= (f) + (n)yt + \pi \\ r &= (r) \\ v_{,}' &= (f') + (n)y't + \pi' \\ r' &= (r') \end{aligned}$$

ubi (f) et (r) ope anomaliae mediae $g = (n)t + (c)$ et elementorum (a) atque (e), (f') vero et (r') ope anomaliae mediae $g' = (n')t + (c')$ et elementorum (a') atque (e') computandae sunt. Quibus praemissis, Δ^2, sicut in Sect. II. art. 25. data est, posita brevitatis causa

$$\begin{aligned} H = {} & \cos^2 \tfrac{1}{2}(I) \cos [(f) - (f') + (n)(y - y' - 2\eta)t + 2k] \\ & + \sin^2 \tfrac{1}{2}(I) \cos [(f) + (f') + (n)(y + y' + 2\alpha)t + 2v] \end{aligned}$$

ubi (I), k, v resp. loco I, K, N substitutae sunt, in approximatione prima abit in

$$\Delta^2 = (r)^2 + (r')^2 - 2(r)(r')H$$

unde nanciscimur

$$\Omega = \frac{m'}{M+m}\left\{\frac{1}{[(r)^2+(r')^2-2(r)(r')H]^{\frac{1}{2}}} - \frac{(r)}{(r')^2}H\right\}$$

Si expressio haec in seriem infinitam secundum potestates ascendentes ipsius $\frac{(r)}{(r')}$ progredientem evoluta fuerit, obtinemus

$$\Omega = \frac{m'}{M+m}\left\{\frac{1}{(r')} + \frac{(r)^2}{(r)'^3}\left[\tfrac{3}{2}H - \tfrac{1}{2}\right] + \frac{(r)^3}{(r)'^4}\left[\tfrac{5}{2}H^3 - \tfrac{3}{2}H\right] + \frac{(r)^4}{(r)'^5}\left[\tfrac{35}{8}H^4 - \tfrac{15}{4}H^2 + \tfrac{3}{8}\right] + \text{etc.}\right\}$$

Perturbationes Lunae e termino secundo et tertio huius expressionis omnes pendent, terminus enim primus in quotientibus differentialibus ipsius Ω requisitis evanescit, et terminus quartus et multo magis termini reliqui ne minimam quidem vim habent, id quod calculo numerico comprobavi.

3.

Brevitatis caussa exinde loco signorum (a), (n), (e), (I), (a'), (n'), (e'), etc. simpliciter signa a, n, e, I, a', n', e', etc. adhibebo, semel animadvertens elementa illa subintelligenda esse. Quibus positis, expressio ipsius H in art. praec. data suppeditat

$$\begin{aligned}
H^2 = {} & \cos^4\tfrac{1}{2}I\cos^2[f-f'+(n)(y-y'-2\eta)t+2k] \\
& +2\cos^2\tfrac{1}{2}I\sin^2\tfrac{1}{2}I\cos[f-f'+(n)(y-y'-2\eta)t+2k]\cos[f+f'+(n)(y+y'+2\alpha)t+2\nu] \\
& +\sin^4\tfrac{1}{2}I\cos^2[f+f'+(n)(y+y'+2\alpha)t+2\nu]
\end{aligned}$$

$$\begin{aligned}
H^3 = {} & \cos^6\tfrac{1}{2}I\cos^3[f-f'+(n)(y-y'-2\eta)t+2k] \\
& +3\cos^4\tfrac{1}{2}I\sin^2\tfrac{1}{2}I\cos^2[f-f'+(n)(y-y'-2\eta)t+2k]\cos[f+f'+(n)(y+y'+2\alpha)t+2\nu] \\
& +3\cos^2\tfrac{1}{2}I\sin^4\tfrac{1}{2}I\cos[f-f'+(n)(y-y'-2\eta)t+2k]\cos^2[f+f'+(n)(y+y'+2\alpha)t+2\nu] \\
& +\sin^6\tfrac{1}{2}I\cos^3[f+f'+(n)(y+y'+2\alpha)t+2\nu]
\end{aligned}$$

unde, transmutatis potestatibus cosinuum in cosinus lineares, neglectoque termino $\sin^6\tfrac{1}{2}I$ cohibenti, qui perturbationes sensibiles proferre nequit, nanciscimur

$$\begin{aligned}
H^2 = {} & [\tfrac{1}{2}-\sin^2\tfrac{1}{2}I+\sin^4\tfrac{1}{2}I] \\
& +[\tfrac{1}{2}-\sin^2\tfrac{1}{2}I+\tfrac{1}{2}\sin^4\tfrac{1}{2}I]\cos 2[f-f'+(n)(y-y'-2\eta)t+2k] \\
& +[\sin^2\tfrac{1}{2}I-\sin^4\tfrac{1}{2}I]\cos 2[f+(n)(y+\alpha-\eta)t+\nu+k] \\
& +[\sin^2\tfrac{1}{2}I-\sin^4\tfrac{1}{2}I]\cos 2[f'+(n)(y'+\alpha+\eta)t+\nu-k] \\
& +\tfrac{1}{2}\sin^4\tfrac{1}{2}I\cos 2[f+f'+(n)(y+y'+2\alpha)t+2\nu]
\end{aligned}$$

$$
\begin{aligned}
H^3 = \; & [\tfrac{3}{4} - \tfrac{9}{4}\sin^2\tfrac{1}{2}I + \tfrac{15}{4}\sin^4\tfrac{1}{2}I]\cos[f - f' + (n)(y - y' - 2\eta)t + 2k] \\
& + [\tfrac{1}{4} - \tfrac{3}{4}\sin^2\tfrac{1}{2}I + \tfrac{3}{4}\sin^4\tfrac{1}{2}I]\cos 3[f - f' + (n)(y - y' - 2\eta)t + 2k] \\
& + [\tfrac{3}{2}\sin^2\tfrac{1}{2}I - 3\sin^4\tfrac{1}{2}I]\cos[f + f' + (n)(y + y' + 2\alpha)t + 2v] \\
& + [\tfrac{3}{4}\sin^2\tfrac{1}{2}I - \tfrac{3}{2}\sin^4\tfrac{1}{2}I]\cos[3f - f' + (n)(3y - y' + 2\alpha - 4\eta)t + 2v + 4k] \\
& + [\tfrac{3}{4}\sin^2\tfrac{1}{2}I - \tfrac{3}{2}\sin^4\tfrac{1}{2}I]\cos[f - 3f' + (n)(y - 3y' - 2\alpha - 4\eta)t - 2v + 4k] \\
& + \tfrac{3}{4}\sin^4\tfrac{1}{2}I\cos[3f + f' + (n)(3y + y' + 4\alpha - 2\eta)t + 4v + 2k] \\
& + \tfrac{3}{4}\sin^4\tfrac{1}{2}I\cos[f + 3f' + (n)(y + 3y' + 4\alpha + 2\eta)t + 4v - 2k]
\end{aligned}
$$

unde

$$
\Omega = \frac{m'}{M+m}\left\{
\begin{aligned}
& \frac{r^2}{r'^3}\left\{
\begin{aligned}
& [\tfrac{1}{4} - \tfrac{3}{2}\sin^2\tfrac{1}{2}I + \tfrac{3}{2}\sin^4\tfrac{1}{2}I] \\
& + [\tfrac{3}{4} - \tfrac{3}{2}\sin^2\tfrac{1}{2}I + \tfrac{3}{4}\sin^4\tfrac{1}{2}I]\cos 2[f - f' + n(y - y' - 2\eta)t + 2k] \\
& + [\tfrac{3}{2}\sin^2\tfrac{1}{2}I - \tfrac{3}{2}\sin^4\tfrac{1}{2}I]\cos 2[f + n(y + \alpha - \eta)t + v + k] \\
& + [\tfrac{3}{2}\sin^2\tfrac{1}{2}I - \tfrac{3}{2}\sin^4\tfrac{1}{2}I]\cos 2[f' + n(y' + \alpha + \eta)t + v - k] \\
& + \tfrac{3}{4}\sin^4\tfrac{1}{2}I\cos 2[f + f' + n(y + y' + 2\alpha)t + 2v]
\end{aligned}\right\} \\
& + \frac{r^3}{r'^4}\left\{
\begin{aligned}
& [\tfrac{3}{8} - \tfrac{33}{8}\sin^2\tfrac{1}{2}I + \tfrac{75}{8}\sin^4\tfrac{1}{2}I]\cos[f - f' + n(y - y' - 2\eta)t + 2k] \\
& + [\tfrac{5}{8} - \tfrac{15}{8}\sin^2\tfrac{1}{2}I + \tfrac{15}{8}\sin^4\tfrac{1}{2}I]\cos 3[f - f' + n(y - y' - 2\eta)t + 2k] \\
& + [\tfrac{9}{4}\sin^2\tfrac{1}{2}I - \tfrac{15}{2}\sin^4\tfrac{1}{2}I]\cos[f + f' + n(y + y' + 2\alpha)t + 2v] \\
& + [\tfrac{15}{8}\sin^2\tfrac{1}{2}I - \tfrac{15}{4}\sin^4\tfrac{1}{2}I]\cos[3f - f' + n(3y - y' + 2\alpha - 4\eta)t + 2v + 4k] \\
& + [\tfrac{15}{8}\sin^2\tfrac{1}{2}I - \tfrac{15}{4}\sin^4\tfrac{1}{2}I]\cos[f - 3f' + n(y - 3y' - 2\alpha - 4\eta)t - 2v + 4k] \\
& + \tfrac{15}{8}\sin^4\tfrac{1}{2}I\cos[3f + f' + n(3y + y' + 4\alpha - 2\eta)t + 4v + 2k] \\
& + \tfrac{15}{8}\sin^4\tfrac{1}{2}I\cos[f + 3f' + n(y + 3y' + 4\alpha + 2\eta)t + 4v - 2k]
\end{aligned}\right\}
\end{aligned}\right\}
$$

Denotantibus i et i' numeros integros, sint

$$
(1)\ldots\ldots\left\{
\begin{aligned}
& \frac{r^2}{a^2}\cos 2f = \Sigma_{-\infty}^{+\infty} Q_c^{(i)}\cos ig\ ; && \frac{r^2}{a^2}\sin 2f = \Sigma_{-\infty}^{+\infty} Q_s^{(i)}\sin ig\ ; \\
& \frac{r^2}{a^2} = \Sigma_{-\infty}^{+\infty} P^{(i)}\cos ig\ ; && \frac{a'^3}{r'^3}\cos 2f' = \Sigma_{-\infty}^{+\infty} G_c^{(i')}\cos i'g'\ ; \\
& \frac{a'^3}{r'^3}\sin 2f' = \Sigma_{-\infty}^{+\infty} G_s^{(i')}\sin i'g'\ ; && \frac{a'^3}{r'^3} = \Sigma_{-\infty}^{+\infty} K^{(i')}\cos i'g'\ ; \\
& \frac{r^3}{a^3}\cos f = \Sigma_{-\infty}^{+\infty} A_c^{(i)}\cos ig\ ; && \frac{r^3}{a^3}\sin f = \Sigma_{-\infty}^{+\infty} A_s^{(i)}\sin ig\ ; \\
& \frac{r^3}{a^3}\cos 3f = \Sigma_{-\infty}^{+\infty} B_c^{(i)}\cos ig\ ; && \frac{r^3}{a^3}\sin 3f = \Sigma_{-\infty}^{+\infty} B_s^{(i)}\sin ig\ ; \\
& \frac{a'^4}{r'^4}\cos f' = \Sigma_{-\infty}^{+\infty} C_c^{(i')}\cos i'g'\ ; && \frac{a'^4}{r'^4}\sin f' = \Sigma_{-\infty}^{+\infty} C_s^{(i')}\sin i'g'\ ; \\
& \frac{a'^4}{r'^4}\cos 3f' = \Sigma_{-\infty}^{+\infty} D_c^{(i')}\cos i'g'\ ; && \frac{a'^4}{r'^4}\sin 3f' = \Sigma_{-\infty}^{+\infty} D_s^{(i')}\sin i'g'\ ;
\end{aligned}\right.
$$

quibus conditiones adiungo has

$$\begin{aligned}
Q_c^{(i)} &= \quad Q_c^{(-i)}\ ;\ G_c^{(i)} = \quad G_c^{(-i)}\ ;\ A_c^{(i)} = \quad A_c^{(-i)}\ ;\ \text{etc.}\\
Q_s^{(i)} &= -Q_s^{(-i)}\ ;\ G_s^{(i)} = -G_s^{(-i)}\ ;\ A_s^{(i)} = -A_s^{(-i)}\ ;\ \text{etc.}\\
P^{(i)} &= \quad P^{(-i)}\ ;\ K^{(i)} = \quad K^{(-i)}
\end{aligned}$$

tum, adhibitis aequationibus generalibus his

$$\left.\begin{aligned}
\Sigma_{-\infty}^{+\infty} E_c^{(i)} \cos ig \times \Sigma_{-\infty}^{+\infty} F_c^{(i')} \cos i'g' &= \Sigma_{-\infty}^{+\infty} \Sigma_{-\infty}^{+\infty} E_c^{(i)} F_c^{(i')} \cos(ig+i'g')\\
\Sigma_{-\infty}^{+\infty} E_c^{(i)} \cos ig \times \Sigma_{-\infty}^{+\infty} F_s^{(i')} \sin i'g' &= \Sigma_{-\infty}^{+\infty} \Sigma_{-\infty}^{+\infty} E_c^{(i)} F_s^{(i')} \sin(ig+i'g')\\
\Sigma_{-\infty}^{+\infty} E_s^{(i)} \sin ig \times \Sigma_{-\infty}^{+\infty} F_s^{(i')} \sin i'g' &= -\Sigma_{-\infty}^{+\infty} \Sigma_{-\infty}^{+\infty} E_s^{(i)} F_s^{(i')} \cos(ig+i'g')
\end{aligned}\right\}\text{........}(2)$$

ubi E et F sunt quantitates quaecunque, quibus conditiones insunt, ut sint

$$E_c^{(i)} = E_c^{(-i)}\ ;\ E_s^{(i)} = -E_s^{(-i)}\ ;\ F_c^{(i)} = F_c^{(-i)}\ ;\ F_s^{(i)} = -F_s^{(-i)}$$

habetur statim

$$\frac{r^2}{r'^3} = \frac{a^2}{a'^3} \Sigma_{-\infty}^{+\infty} \Sigma_{-\infty}^{+\infty} P^{(i)} K^{(i')} \cos(ig+i'g')$$

porro, quum sit

$$\begin{aligned}
&\cos 2\,[f-f'+n(y-y'-2\eta)t+2k]\\
&\qquad = [\cos 2f \cos 2f' + \sin 2f \sin 2f']\cos 2\,[n(y-y'-2\eta)t+2k]\\
&\qquad + [\cos 2f \sin 2f' - \sin 2f \cos 2f']\sin 2\,[n(y-y'-2\eta)t+2k]
\end{aligned}$$

habetur primum

$$\begin{aligned}
&\frac{r^2}{r'^3}\cos 2\,[f-f'+n(y-y'-2\eta)t+2k]\\
&= \frac{a^2}{a'^3}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{Q_c^{(i)}G_c^{(i')}-Q_s^{(i)}G_s^{(i')}\}\cos(ig+i'g')\cos 2\,[n(y-y'-2\eta)t+2k]\\
&+ \frac{a^2}{a'^3}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{Q_c^{(i)}G_s^{(i')}-Q_s^{(i)}G_c^{(i')}\}\sin(ig+i'g')\sin 2\,[n(y-y'-2\eta)t+2k]\\
&= \tfrac{1}{2}\frac{a^2}{a'^3}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{Q_c^{(i)}G_c^{(i')}-Q_s^{(i)}G_s^{(i')}-Q_c^{(i)}G_s^{(i')}+Q_s^{(i)}G_c^{(i')}\}\cos[ig+i'g'+n(2y-2y'-4\eta)t+4k]\\
&+ \tfrac{1}{2}\frac{a^2}{a'^3}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{Q_c^{(i)}G_c^{(i')}-Q_s^{(i)}G_s^{(i')}+Q_c^{(i)}G_s^{(i')}-Q_s^{(i)}G_c^{(i')}\}\cos[-ig-i'g'+n(2y-2y'-4\eta)t+4k]\\
&= \tfrac{1}{2}\frac{a^2}{a'^3}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{Q_c^{(i)}+Q_s^{(i)}\}\{G_c^{(i')}-G_s^{(i')}\}\cos[ig+i'g'+n(2y-2y'-4\eta)t+4k]\\
&+ \tfrac{1}{2}\frac{a^2}{a'^3}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{Q_c^{(i)}-Q_s^{(i)}\}\{G_c^{(i')}+G_s^{(i')}\}\cos[-ig-i'g'+n(2y-2y'-4\eta)t+4k]
\end{aligned}$$

sed prior horum coefficientium, scriptis $-i$ loco i et $-i'$ loco i', transit in hunc

$$\{Q_c^{(i)} - Q_s^{(i)}\}\{G_c^{(i')} + G_s^{(i')}\}$$

qui posteriori, et posterior coefficiens, iisdem mutationibus factis, abit in hunc

$$\{Q_c^{(i)} + Q_s^{(i)}\}\{G_c^{(i')} - G_s^{(i')}\}$$

qui priori aequalis est. Hinc sequitur ut sit

$$\frac{r^2}{r'^3}\cos 2\,[f-f'+n(y-y'-2\eta)t+2k] =$$

$$\frac{a^2}{a'^3}\,\Sigma_{-\infty}^{+\infty}\,\Sigma_{-\infty}^{+\infty}\,\{Q_c^{(i)}+Q_s^{(i)}\}\{G_c^{(i')}-G_s^{(i')}\}\cos[ig+i'g'+n(2y-2y'-4\eta)t+4k]$$

et eodem modo termini reliqui ipsius Ω in series infinitas evolvuntur. Quibus factis, substitutoque valore ipsius $\frac{m'}{M+m}\cdot\frac{a^3}{a'^3}$ in art. 20. Sect. II. dato, et posita

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 + \Omega_8 + \Omega_9 + \Omega_{10}$$

et

$$\frac{(n')}{(n)} = u$$

invenitur

$$a\Omega_1 = \frac{u^2}{1+\frac{M}{m'}}\left[\tfrac{1}{4}-\tfrac{3}{2}\sin^2\tfrac{1}{2}I+\tfrac{3}{2}\sin^4\tfrac{1}{2}I\right]P^{(i)}K^{(i')}\cos(ig+i'g')$$

$$a\Omega_2 = \frac{u^2}{1+\frac{M}{m'}}\left[\tfrac{3}{4}-\tfrac{3}{2}\sin^2\tfrac{1}{2}I+\tfrac{3}{4}\sin^4\tfrac{1}{2}I\right]\{Q_c^{(i)}+Q_s^{(i)}\}\{G_c^{(i')}-G_s^{(i')}\}\cos[ig+i'g'+n(2y-2y'-4\eta)t+4k]$$

$$a\Omega_3 = \frac{u^2}{1+\frac{M}{m'}}\left[\tfrac{3}{2}\sin^2\tfrac{1}{2}I-\tfrac{3}{2}\sin^4\tfrac{1}{2}I\right]\{Q_c^{(i)}+Q_s^{(i)}\}K^{(i')}\cos[ig+i'g'+n(2y+2\alpha-2\eta)t+2v+2k]$$

$$a\Omega_4 = \frac{u^2}{1+\frac{M}{m'}}\left[\tfrac{3}{2}\sin^2\tfrac{1}{2}I-\tfrac{3}{2}\sin^4\tfrac{1}{2}I\right]P^{(i)}\{G_c^{(i')}+G_s^{(i')}\}\cos[ig+i'g'+n(2y'+2\alpha+2\eta)t+2v-2k]$$

$$a\Omega_5 = \frac{u^2}{1+\frac{M}{m'}}\cdot\tfrac{3}{4}\sin^4\tfrac{1}{2}I\{Q_c^{(i)}+Q_s^{(i)}\}\{G_c^{(i')}+G_s^{(i')}\}\cos[ig+i'g'+n(2y+2y'+4\alpha)t+4v]$$

$$a\Omega_6 = \frac{u^2}{1+\frac{M}{m'}}\left(\frac{a}{a'}\right)\left[\tfrac{3}{8}-\tfrac{33}{8}\sin^2\tfrac{1}{2}I+\tfrac{75}{8}\sin^4\tfrac{1}{2}I\right]\{A_c^{(i)}+A_s^{(i)}\}\{C_c^{(i')}-C^{(i')}\}\cos[ig+i'g'+n(y-y'-2\eta)t+2k]$$

$$a\Omega_7 = \frac{u^2}{1+\frac{M}{m'}}\left(\frac{a}{a'}\right)\left[\tfrac{5}{8}-\tfrac{15}{8}\sin^2\tfrac{1}{2}I+\tfrac{15}{8}\sin^4\tfrac{1}{2}I\right]\{B_c^{(i)}+B_s^{(i)}\}\{D_c^{(i')}-D_s^{(i')}\}\cos[ig+i'g'+n(3y-3y'-6\eta)t+6k]$$

$$a\Omega_8 = \frac{u^2}{1+\frac{M}{m'}}\left(\frac{a}{a'}\right)[\tfrac{9}{4}\sin^2\tfrac{1}{2}I - \tfrac{15}{2}\sin^4\tfrac{1}{2}I]\{A_c^{(i)}+A_s^{(i)}\}\{C_c^{(i')}+C_s^{(i')}\}\cos[ig+i'g'+n(y+y'+2\alpha)t+2v]$$

$$a\Omega_9 = \frac{u^2}{1+\frac{M}{m'}}\left(\frac{a}{a'}\right)[\tfrac{15}{8}\sin^2\tfrac{1}{2}I - \tfrac{15}{4}\sin^4\tfrac{1}{2}I]\{B_c^{(i)}+B_c^{(i)}\}\{C_c^{(i')}-C_s^{(i')}\}\cos[ig+i'g'+n(3y-y'+2\alpha-4\eta)t+2v+4k]$$

$$a\Omega_{10} = \frac{u^2}{1+\frac{M}{m'}}\left(\frac{a}{a'}\right)[\tfrac{15}{8}\sin^2\tfrac{1}{2}I - \tfrac{15}{4}\sin^4\tfrac{1}{2}I]\{A_c^{(i)}+A_s^{(i)}\}\{D_c^{(i')}-D_s^{(i')}\}\cos[ig+i'g'+n(y-3y'-2\alpha-4\eta)t-2v+4k]$$

ubi ultimos duos per $\sin^4\frac{1}{2}I$ multiplicatos terminos ipsius Ω omisi, quia perturbationes sensibiles proferre nequeunt, id quod calculo numerico comperi. Summationes et respectu i et respectu i', quarum signa brevitatis caussa suppressi, in expressionibus praecedentibus a $-\infty$ usque ad $+\infty$ per omnes numeros integros, inclusa cifra, ubique sunt extendendae.

4.

Quantitatem igitur Ω in seriem infinitam secundum cosinus multiplicium anomaliarum mediarum progredientem evolvimus, cuius seriei coefficientes facillima opera accuratissime computari poterunt, si expressiones generales transcendentium $G_c^{(i)}$, $G_s^{(i)}$, $Q_c^{(i)}$, $Q_s^{(i)}$, etc. notae fuerint. Iam quum expressio $\cos nf$, ubi n est integer, ad seriem finitam huius formae $A\cos^i f$, ubi i quoque est integer et A constans, reduci possit, et $\cos f$ hunc $-\frac{1}{e}+a\frac{1-e^2}{er}$ habeat valorem, evidens est expressionem $r^h\cos nf$ ad seriem finitam huius formae Br^m, ubi m integer et B constans, reduci posse. Quum porro expressio $\sin nf$ in seriem finitam huius formae $A\cos^i f\sin f$ transmutari possit, et $\sin f$ expressione hac $\frac{\sqrt{1-e^2}}{ae}\cdot\frac{dr}{dg}$ reddi possit, manifestum est expressionem $r^h\sin nf$ ad seriem finitam huius formae $B\frac{d.r^m}{dg}$ reduci posse. Quos calculos peragamus.

Aequationes

$$\frac{r}{a}=\frac{1-e^2}{1+e\cos f}\,,\quad \frac{dr}{adg}=\frac{e\sin f}{\sqrt{1-e^2}}$$

suppeditant

$$\cos f=-\frac{1}{e}+a\frac{1-e^2}{er}\,,\quad \sin f=\frac{\sqrt{1-e^2}}{ae}\cdot\frac{dr}{dg}$$

itaque quum sit

$$\frac{r^2}{a^2}\cos 2f = 2\,\frac{r^2}{a^2}\cos^2 f - \frac{r^2}{a^2}\;;\;\; \frac{r^2}{a^2}\sin 2f = 2\,\frac{r^2}{a^2}\sin f\cos f$$

nanciscimur

$$\frac{r^2}{a^2}\cos 2f = 2\,\frac{(1-e^2)^2}{e^2} - 4\,\frac{1-e^2}{e^2}\cdot\frac{r}{a} + \frac{2-e^2}{e^2}\cdot\frac{r^2}{a^2}$$

$$\frac{r^2}{a^2}\sin 2f = \frac{(1-e^2)^{\frac{1}{2}}}{a^2e^2}\cdot\frac{d.r^2}{dg} - \frac{2(1-e^2)^{\frac{1}{2}}}{3a^3e^2}\cdot\frac{d.r^3}{dg}$$

porro ex

$$\frac{a^3}{r^3}\cos 2f = 2\,\frac{a^3}{r^3}\cos^2 f - \frac{a^3}{r^3}\;;\;\; \frac{a^3}{r^3}\sin 2f = 2\,\frac{a^3}{r^3}\cos f\sin f$$

nanciscimur

$$\frac{a^3}{r^3}\cos 2f = 2\,\frac{a^5(1-e^2)^2}{e^2r^5} - 4\,\frac{a^4(1-e^2)}{e^2r^4} + \frac{a^3(2-e^2)}{e^2r^3}$$

$$\frac{a^3}{r^3}\sin 2f = -\frac{2a^3(1-e^2)^{\frac{1}{2}}}{3e^2}\cdot\frac{d.r^{-3}}{dg} + \frac{a^2(1-e^2)^{\frac{1}{2}}}{e^2}\cdot\frac{d.r^{-2}}{dg}$$

porro

$$\frac{r^3}{a^3}\cos f = -\frac{r^3}{a^3e} + \frac{r^2(1-e^2)}{a^2e}$$

$$\frac{r^3}{a^3}\sin f = \frac{\sqrt{1-e^2}}{4a^4e}\cdot\frac{d.r^4}{dg}$$

$$\frac{a^4}{r^4}\cos f = -\frac{a^4}{er^4} + \frac{a^5(1-e^2)}{er^5}$$

$$\frac{a^4}{r^4}\sin f = -\frac{a^2\sqrt{1-e^2}}{3e}\cdot\frac{d.r^{-3}}{dg}$$

denique

$$\frac{r^3}{a^3}\cos 3f = 4\,\frac{r^3}{a^3}\cos^3 f - 3\,\frac{r^3}{a^3}\cos f\;;\;\; \frac{r^3}{a^3}\sin 3f = 4\,\frac{r^3}{a^3}\cos^2 f\sin f - \frac{r^3}{a^3}\sin f$$

$$\frac{a^4}{r^4}\cos 3f = 4\,\frac{a^4}{r^4}\cos^3 f - 3\,\frac{a^4}{r^4}\cos f\;;\;\; \frac{a^4}{r^4}\sin 3f = 4\,\frac{a^4}{r^4}\cos^2 f\sin f - \frac{a^4}{r^4}\sin f$$

suppeditant

$$\frac{r^3}{a^3}\cos 3f = 4\,\frac{(1-e^2)^3}{e^3} - 12\,\frac{(1-e^2)^2}{ae^3}\,r + 3\,\frac{4-5e^2+e^4}{a^2e^3}\,r^2 - \frac{4-3e^2}{a^3e^3}\,r^3$$

$$\frac{r^3}{a^3}\sin 3f = \sqrt{1-e^2}\left\{\frac{2(1-e^2)^2}{a^2e^3}\cdot\frac{d.r^2}{dg} - \frac{8(1-e^2)}{3a^3e^3}\cdot\frac{d.r^3}{dg} + \frac{4-e^2}{4a^4e^3}\cdot\frac{d.r^4}{dg}\right\}$$

$$\frac{a^4}{r^4}\cos 3f = 4\,\frac{a^7(1-e^2)^3}{e^3r^7} - 12\,\frac{a^6(1-e^2)^2}{e^3r^6} + 3\,\frac{a^5(4-5e^2+e^4)}{e^3r^5} - \frac{a^4(4-3e^2)}{e^3r^4}$$

$$\frac{a^4}{r^4}\sin 3f = -\sqrt{1-e^2}\left\{\frac{4a^5(1-e^2)^2}{5e^3}\cdot\frac{d.r^{-5}}{dg} - \frac{2a^4(1-e^2)}{e^3}\cdot\frac{d.r^{-4}}{dg} + \frac{a^3(4-e^2)}{3e^3}\cdot\frac{d.r^{-3}}{dg}\right\}$$

quibus formulis transcendentes nostrae ad potestates radii vectoris quotientesque earum differentiales respectu anomaliae mediae reductae sunt. Iam sit

$$\left(\frac{r}{a}\right)^n = \Sigma_{-\infty}^{+\infty} R_n^{(i)} \cos ig$$

ubi

$$R_n^{(i)} = R_n^{(-i)}$$

supponitur, hinc est

$$\frac{d.\left(\frac{r}{a}\right)^n}{dg} = -\Sigma_{-\infty}^{+\infty} iR_n^{(i)} \sin ig$$

et formulae modo inventae transeunt in has

$$\frac{r^2}{a^2}\cos 2f = 2\frac{(1-e^2)^2}{e^2} + \Sigma_{-\infty}^{+\infty}\left\{\frac{2-e^2}{e^2}R_2^{(i)} - 4\frac{1-e^2}{e^2}R_1^{(i)}\right\}\cos ig$$

$$\frac{r^2}{a^2}\sin 2f = \sqrt{1-e^2}\,\Sigma_{-\infty}^{+\infty} i\left\{\frac{2}{3e^2}R_3^{(i)} - \frac{1-e^2}{e^2}R_2^{(i)}\right\}\sin ig$$

$$\frac{a^3}{r^3}\cos 2f = \Sigma_{-\infty}^{+\infty}\left\{\frac{2(1-e^2)^2}{e^2}R_{-5}^{(i)} - \frac{4(1-e^2)}{e^2}R_{-4}^{(i)} + \frac{2-e^2}{e^2}R_{-3}^{(i)}\right\}\cos ig$$

$$\frac{a^3}{r^3}\sin 2f = \sqrt{1-e^2}\,\Sigma_{-\infty}^{+\infty} i\left\{\frac{2(1-e^2)}{3e^2}R_{-3}^{(i)} - \frac{1}{e^2}R_{-2}^{(i)}\right\}\sin ig$$

$$\frac{r^3}{a^3}\cos f = \Sigma_{-\infty}^{+\infty}\left\{\frac{1-e^2}{e}R_2^{(i)} - \frac{1}{e}R_3^{(i)}\right\}\cos ig$$

$$\frac{r^3}{a^3}\sin f = -\frac{\sqrt{1-e^2}}{4e}\Sigma_{-\infty}^{+\infty} i\,R_4^{(i)}\sin ig$$

$$\frac{a^4}{r^4}\cos f = \Sigma_{-\infty}^{+\infty}\left\{\frac{1-e^2}{e}R_{-5}^{(i)} - \frac{1}{e}R_{-4}^{(i)}\right\}\cos ig$$

$$\frac{a^4}{r^4}\sin f = \frac{\sqrt{1-e^2}}{3e}\Sigma_{-\infty}^{+\infty} i\,R_{-3}^{(i)}\sin ig$$

$$\frac{r^3}{a^3}\cos 3f = 4\,\frac{(1-e^2)^3}{e^3} + \Sigma_{-\infty}^{+\infty}\left\{\frac{3(4-5e^2+e^4)}{e^3}R_2^{(i)} - \frac{12(1-e^2)^2}{e^3}R_1^{(i)} - \frac{4-3e^2}{e^3}R_3^{(i)}\right\}\cos ig$$

$$\frac{r^3}{a^3}\sin 3f = \sqrt{1-e^2}\,\Sigma_{-\infty}^{+\infty} i\left\{\frac{8(1-e^2)}{3e^3}R_3^{(i)} - \frac{2(1-e^2)^2}{e^3}R_2^{(i)} - \frac{4-e^2}{4e^3}R_4^{(i)}\right\}\sin ig$$

$$\frac{a^4}{r^4}\cos 3f = \Sigma_{-\infty}^{+\infty}\left\{\frac{4(1-e^2)^3}{e^3}R_{-7}^{(i)} - \frac{12(1-e^2)^2}{e^3}R_{-6}^{(i)} + \frac{3(4-5e^2+e^4)}{e^3}R_{-5}^{(i)} - \frac{4-3e^2}{e^3}R_{-4}^{(i)}\right\}\cos ig$$

$$\frac{a^4}{r^4}\sin 3f = \sqrt{1-e^2}\,\Sigma_{-\infty}^{+\infty} i\left\{\frac{4(1-e^2)^2}{5e^3}R_{-5}^{(i)} - \frac{2(1-e^2)}{e^3}R_{-4}^{(i)} + \frac{4-e^2}{3e^3}R_{-3}^{(i)}\right\}\sin ig$$

Comparatis his formulis cum (1), elicitur

$$Q_c^{(i)} = \frac{2-e^2}{e^2} R_2^{(i)} - 4\,\frac{1-e^2}{e^2} R_1^{(i)};\ \text{si excipis}\ Q_c^{(o)} = 2\,\frac{(1-e^2)^2}{e^2} + \frac{2-e^2}{e^2} R_2^{(o)} - 4\,\frac{1-e^2}{e^2} R_1^{(o)}$$

$$Q_s^{(i)} = i\sqrt{1-e^2}\left\{\frac{2}{3e^2} R_3^{(i)} - \frac{1-e^2}{e^2} R_2^{(i)}\right\}$$

$$G_c^{(i)} = \frac{2(1-e^2)^2}{e^2} R_{-5}^{(i)} - \frac{4(1-e^2)}{e^2} R_{-4}^{(i)} + \frac{2-e^2}{e^2} R_{-3}^{(i)}$$

$$G_s^{(i)} = i\sqrt{1-e^2}\left\{\frac{2(1-e^2)}{3e^2} R_{-3}^{(i)} - \frac{1}{e^2} R_{-2}^{(i)}\right\}$$

$$P^{(i)} = R_2^{(i)}$$

$$K^{(i)} = R_{-3}^{(i)}$$

$$A_c^{(i)} = \frac{1-e^2}{e} R_2^{(i)} - \frac{1}{e} R_3^{(i)}$$

$$A_s^{(i)} = -i\,\frac{\sqrt{1-e^2}}{4e} R_4^{(i)}$$

$$C_c^{(i)} = \frac{1-e^2}{e} R_{-5}^{(i)} - \frac{1}{e} R_{-4}^{(i)}$$

$$C_s^{(i)} = i\,\frac{\sqrt{1-e^2}}{3e} R_{-3}^{(i)}$$

$$B_c^{(i)} = -\frac{12(1-e^2)^2}{e^3} R_1^{(i)} + \frac{3(4-5e^2+e^4)}{e^3} R_2^{(i)} - \frac{4-3e^2}{e^3} R_3^{(i)}$$

$$\text{si excipis}\ B_c^{(o)} = \frac{4(1-e^2)^3}{e^3} - \frac{12(1-e^2)^2}{e^3} R_1^{(o)} + \frac{3(4-5e^2+e^4)}{e^3} R_2^{(o)} - \frac{4-3e^2}{e^3} R_3^{(o)}$$

$$B_s^{(i)} = -i\sqrt{1-e^2}\left\{\frac{2(1-e^2)^2}{e^3} R_2^{(i)} - \frac{8(1-e^2)}{3e^3} R_3^{(i)} + \frac{4-e^2}{4e^3} R_4^{(i)}\right\}$$

$$D_c^{(i)} = \frac{4(1-e^2)^3}{e^3} R_{-7}^{(i)} - \frac{12(1-e^2)^2}{e^3} R_{-6}^{(i)} + \frac{3(4-5e^2+e^4)}{e^3} R_{-5}^{(i)} - \frac{4-3e^2}{e^3} R_{-4}^{(i)}$$

$$D_s^{(i)} = i\sqrt{1-e^2}\left\{\frac{4(1-e^2)^2}{5e^3} R_{-5}^{(i)} - \frac{2(1-e^2)}{e^3} R_{-4}^{(i)} + \frac{4-e^2}{3e^3} R_{-3}^{(i)}\right\}$$

quarum Q, P, A et B ad indicem i pertinentes ope excentricitatis Lunae, G, K, C et D vero ad indicem i' spectantes ope excentricitatis Terrae computandae sunt.

5.

Quantitates omnes $R_n^{(i)}$ in art. praec. introductae ad quantitates $R_2^{(i)}$, $R_{-2}^{(i)}$, $\frac{dR_2^{(i)}}{de}$ et $\frac{dR_{-2}^{(i)}}{de}$ reduci possunt. Quem in finem differentiata aequatio haec

$$\frac{d.\left(\frac{r}{a}\right)^n}{dg} = n\left(\frac{r}{a}\right)^{n-1} \cdot \frac{e\sin f}{\sqrt{1-e^2}}$$

suppeditat

$$\frac{d^2.\left(\frac{r}{a}\right)^n}{dg^2}=n\,(n\text{-}1)\left(\frac{r}{a}\right)^{n-2}\cdot\frac{e^2}{1\text{-}e^2}-n\,(n\text{-}1)\left(\frac{r}{a}\right)^{n-2}\frac{e^2}{1\text{-}e^2}\cos^2 f+n\left(\frac{r}{a}\right)^{n\text{-}3}e\cos f$$

quae, eliminato cos f ope valoris eius in art. praec. dati, abit in hanc

$$\frac{d^2.\left(\frac{r}{a}\right)^n}{dg^2}=n\,(2\text{-}n)\left(\frac{r}{a}\right)^{n-4}(1\text{-}e^2)+n(2n\text{-}3)\left(\frac{r}{a}\right)^{n-3}\text{-}n(n\text{-}1)\left(\frac{r}{a}\right)^{n-2}$$

Signa vero in art. praec. introducta praebent

$$\frac{d^2.\left(\frac{r}{a}\right)^n}{dg^2}=-\Sigma_{-\infty}^{+\infty}i^2 R_n^{(i)}\cos ig$$

unde aequatio praecedens subministrat aequationem ab ill. Bessel primo datam hanc

$$-i^2 R_n^{(i)}=n\,(2-n)\,R_{n-4}^{(i)}(1-e^2)+n(2n-3)\,R_{n-3}^{(i)}-n\,(n-1)\,R_{n-2}^{(i)}$$

quae in casu peculiari, ubi $i=o$, abit in

$$o=(2-n)\,R_{n-4}^{(o)}(1-e^2)+(2n-3)\,R_{n-3}^{(o)}-(n-1)\,R_{n-2}^{(o)}$$

Positis deinceps $n=-3$, $n=-2$, $n=-1$, $n=o$, $n=1$, $n=2$, $n=3$, $n=4$, aequationes modo evolutae praebent

$$\begin{aligned}
-i^2 R_{-3}^{(i)} &= -15\,(1-e^2)\,R_{-7}^{(i)}+27\,R_{-6}^{(i)}-12\,R_{-5}^{(i)}\\
-i^2 R_{-2}^{(i)} &= -\ \ 8\,(1-e^2)\,R_{-6}^{(i)}+14\,R_{-5}^{(i)}-\ \ 6\,R_{-4}^{(i)}\\
-i^2 R_{-1}^{(i)} &= -\ \ 3\,(1-e^2)\,R_{-5}^{(i)}+\ \ 5\,R_{-4}^{(i)}-\ \ 2\,R_{-3}^{(i)}\\
-i^2 R_{1}^{(i)} &= \qquad (1-e^2)\,R_{-3}^{(i)}-R_{-2}^{(i)}\\
-i^2 R_{2}^{(i)} &= \qquad\qquad\qquad\qquad 2\,R_{-1}^{(i)}-\ \ 2\,R_{o}^{(i)}\\
-i^2 R_{3}^{(i)} &= -\ \ 3\,(1-e^2)\,R_{-1}^{(i)}+\ \ 9\,R_{o}^{(i)}-\ \ 6\,R_{1}^{(i)}\\
-i^2 R_{4}^{(i)} &= -\ \ 8\,(1-e^2)\,R_{o}^{(i)}+20\,R_{1}^{(i)}-12\,R_{2}^{(i)}
\end{aligned}$$

quae omnes etiam in casu, ubi $i=o$, locum habent; in hoc vero casu accedit aequatio haec

$$o=2\,R_{-4}^{(o)}(1-e^2)-3\,R_{-3}^{(o)}+R_{-2}^{(o)}$$

Quum statim habeatur

$$R_{o}^{(i)}=o\,;\ \text{si excipis}\ R_{o}^{(o)}=1$$

aequationes hae monstrant omnes $R_n^{(i)}$ computari posse, si $R_1^{(i)}$, $R_2^{(i)}$, $R_{-2}^{(i)}$ et $R_{-4}^{(i)}$ notas esse supponitur. Resolutis igitur aequationibus praecedentibus respectu harum quantitatum, emergunt

$$R_{-7}^{(i)} = -\frac{i^4(47+16e^2)}{120(1-e^2)^3}R_2^{(i)} + \frac{35i^2(1-e^2)^2-188-64e^2}{120(1-e^2)^4}R_{-2}^{(i)} + \frac{47i^2+16i^2e^2-2i^4(1-e^2)^2}{30(1-e^2)^4}R_1^{(i)} + \frac{154+161e^2}{60(1-e^2)^8}R_{-4}^{(i)}$$

$$R_{-6}^{(i)} = -\frac{7i^4}{24(1-e^2)^2}R_2^{(i)} + \frac{3i^2(1-e^2)^2-28}{24(1-e^2)^3}R_{-2}^{(i)} + \frac{7i^2}{6(1-e^2)^3}R_1^{(i)} + \frac{35-9(1-e^2)}{12(1-e^2)^2}R_{-4}^{(i)}$$

$$R_{-5}^{(i)\prime} = -\frac{i^4}{6(1-e^2)}R_2^{(i)} - \frac{2}{3(1-e^2)^2}R_{-2}^{(i)} + \frac{2i^2}{3(1-e^2)^2}R_1^{(i)} + \frac{5}{3(1-e^2)}R_{-4}^{(i)}$$

$$R_{-3}^{(i)} = \frac{1}{1-e^2}R_{-2}^{(i)} - \frac{i^2}{1-e^2}R_1^{(i)}$$

$$R_{-1}^{(i)} = -\frac{i^2}{2}R_2^{(i)}$$

$$R_3^{(i)} = -\frac{3(1-e^2)}{2}R_2^{(i)} + \frac{6}{i^2}R_1^{(i)}$$

$$R_4^{(i)} = \frac{12}{i^2}R_2^{(i)} - \frac{20}{i^2}R_1^{(i)}$$

ubi casus $i = o$ excipitur, pro quo aequationes praecedentes resolutae praebent

$$R_{-7}^{(o)} = \frac{8+40e^2+15e^4}{8(1-e^2)^5}R_{-2}^{(o)}$$

$$R_{-6}^{(o)} = \frac{8+24e^2+3e^4}{8(1-e^2)^4}R_{-2}^{(o)}$$

$$R_{-5}^{(o)} = \frac{2+3e^2}{2(1-e^2)^3}R_{-2}^{(o)}$$

$$R_{-4}^{(o)} = \frac{2+e^2}{2(1-e^2)^2}R_{-2}^{(o)}$$

$$R_{-3}^{(o)} = \frac{1}{1-e^2}R_{-2}^{(o)}$$

$$R_{-1}^{(o)} = 1$$

$$R_1^{(o)} = 1 + \tfrac{1}{2}e^2$$

$$R_2^{(o)} = 1 + \tfrac{3}{2}e^2$$

$$R_3^{(o)} = 1 + 3e^2 + \tfrac{3}{8}e^4$$

Ipsis $R_1^{(i)}$ et $R_{-4}^{(i)}$ ex his aequationibus eliminandis inserviunt aequationes hae

$$\frac{d.\left(\frac{r}{a}\right)^2}{de} = -2\,\frac{r}{a}\cos f$$

$$\frac{d.\left(\frac{r}{a}\right)^{-2}}{de} = 2\,\frac{a^3}{r^3}\cos f$$

quae, substituto valore ipsius $\cos f$ supra dato, evadunt

$$\frac{d.\left(\frac{r}{a}\right)^2}{de} = \frac{2r}{ea} - \frac{2(1-e^2)}{e}$$

$$\frac{d.\left(\frac{r}{a}\right)^{-2}}{de} = -\frac{2a^3}{er^3} + \frac{2a^4(1-e^2)}{er^4}$$

unde

$$\frac{dR^{(i)}_2}{de} = \frac{2}{e}R^{(i)}_1$$

$$\frac{dR^{(i)}_{-2}}{de} = -\frac{2}{e}R^{(i)}_{-3} + \frac{2(1-e^2)}{e}R^{(i)}_{-4}$$

quae, excluso valore $i = o$, suppeditant

$$R^{(i)}_1 = \frac{e}{2}\,\frac{dR^{(i)}_2}{de}$$

$$R^{(i)}_{-4} = \frac{1}{(1-e^2)^2}R^{(i)}_{-2} + \frac{e}{2(1-e^2)}\,\frac{dR^{(i)}_{-2}}{de} - \frac{ei^2}{2(1-e^2)^2}\,\frac{dR^{(i)}_2}{de}$$

quibuscum $R^{(i)}_1$ et $R^{(i)}_{-4}$ eliminatis, emergunt

$$R^{(i)}_{-7} = -\frac{i^4(47+16e^2)}{120(1-e^2)^3}R^{(i)}_2 - \frac{i^2e(60+223e^2+32e^4)+4i^4e(1-e^2)^3}{120(1-e^2)^5}\,\frac{dR^{(i)}_2}{de} + \frac{120+446e^2+64e^4+35i^2(1-e^2)^3}{120(1-e^2)^5}R^{(i)}_{-2}$$
$$+\frac{154e+161e^3}{120(1-e^2)^4}\,\frac{dR^{(i)}_{-2}}{de}$$

$$R^{(i)}_{-6} = -\frac{7i^4}{24(1-e^2)^2}R^{(i)}_2 - \frac{i^2e(12+23e^2)}{24(1-e^2)^4}\,\frac{dR^{(i)}_2}{de} + \frac{24+46e^2+3i^2(1-e^2)^3}{24(1-e^2)^4}R^{(i)}_{-2} + \frac{26e+9e^3}{24(1-e^2)^3}\,\frac{dR^{(i)}_{-2}}{de}$$

$$R^{(i)}_{-5} = -\frac{i^4}{6(1-e^2)}R^{(i)}_2 - \frac{i^2e(3+2e^2)}{6(1-e^2)^3}\,\frac{dR^{(i)}_2}{de} + \frac{3+2e^2}{3(1-e^2)^3}R^{(i)}_{-2} + \frac{5e}{6(1-e^2)^2}\,\frac{dR^{(i)}_{-2}}{de}$$

$$R^{(i)}_{-4} = -\frac{i^2e}{2(1-e^2)^2}\,\frac{dR^{(i)}_2}{de} + \frac{1}{(1-e^2)^2}R^{(i)}_{-2} + \frac{e}{2(1-e^2)}\,\frac{dR^{(i)}_{-2}}{de}$$

$$R^{(i)}_{-3} = -\frac{i^2e}{2(1-e^2)}\,\frac{dR^{(i)}_2}{de} + \frac{1}{1-e^2}R^{(i)}_{-2}$$

$$R^{(i)}_{-1} = -\frac{i^2}{2}R^{(i)}_2$$

$$R^{(i)}_1 = \frac{e}{2}\,\frac{dR^{(i)}_2}{de}$$

$$R_3^{(i)} = -\frac{3(1-e^2)}{2} R_2^{(i)} + \frac{3e}{i^2}\frac{dR_2^{(i)}}{de}$$

$$R_4^{(i)} = \frac{12}{i^2} R_2^{(i)} - \frac{10e}{i^2}\frac{dR_2^{(i)}}{de}$$

ubi tamen casus $i = o$ excipiendus est. His quidem aequationibus omnes $R_n^{(i)}$ ad $R_2^{(i)}$, $R_{-2}^{(i)}$ et ad quotientes earum differentiales respectu excentricitatis reductae sunt.

Substitutis his valoribus ipsarum $R_n^{(i)}$ in valoribus transcendentium $Q_c^{(i)}$, $Q_s^{(i)}$, $G_c^{(i')}$, etc. in art. praec. datis, nanciscimur denique

$$Q_c^{(i)} = \frac{2-e^2}{e^2} R_2^{(i)} - \frac{2(1-e^2)}{e}\frac{dR_2^{(i)}}{de}\text{; si excipis } Q_c^{(o)} = \tfrac{1}{2}e^2$$

$$Q_s^{(i)} = \sqrt{1-e^2}\left\{\frac{2}{ie}\frac{dR_2^{(i)}}{de} - \frac{2i(1-e^2)}{e^2} R_2^{(i)}\right\}\text{; si excipis } Q_s^{(o)} = o$$

$$P^{(i)} = R_2^{(i)}$$

$$A_c^{(i)} = \frac{5(1-e^2)}{2e} R_2^{(i)} - \frac{3}{i^2}\frac{dR_2^{(i)}}{de}\text{; si excipis } A_c^{(o)} = -\tfrac{5}{2}e - \tfrac{15}{8}e^3$$

$$A_s^{(i)} = \sqrt{1-e^2}\left\{\frac{5}{2i}\frac{dR_2^{(i)}}{de} - \frac{3}{ie} R_2^{(i)}\right\}\text{; si excipis } A_s^{(o)} = o$$

$$B_c^{(i)} = \frac{36-51e^2+15e^4}{2e^3} R_2^{(i)} - \frac{12-9e^2+6i^2(1-e^2)^2}{i^2 e^2}\frac{dR_2^{(i)}}{de}\text{; si excipis } B_c^{(o)} = -\tfrac{35}{8}e^3$$

$$B_s^{(i)} = \sqrt{1-e^2}\left\{\frac{36-21e^2}{2ie^2}\frac{dR_2^{(i)}}{de} - \frac{12-3e^2+6i^2(1-e^2)^2}{ie^3} R_2^{(i)}\right\}\text{; si excipis } B_s^{(o)} = o$$

quae omnes ope excentricitatis Lunae computandae sunt, deinde

$$G_c^{(i')} = -\frac{i'^4(1-e^2)}{3e^2} R_2^{(i')} - \frac{i'^2 e}{6(1-e^2)}\frac{dR_2^{(i')}}{de} + \frac{1}{3(1-e^2)} R_{-2}^{(i')} - \frac{1}{3e}\frac{dR_{-2}^{(i')}}{de}\text{; si excipis } G_c^{(o)} = o$$

$$G_s^{(i')} = -\sqrt{1-e^2}\left\{\frac{i'^3}{3e}\frac{dR_2^{(i')}}{de} + \frac{i'}{3e^2} R_{-2}^{(i')}\right\}$$

$$K^{(i')} = -\frac{i'^2 e}{2(1-e^2)}\frac{dR_2^{(i')}}{de} + \frac{1}{1-e^2} R_{-2}^{(i')}\text{; si excipis } K^{(o)} = \frac{1}{(1-e^2)^{\frac{3}{2}}}$$

$$C_c^{(i')} = -\frac{i'^4}{6e} R_2^{(i')} - \frac{i'^2 e^2}{3(1-e^2)^2}\frac{dR_2^{(i')}}{de} + \frac{2e}{3(1-e^2)^2} R_{-2}^{(i')} + \frac{1}{3(1-e^2)}\frac{dR_{-2}^{(i')}}{de}\text{; si excipis } C_c^{(o)} = \frac{e}{(1-e^2)^{\frac{3}{2}}}$$

$$C_s^{(i')} = -\sqrt{1-e^2}\left\{\frac{i'^3}{6(1-e^2)}\frac{dR_2^{(i')}}{de} - \frac{i'}{3e(1-e^2)} R_{-2}^{(i')}\right\}$$

$$D_c^{(i')} = -\frac{i'^4(2+e^2)}{30e^3} R_2^{(i')} + \frac{i'^2 e^2 - 2i'^4(1-e^2)^2}{15e^2(1-e^2)}\frac{dR_2^{(i')}}{de} - \frac{2e^2+5i'^2(1-e^2)^2}{15e^3(1-e^2)} R_{-2}^{(i')} + \frac{2}{15e^2}\frac{dR_{-2}^{(i')}}{de}\text{; si excipis } D_c^{(o)} = o$$

$$D_s^{(i')} = -\sqrt{1-e^2}\left\{\frac{2i'^5(1-e^2)}{15e^3} R_2^{(i')} - \frac{i'^3(2+3e^2)}{30e^2(1-e^2)}\frac{dR_2^{(i')}}{de} + \frac{i'(2+3e^2)}{15e^3(1-e^2)} R_{-2}^{(i')} - \frac{i'}{3e^2}\frac{dR_{-2}^{(i')}}{de}\right\}$$

quae omnes ope excentricitatis Terrae computandae sunt.

Habetur vero

$$R_2^{(i)} = -\frac{\frac{2}{i^2}\left(\frac{ie}{2}\right)^i}{1.2.3\ldots i}\left\{1 - \frac{1}{i+1}\left(\frac{ie}{2}\right)^2 + \frac{1}{1.2.(i+1)(i+2)}\left(\frac{ie}{2}\right)^4 - \frac{1}{1.2.3\,(i+1)(i+2)(i+3)}\left(\frac{ie}{2}\right)^6 \pm \text{etc.}\right\}$$

ubi tamen $i = o$ excipienda est. Ad coefficientes $R_{-2}^{(i)}$ investigandos habemus

$$\frac{df}{dg} = \frac{a^2}{r^2}\sqrt{1-e^2}$$

unde

$$\frac{a^2}{r^2} = \frac{1}{\sqrt{1-e^2}}\,\frac{df}{dg}$$

Si sit

$$f = g + 2\Sigma_1^\infty\, L^{(i)} \sin ig$$

ubi igitur $2L^{(i)}$ coefficientes notos aequationis centri designant, erit

$$\frac{df}{dg} = 1 + 2\Sigma_1^\infty\, i L^{(i)} \cos ig$$

unde

$$R_{-2}^{(i)} = \frac{\pm i}{\sqrt{1-e^2}}\, L^{(i)}$$

ubi signum superius pro positivis, et signum inferius pro negativis ipsius i valoribus locum habet. Eaedem aequationes suppeditant

$$R_{-2}^{(o)} = \frac{1}{\sqrt{1-e^2}}$$

quo evolutio transcendentium nostrarum peracta est.

6.

Per analysin artt. praecc. coefficientes evolutae quantitatis Ω ad expressiones finitas reduximus, quibus coefficientes illi computari possunt, quantaecunque sunt excentricitates. Quum vero excentricitates et Terrae et Lunae minores sint, evolutiones harum transcendentium in series infinitas secundum potestates excentricitatis progredientes in calculo numerico perturbationum Lunae adhibere praestat. Quem in finem transcendentes $R_2^{(i)}$ et $R_{-2}^{(i)}$ evolutas primum appono,

$$R_2^{(0)} = 1 + \frac{3}{2} e^2$$
$$R_2^{(1)} = -e + \frac{1}{8} e^3 - \frac{1}{192} e^5 + \frac{1}{9216} e^7 \mp \text{etc.}$$
$$R_2^{(2)} = -\frac{1}{4} e^2 + \frac{1}{12} e^4 - \frac{1}{96} e^6 + \frac{1}{1440} e^8 \mp \text{etc.}$$
$$R_2^{(3)} = -\frac{1}{8} e^3 + \frac{9}{128} e^5 - \frac{81}{5120} e^7 \pm \text{etc.}$$
$$R_2^{(4)} = -\frac{1}{12} e^4 + \frac{1}{15} e^6 - \frac{1}{45} e^8 \pm \text{etc.}$$
$$R_2^{(5)} = -\frac{25}{384} e^5 + \frac{625}{9216} e^7 \mp \text{etc.}$$
$$R_2^{(6)} = -\frac{9}{160} e^6 + \frac{81}{1120} e^8 \mp \text{etc.}$$
$$R_2^{(7)} = -\frac{2401}{46080} e^7 \pm \text{etc.}$$
$$R_2^{(8)} = -\frac{16}{315} e^8 \pm \text{etc.}$$

e quibus differentiatis emergunt

$$\frac{dR_2^{(0)}}{de} = 3e$$
$$\frac{dR_2^{(1)}}{de} = -1 + \frac{3}{8} e^2 - \frac{5}{192} e^4 + \frac{7}{9216} e^6 \mp \text{etc.}$$
$$\frac{dR_2^{(2)}}{de} = -\frac{1}{2} e + \frac{1}{3} e^3 - \frac{1}{16} e^5 + \frac{1}{180} e^7 \mp \text{etc.}$$
$$\frac{dR_2^{(3)}}{de} = -\frac{3}{8} e^2 + \frac{45}{128} e^4 - \frac{567}{5120} e^6 \pm \text{etc.}$$
$$\frac{dR_2^{(4)}}{de} = -\frac{1}{3} e^3 + \frac{2}{5} e^5 - \frac{8}{45} e^7 \pm \text{etc.}$$
$$\frac{dR_2^{(5)}}{de} = -\frac{125}{384} e^4 + \frac{4375}{9216} e^6 \mp \text{etc.}$$
$$\frac{dR_2^{(6)}}{de} = -\frac{27}{80} e^5 + \frac{18}{140} e^7 \mp \text{etc.}$$
$$\frac{dR_2^{(7)}}{de} = -\frac{16807}{46080} e^6 \pm \text{etc.}$$
$$\frac{dR_2^{(8)}}{de} = -\frac{128}{315} e^7 \pm \text{etc.}$$

deinde ex notis aequationis centri coefficientibus computavi

$$R^{(0)}_{-2} = \frac{1}{\sqrt{1-e^2}}$$

$$R^{(1)}_{-2} = e + \frac{3}{8} e^3 + \frac{65}{192} e^5 + \frac{2675}{9216} e^7 + \text{etc.}$$

$$R^{(2)}_{-2} = \frac{5}{4} e^2 + \frac{1}{6} e^4 + \frac{21}{64} e^6 + \text{etc.}$$

$$R^{(3)}_{-2} = \frac{13}{8} e^3 - \frac{25}{128} e^5 + \frac{393}{1024} e^7 + \text{etc.}$$

$$R^{(4)}_{-2} = \frac{103}{48} e^4 - \frac{387}{480} e^6 + \text{etc.}$$

$$R^{(5)}_{-2} = \frac{1097}{384} e^5 - \frac{16621}{9216} e^7 + \text{etc.}$$

$$R^{(6)}_{-2} = \frac{1223}{320} e^6 \mp \text{etc.}$$

$$R^{(7)}_{-2} = \frac{47273}{9216} e^7 \mp \text{etc.}$$

e quibus differentiatis emergunt

$$\frac{dR^{(0)}_{-2}}{de} = \frac{e}{(1-e^2)^{\frac{3}{2}}}$$

$$\frac{dR^{(1)}_{-2}}{de} = 1 + \frac{9}{8} e^2 + \frac{325}{192} e^4 + \frac{18725}{9216} e^6 + \text{etc.}$$

$$\frac{dR^{(2)}_{-2}}{de} = \frac{5}{2} e + \frac{2}{3} e^3 + \frac{63}{32} e^5 + \text{etc.}$$

$$\frac{dR^{(3)}_{-2}}{de} = \frac{39}{8} e^2 - \frac{125}{128} e^4 + \frac{2751}{1024} e^6 + \text{etc.}$$

$$\frac{dR^{(4)}_{-2}}{de} = \frac{103}{12} e^3 - \frac{387}{80} e^5 + \text{etc.}$$

$$\frac{dR^{(5)}_{-2}}{de} = \frac{5485}{384} e^4 - \frac{116347}{9216} e^6 + \text{etc.}$$

$$\frac{dR^{(6)}_{-2}}{de} = \frac{3669}{160} e^5 \mp \text{etc.}$$

$$\frac{dR^{(7)}_{-2}}{de} = \frac{330911}{9216} e^6 \mp \text{etc.}$$

qui quidem termini in valoribus harum transcendentium appositi ad perturbationes Lunae quam accuratissime obtinendas sufficiunt. Substitutis his valoribus in expressionibus ipsarum $Q^{(i)}_c$, $Q^{(i)}_s$, $G^{(i')}_c$, etc. in art. praéc. datis, emergunt

$$Q_c^{(0)} = \frac{5}{2} e^2$$

$$Q_c^{(1)} = -\frac{3}{2} e + \frac{2}{3} e^3 - \frac{37}{768} e^5 \pm \text{etc.}$$

$$Q_c^{(2)} = \frac{1}{2} - \frac{5}{4} e^2 + \frac{11}{16} e^4 - \frac{179}{1440} e^6 \pm \text{etc.}$$

$$Q_c^{(3)} = \frac{1}{2} e - \frac{19}{16} e^3 + \frac{1053}{1280} e^5 \mp \text{etc.}$$

$$Q_c^{(4)} = \frac{1}{2} e^2 - \frac{5}{4} e^4 + \frac{47}{45} e^6 \mp \text{etc.}$$

$$Q_c^{(5)} = \frac{25}{48} e^3 - \frac{1075}{768} e^5 \pm \text{etc.}$$

$$Q_c^{(6)} = \frac{9}{16} e^4 - \frac{261}{160} e^6 \pm \text{etc.}$$

$$Q_c^{(7)} = \frac{2401}{3840} e^5 \mp \text{etc.}$$

$$Q_c^{(8)} = \frac{32}{45} e^6 \mp \text{etc.}$$

$$Q_c^{(-i)} = Q_c^{(i)}$$

$$Q_s^{(0)} = o$$

$$Q_s^{(1)} = -\frac{3}{2} e + \frac{33}{24} e^3 + \frac{19}{256} e^5 \pm \text{etc.}$$

$$Q_s^{(2)} = \frac{1}{2} - \frac{5}{4} e^2 + \frac{3}{4} e^4 - \frac{73}{720} e^6 \pm \text{etc.}$$

$$Q_s^{(3)} = \frac{1}{2} e - \frac{19}{16} e^3 + \frac{1087}{1280} e^5 \mp \text{etc.}$$

$$Q_s^{(4)} = \frac{1}{2} e^2 - \frac{5}{4} e^4 + \frac{763}{720} e^6 \mp \text{etc.}$$

$$Q_s^{(5)} = \frac{25}{48} e^3 - \frac{1075}{768} e^5 \pm \text{etc.}$$

$$Q_s^{(6)} = \frac{9}{16} e^4 - \frac{261}{160} e^6 \pm \text{etc.}$$

$$Q_s^{(7)} = \frac{2401}{3840} e^5 \mp \text{etc.}$$

$$Q_s^{(8)} = \frac{32}{45} e^6 \mp \text{etc.}$$

$$Q_s^{(-i)} = -Q_s^{(i)}$$

$$P^{(0)} = 1 + \frac{3}{2} e^2$$

$$P^{(1)} = -e + \frac{1}{8} e^3 - \frac{1}{192} e^5 \pm \text{etc.}$$

$$P^{(2)} = -\frac{1}{4} e^2 + \frac{1}{12} e^4 - \frac{1}{96} e^6 \pm \text{etc.}$$

$$P^{(3)} = -\frac{1}{8} e^3 + \frac{9}{128} e^5 \mp \text{etc.}$$

$$P^{(4)} = -\frac{1}{12} e^4 + \frac{1}{15} e^6 \mp \text{etc.}$$

$$P^{(5)} = -\frac{25}{384} e^5 \pm \text{etc.}$$

$$P^{(6)} = -\frac{9}{160} e^6 \pm \text{etc.}$$

$$P^{(-i)} = P^{(i)}$$

$$A_c^{(0)} = -\frac{5}{2} e - \frac{15}{8} e^3$$

$$A_c^{(1)} = \frac{1}{2} + \frac{27}{16} e^2 - \frac{95}{384} e^4 \pm \text{etc.}$$

$$A_c^{(2)} = -\frac{1}{4} e + \frac{7}{12} e^3 \mp \text{etc.}$$

$$A_c^{(3)} = -\frac{3}{16} e^2 + \frac{95}{256} e^4 \mp \text{etc.}$$

$$A_c^{(4)} = -\frac{7}{48} e^3 \pm \text{etc.}$$

$$A_c^{(5)} = -\frac{95}{768} e^4 \pm \text{etc.}$$

$$A_c^{(-i)} = A_c^{(i)}$$

$$A_s^{(0)} = o$$

$$A_s^{(1)} = \frac{1}{2} + \frac{5}{16} e^2 - \frac{151}{384} e^4 \pm \text{etc.}$$

$$A_s^{(2)} = -\frac{1}{4} e + \frac{5}{12} e^3 \mp \text{etc.}$$

$$A_s^{(3)} = -\frac{3}{16} e^2 + \frac{81}{256} e^4 \mp \text{etc.}$$

$$A_s^{(4)} = -\frac{7}{48} e^3 \pm \text{etc.}$$

$$A_s^{(5)} = -\frac{95}{768} e^4 \pm \text{etc.}$$

$$A_s^{(-i)} = -A_s^{(i)}$$

$$B_c^{(0)} = -\frac{35}{8}e^3$$
$$B_c^{(1)} = \frac{57}{16}e^2 - \frac{445}{256}e^4 \pm \text{etc.}$$
$$B_c^{(2)} = -\frac{9}{4}e + \frac{33}{8}e^3 \mp \text{etc.}$$
$$B_c^{(3)} = \frac{1}{2} - 3e^2 + \frac{591}{128}e^4 \mp \text{etc.}$$
$$B_c^{(4)} = \frac{3}{4}e - \frac{57}{16}e^3 \pm \text{etc.}$$
$$B_c^{(5)} = \frac{15}{16}e^2 - \frac{135}{32}e^4 \pm \text{etc.}$$
$$B_c^{(6)} = \frac{9}{8}e^3 \mp \text{etc.}$$
$$B_c^{(7)} = \frac{343}{256}e^4 \mp \text{etc.}$$
$$B_c^{(-i)} = B_c^{(i)}$$

$$B_s^{(0)} = o$$
$$B_s^{(1)} = \frac{57}{16}e^2 - \frac{595}{256}e^4 \pm \text{etc.}$$
$$B_s^{(2)} = -\frac{9}{4}e + \frac{33}{8}e^3 \mp \text{etc.}$$
$$B_s^{(3)} = \frac{1}{2} - 3e^2 + \frac{591}{128}e^4 \mp \text{etc.}$$
$$B_s^{(4)} = \frac{3}{4}e - \frac{57}{16}e^3 \pm \text{etc.}$$
$$B_s^{(5)} = \frac{15}{16}e^2 - \frac{135}{32}e^4 \pm \text{etc.}$$
$$B_s^{(6)} = \frac{9}{8}e^3 \mp \text{etc.}$$
$$B_s^{(7)} = \frac{343}{256}e^4 \mp \text{etc.}$$
$$B_s^{(-i)} = -B_s^{(i)}$$

quae omnes ad indicem i pertinent et ope excentricitatis Lunae computandae sunt; deinde

$$G_c^{(0)} = o$$

$$G_c^{(1)} = -\frac{1}{4}\,e' + \frac{1}{24}\,e'^3 \mp \text{etc.}$$

$$G_c^{(2)} = \frac{1}{2} - \frac{5}{4}\,e'^2 + \frac{41}{96}\,e'^4 \mp \text{etc.}$$

$$G_c^{(3)} = \frac{7}{4}\,e' - \frac{123}{32}\,e'^3 \pm \text{etc.}$$

$$G_c^{(4)} = \frac{17}{4}\,e'^2 - \frac{115}{12}\,e'^4 \pm \text{etc.}$$

$$G_c^{(5)} = \frac{845}{96}\,e'^3 \mp \text{etc.}$$

$$G_c^{(6)} = \frac{533}{32}\,e'^4 \mp \text{etc.}$$

$$G_c^{(7)} = \frac{228347}{7680}\,e'^5 \mp \text{etc.}$$

$$G_c^{(-i')} = G_c^{(i')}$$

$$G_s^{(0)} = o$$

$$G_s^{(1)} = -\frac{1}{4}\,e' + \frac{1}{48}\,e'^3 \mp \text{etc.}$$

$$G_s^{(2)} = \frac{1}{2} - \frac{5}{4}\,e'^2 + \frac{37}{96}\,e'^4 \mp \text{etc.}$$

$$G_s^{(3)} = \frac{7}{4}\,e' - \frac{123}{32}\,e'^3 \pm \text{etc.}$$

$$G_s^{(4)} = \frac{17}{4}\,e'^2 - \frac{115}{12}\,e'^4 \pm \text{etc.}$$

$$G_s^{(5)} = \frac{845}{96}\,e'^3 \mp \text{etc.}$$

$$G_s^{(6)} = \frac{533}{32}\,e'^4 \mp \text{etc.}$$

$$G_s^{(7)} = \frac{228347}{7680}\,e'^5 \mp \text{etc.}$$

$$G_s^{(-i')} = -\,G_s^{(i')}$$

$$K^{(0)} = \frac{1}{(1-e'^2)^{\frac{3}{2}}}$$

$$K^{(1)} = \frac{3}{2}\,e' + \frac{27}{16}\,e'^3 + \text{etc.}$$

$$K^{(2)} = \frac{9}{4}\,e'^2 + \frac{7}{4}\,e'^4 + \text{etc.}$$

$$K^{(3)} = \frac{53}{16} e'^3 + \text{etc.}$$

$$K^{(4)} = \frac{231}{48} e'^4 + \text{etc.}$$

$$K^{(5)} = \frac{1773}{256} e'^5 + \text{etc.}$$

$$K^{(-i')} = K^{(i')}$$

$$C_c^{(0)} = \frac{e'}{(1-e'^2)^{\frac{1}{2}}}$$

$$C_c^{(1)} = \frac{1}{2} + \frac{27}{16} e'^2 + \text{etc.}$$

$$C_c^{(2)} = \frac{3}{2} e' + \frac{7}{3} e'^3 + \text{etc.}$$

$$C_c^{(3)} = \frac{53}{16} e'^2 + \text{etc.}$$

$$C_c^{(4)} = \frac{77}{12} e'^3 + \text{etc.}$$

$$C_c^{(-i')} = C_c^{(i')}$$

$$C_s^{(0)} = o$$

$$C_s^{(1)} = \frac{1}{2} + \frac{5}{16} e'^2 + \text{etc.}$$

$$C_s^{(2)} = \frac{3}{2} e' + \frac{5}{12} e'^3 + \text{etc.}$$

$$C_s^{(3)} = \frac{53}{16} e'^2 + \text{etc.}$$

$$C_s^{(4)} = \frac{77}{12} e'^3 + \text{etc.}$$

$$C_s^{(-i')} = - C_s^{(i')}$$

$$D_c^{(0)} = o$$

$$D_c^{(1)} = \frac{1}{16} e'^2 + \text{etc.}$$

$$D_c^{(2)} = - \frac{1}{2} e' + \frac{5}{8} e'^3 + \text{etc.}$$

$$D_c^{(3)} = \frac{1}{2} - 3e'^2 \mp \text{etc.}$$

$$D_c^{(4)} = \frac{5}{2} e' - 11 e'^3 - \text{etc.}$$

$$D_c^{(5)} = \frac{127}{16}\, e'^2 - \text{etc.}$$

$$D_c^{(6)} = \frac{163}{8}\, e'^3 - \text{etc.}$$

$$D_c^{(-i')} = D_c^{(i')}$$

$$D_s^{(0)} = o$$

$$D_s^{(1)} = \frac{1}{16}\, e'^2 + \text{etc.}$$

$$D_s^{(2)} = -\frac{1}{2}\, e' + \frac{5}{8}\, e'^3 + \text{etc.}$$

$$D_s^{(3)} = \frac{1}{2} - 3e'^2 - \text{etc.}$$

$$D_s^{(4)} = \frac{5}{2}\, e' - 11e'^3 - \text{etc.}$$

$$D_s^{(5)} = \frac{127}{16}\, e'^2 - \text{etc.}$$

$$D_s^{(6)} = \frac{163}{8}\, e'^3 - \text{etc.}$$

$$D_s^{(-i')} = -D_s^{(i')}$$

quae omnes ad indicem i' pertinent et ope excentricitatis Terrae computandae sunt. Termini harum serierum hic appositi ad perturbationes Lunae usque ad millesimam minutae secundae partem computandas satis superque sufficiunt.

7.

Secundum praecedentia quantitas perturbatrix Ω formam induit hanc

$$a\Omega = [i,\, i']_x \cos\,(ig + i'g' + H_x)$$

ubi x index est ad quantitates decem Ω_1, Ω_2, etc. in art. 3. introductas deinceps referendus, et H_x arcus decem, qui ipsi $ig + i'g'$ in valoribus ipsarum Ω_1, Ω_2, etc. adiuncti sunt, deinceps denotat. Est igitur

$$[i,\, i']_1 = \frac{u^2}{1 + \frac{M}{m'}}\, [\tfrac{1}{4} - \tfrac{3}{2}\sin^2\tfrac{1}{2}I + \tfrac{3}{2}\sin^4\tfrac{1}{2}I]\, P^{(i)}\, K^{(i')}$$

$$[i,\, i']_2 = \frac{u^2}{1 + \frac{M}{m'}}\, [\tfrac{3}{4} - \tfrac{3}{2}\sin^2\tfrac{1}{2}I + \tfrac{3}{4}\sin^4\tfrac{1}{2}I]\, \{Q_c^{(i)} + Q_s^{(i)}\}\{G_c^{(i')} - G_s^{(i')}\}$$

etc.

$$H_1 = o$$
$$H_2 = n(2y - 2y' - 4\eta)t + 4k$$
etc.

Quum in expressione finita ipsius Ω arcui $v_,$ semper quantitas nyt adiuncta sit, erit

$$a\left(\frac{d\Omega}{dv_,}\right) = -f_x[i, i']_x \sin(ig + i'g' + H_x)$$

ubi valores decem ipsius f_x sunt

$$f_1 = o, \quad f_2 = 2, \quad f_3 = 2, \quad f_4 = o, \quad f_5 = 2$$
$$f_6 = 1, \quad f_7 = 3, \quad f_8 = 1, \quad f_9 = 3, \quad f_{10} = 1$$

praeterea quum prior ipsius Ω pars per r^2 et posterior per r^3 multiplicata sit, habemus

$$ar\left(\frac{d\Omega}{dr}\right) = h_x[i, i']_x \cos(ig + i'g' + H_x)$$

ubi valores decem ipsius h_x sunt

$$h_1 = h_2 = h_3 = h_4 = h_5 = 2$$
$$h_6 = h_7 = h_8 = h_9 = h_{10} = 3$$

8.

Revertamur ad expressionem ipsius T hanc

$$T = \frac{n}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{r}\cos(f-\varphi) - 1 + \frac{2\varrho}{a(1-e^2)}\cos(f-\varphi) - \frac{2\varrho}{a(1-e^2)}\right\} a\left(\frac{d\Omega}{dv_,}\right)$$
$$+ \frac{n}{\sqrt{1-e^2}}\, 2\frac{\varrho}{r}\sin(f-\varphi)\, ar\left(\frac{d\Omega}{dr}\right) - \frac{ny}{\sqrt{1-e^2}} \cdot \frac{d.\frac{\varrho^2}{a^2}}{dy}$$

ubi, designantibus f anomaliam veram Lunae adiumento t, et φ eandem adiumento τ computandam, $f + nyt + \pi$ loco $v_,$ et $\varphi + nyt + \pi$ loco λ substitui. Positis

$$\frac{1}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{a}\cos\varphi\left[\frac{a}{r}\cos f + \frac{\cos f}{1-e^2}\right] + 2\frac{\varrho}{a}\sin\varphi\left[\frac{a}{r}\sin f + \frac{\sin f}{1-e^2}\right] - \frac{2\varrho}{a(1-e^2)} - 1\right\} = \Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty} X_c^{\varkappa, k}\cos(\varkappa\gamma + kg)$$
$$\frac{1}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{a}\cos\varphi \cdot \frac{a}{r}\sin f - 2\frac{\varrho}{a}\sin\varphi \cdot \frac{a}{r}\cos f\right\} = \Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty} X_s^{\varkappa, k}\sin(\varkappa\gamma + kg)$$

ubi $\varkappa$ et k numeri integri sunt, habetur

$$T = n\,\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty} X_{c}^{\varkappa,k} \cos(\varkappa\gamma + kg)\, a\left(\frac{d\Omega}{dv_{,}}\right) + n\,\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty} X_{s}^{\varkappa,k} \sin(\varkappa\gamma + kg)\, ar\left(\frac{d\Omega}{dr}\right) - \frac{ny}{\sqrt{1-e^2}}\frac{d.\frac{\varrho^2}{a^2}}{d\gamma}$$

Si in hac expressione valores ipsarum $a\left(\frac{d\Omega}{dv_{,}}\right)$ et $ar\left(\frac{d\Omega}{dr}\right)$ ex art. praec. petendi, et valor ipsius $\frac{d.\left(\frac{\varrho}{a}\right)^2}{d\gamma}$, mutata t in τ, ex art. 4. sumendus substituti fuerint, emerget adiumento aequationum (2)

$$T = n\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{X_{s}^{\varkappa,k} h_x - X_{c}^{\varkappa,k} f_x\}[i,i']_x \sin[\varkappa\gamma + (k+i)g + i'g' + H_x] + n\frac{y}{\sqrt{1-e^2}}\Sigma_{-\infty}^{+\infty}\varkappa R_2^{(\varkappa)} \sin\varkappa\gamma \;\ldots\ldots(3)$$

Ad $X_{c}^{\varkappa,k}$ et $X_{s}^{\varkappa,k}$ explicandas pono

$$2\frac{\varrho}{a}\cos\varphi = \Sigma_{-\infty}^{+\infty} S_c^{(\varkappa)} \cos\varkappa\gamma$$

$$2\frac{\varrho}{a}\sin\varphi = \Sigma_{-\infty}^{+\infty} S_s^{(\varkappa)} \sin\varkappa\gamma$$

$$\frac{a}{r}\cos f = \Sigma_{-\infty}^{+\infty} N_c^{(k)} \cos kg$$

$$\frac{a}{r}\sin f = \Sigma_{-\infty}^{+\infty} N_s^{(k)} \sin kg$$

$$\frac{\cos f}{1-e^2} = \Sigma_{-\infty}^{+\infty} U_c^{(k)} \cos kg$$

$$\frac{\sin f}{1-e^2} = \Sigma_{-\infty}^{+\infty} U_s^{(k)} \sin kg$$

$$2\frac{\varrho}{a(1-e^2)} = \Sigma_{-\infty}^{+\infty} F^{(\varkappa)} \cos\varkappa\gamma$$

hinc nanciscimur

$$\left\{2\frac{\varrho}{a}\cos\varphi\left[\frac{a}{r}\cos f + \frac{\cos f}{1-e^2}\right] + 2\frac{\varrho}{a}\sin\varphi\left[\frac{a}{r}\sin f + \frac{\sin f}{1-e^2}\right] - \frac{2\varrho}{a(1-e^2)} - 1\right\} =$$
$$\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{S_c^{(\varkappa)}[N_c^{(k)} + U_c^{(k)}] - S_s^{(\varkappa)}[N_s^{(k)} + U_s^{(k)}]\}\cos(\varkappa\gamma + kg) - \Sigma_{-\infty}^{+\infty} F^{(\varkappa)}\cos\varkappa\gamma - 1$$

$$\left\{2\frac{\varrho}{a}\cos\varphi.\frac{a}{r}\sin f - 2\frac{\varrho}{a}\sin\varphi.\frac{a}{r}\cos f\right\} = \Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\{S_c^{(\varkappa)} N_s^{(k)} - S_s^{(\varkappa)} N_c^{(k)}\}\sin(\varkappa\gamma + kg)$$

Si expressiones hae cum praecedentibus expressionibus ipsarum $X_{c}^{\varkappa,k}$ et $X_{s}^{\varkappa,k}$ comparatae erunt, emergit

$$X_{s}^{\varkappa,k} = \frac{1}{\sqrt{1-e^2}}\left\{S_c^{(\varkappa)} N_s^{(k)} - S_s^{(\varkappa)} N_c^{(k)}\right\}$$
$$X_{c}^{\varkappa,k} = \frac{1}{\sqrt{1-e^2}}\left\{S_c^{(\varkappa)}[N_c^{(k)} + U_c^{(k)}] - S_s^{(\varkappa)}[N_s^{(k)} + U_s^{(k)}]\right\}$$

si excipis

$$X_c^{\varkappa,0} = \frac{1}{\sqrt{1-e^2}} \left\{ S_c^{(\varkappa)} [N_c^{(0)} + U_c^{(0)}] - F^{(\varkappa)} \right\}$$

et si denuo excipis

$$X_c^{0,0} = \frac{1}{\sqrt{1-e^2}} \left\{ S_c^{(0)} [N_c^{(0)} + U_c^{(0)}] - F^{(0)} - 1 \right\}$$

Ad transcendentes has evolvendas habemus

$$2\,\frac{\varrho}{a} \cos\varphi = -\,\frac{1}{a}\,\frac{d.\varrho^2}{de}$$

$$2\,\frac{\varrho}{a} \sin\varphi = \frac{\sqrt{1-e^2}}{ea}\,\frac{d.\varrho^2}{d\gamma}$$

unde statim nanciscimur

$$S_c^{(\varkappa)} = -\frac{dR_1^{(\varkappa)}}{de}$$

$$S_s^{(\varkappa)} = -\frac{\sqrt{1-e^2}}{e}\,\varkappa R_2^{(\varkappa)}$$

deinde secundum art. 5.

$$F^{(\varkappa)} = \frac{e}{1-e^2}\,\frac{dR_1^{(\varkappa)}}{de};\ \text{si excipis}\ F^{(0)} = \frac{2+e^2}{1-e^2}$$

Ad ceteras transcendentes obtinendas animadverto esse

$$\frac{a}{r}\cos f = \frac{a^2}{r^2}\,\frac{1-e^2}{e} - \frac{a}{re} = \frac{1-e^2}{e}\,\Sigma_{-\infty}^{+\infty} R_{-2}^{(k)} \cos kg - \frac{1}{e}\,\Sigma_{-\infty}^{+\infty} R_{-1}^{(\varkappa)}$$

et eodem modo, uti in initio art. 5., differentiatam quantitatem $\frac{dlr}{dg}$ suppeditare

$$\frac{d^2lr}{dg^2} = 2\,\frac{a^4}{r^4}(1-e^2) - 3\,\frac{a^3}{r^3} + \frac{a^2}{r^2} = \Sigma_{-\infty}^{+\infty} \left\{ 2(1-e^2) R_{-4}^{(k)} - 3R_{-3}^{(k)} + R_{-2}^{(k)} \right\} \cos kg$$

unde

$$\frac{a}{r}\sin f = \frac{\sqrt{1-e^2}}{e} \cdot \frac{dlr}{dg} = \frac{\sqrt{1-e^2}}{e}\,\Sigma_{-\infty}^{+\infty}\,\frac{1}{k} \left\{ 2(1-e^2) R_{-4}^{(k)} - 3R_{-3}^{(k)} + R_{-2}^{(k)} \right\} \sin kg$$

cui integrali constans non est addenda. Denique

$$\frac{\cos f}{1-e^2} = \frac{a}{er} - \frac{1}{e(1-e^2)} = \frac{1}{e}\,\Sigma_{-\infty}^{+\infty} R_{-1}^{(k)} \cos kg - \frac{1}{e(1-e^2)}$$

$$\frac{\sin f}{1-e^2} = \frac{1}{ae\sqrt{1-e^2}} \cdot \frac{dr}{dg} = -\frac{1}{e\sqrt{1-e^2}}\,\Sigma_{-\infty}^{+\infty} k R_1^{(k)} \sin kg$$

itaque ope formularum art. 5. expeditur

$$N_c^{(k)} = \frac{1-e^2}{e} R_{-2}^{(k)} + \frac{k^2}{2e} R_2^{(k)}; \text{ si excipis } N_c^{(o)} = -\frac{e}{1+\sqrt{1-e^2}}$$

$$N_s^{(k)} = \frac{k}{2\sqrt{1-e^2}} \frac{dR_2^{(k)}}{de} - \frac{e}{k\sqrt{1-e^2}} R_{-2}^{(k)} + \frac{\sqrt{1-e^2}}{k} \frac{dR_{-2}^{(k)}}{de}; \text{ si excipis } N_s^{(o)} = o$$

$$U_c^{(k)} = -\frac{k^2}{2e} R_2^{(k)}; \text{ si excipis } U_c^{(o)} = -\frac{e}{1-e^2}$$

$$U_s^{(k)} = -\frac{k}{2\sqrt{1-e^2}} \frac{dR_2^{(k)}}{de}$$

quibus formulis transcendentes nostrae ad coefficientes evolutionis ipsarum r^2 et r^{-2} et earum quotientes differentiales respectu excentricitatis reductae sunt. Substitutis his formulis, quantitates $X_c^{\varkappa, k}$ et $X_s^{\varkappa, k}$ per easdem transcendentes expressae erunt. His vero quantitatibus forma concinnior attribui potest. Quum sit

$$-\frac{a}{r} \cos f = \frac{dlr}{de} \text{ et } \frac{a}{r} \frac{\sin f}{\sqrt{1-e^2}} = \frac{1}{e} \frac{d.lr}{dg}$$

pono

$$\frac{eN_s^{(k)}}{k\sqrt{1-e^2}} = W^{(k)}$$

ubi casus $k = o$ excipiendus est. Hinc habetur

$$N_c^{(k)} = \frac{dW^{(k)}}{de}$$

si excipis

$$N_c^{(o)} = -\frac{e}{1+\sqrt{1-e^2}}$$

Porro formulae praecedentes suppeditant

$$\frac{N_c^{(k)} + U_c^{(k)}}{\sqrt{1-e^2}} = \frac{\sqrt{1-e^2}}{e} R_{-2}^{(k)}$$

$$N_s^{(k)} + U_s^{(k)} = -\frac{e}{k\sqrt{1-e^2}} R_{-2}^{(k)} + \frac{\sqrt{1-e^2}}{k} \frac{dR_{-2}^{(k)}}{de}$$

itaque posita

$$e \frac{N_c^{(k)} + U_c^{(k)}}{k\sqrt{1-e^2}} = V^{(k)}$$

habetur

$$\frac{dV^{(k)}}{de} = N^{(k)}_{s} + U^{(k)}_{s}$$

si excipis casum quo $k = o$. Hinc colligitur esse

$$X^{\varkappa, k}_{s} = \varkappa\,\frac{R^{(\varkappa)}_{2}}{e}\left(\frac{dW^{(k)}}{de}\right) - \left(\frac{dR^{(\varkappa)}_{2}}{de}\right) k\,\frac{W^{(k)}}{e}$$

$$X^{\varkappa, k}_{o} = \varkappa\,\frac{R^{(\varkappa)}_{2}}{e}\left(\frac{dV^{(k)}}{de}\right) - \left(\frac{dR^{(\varkappa)}_{2}}{de}\right) k\,\frac{V^{(k)}}{e}$$

si excipis

$$X^{\varkappa, o}_{o} = \frac{e}{1 - e^2 + \sqrt{1-e^2}}\left(\frac{dR^{(\varkappa)}_{2}}{de}\right)$$

et si denuo excipis

$$X^{o, o}_{o} = -\frac{1}{\sqrt{1-e^2}}\left\{3 - \frac{3e^2}{1+\sqrt{1-e^2}}\right\}$$

quibus in formulis nobis est

$$V^{(k)} = \frac{\sqrt{1-e^2}}{k}\,R^{(k)}_{-2}$$

ubi casus $k = o$ omnino excipitur, porro

$$W^{(k)} = \frac{e}{2(1-e^2)}\,\frac{dR^{(k)}_{2}}{de} - \frac{e^2}{k^2(1-e^2)}\,R^{(k)}_{-2} + \frac{e}{k^2}\,\frac{dR^{(k)}_{-2}}{de}\,.$$

sive, id quod ad eius evolutionem in seriem commodissimum est

$$W^{(k)} = \int\left\{\frac{1-e^2}{e}\,R^{(k)}_{-2} + \frac{k^2}{2e}\,R^{(k)}_{2}\right\} de$$

si excipis

$$W^{(o)} = -\int\frac{e}{1+\sqrt{1-e^2}}\,de$$

quibus integralibus constans addenda non est.

Transformationem quidem ipsius T in theoria nostra Iovis atque Saturni explicatam in theoria quoque Lunae in usum vocare nobis licuisset, sed quum illa ad formam hanc

$$\left\{i\,C^{\varkappa, k} + f_x\,D^{\varkappa, k}\right\}[i, i']_x$$

coefficientium ipsius T perduceret, coefficientes igitur ipsius $[i, i']_x$ functiones indicum $\varkappa$, k, x et i forent: calculus supra adhibitus, qui coefficientes hos functiones indicum $\varkappa$, k et x absque i suppeditat, hoc loco

praeferendus erat. In theoria vero planetarum, ubi ratio illa simplex inter $\left(\frac{d\Omega}{dv_{\prime}}\right)$ et $r\left(\frac{d\Omega}{dr}\right)$, scilicet $-hx : fx$, locum non habet, et ubi $\left(\frac{d\Omega'}{dt'}\right)$ et $r'\left(\frac{d\Omega'}{dr'}\right)$, quoties reciprocae duorum planetarum perturbationes computandae sunt, ex Ω et $r\left(\frac{d\Omega}{dr}\right)$ facillima opera obtinentur, transformatio illa praeferenda est.

9.

Theorema illud in art. 8. theoriae nostrae Iovis atque Saturni demonstratum, secundum quod termini aut ipsius T aut ipsius $\int T\,dt$, etc., in quibus sine respectu signi algebraici $\varkappa$ maior est quam 1, ex terminis facillime computantur, in quibus $\varkappa = 1$ et $\varkappa = -1$, etiam in Lunae theoria locum habet, et ad calculi perturbationum tum primi ordinis tum ordinum altiorum compendium maxime confert. Loco huius theorematis, cuius in theoria planetarum adhibendi demonstratio loco excitato invenitur, hoc loco theorema generalius demonstrabimus.

Sit Γ functio huius formae

$$\Gamma = G\frac{\varrho}{a}\cos\varphi + H\frac{\varrho}{a}\sin\varphi + L$$

ubi G, H atque L functiones solius variabilis t sunt, quas in series infinitas secundum sinus et cosinus arcuum $\alpha t + \beta$, $\alpha' t + \beta'$, etc. progredientes, denotantibus α, β, α', β', etc. constantes, evolvere licet. Si in hac aequatione loco $\frac{\varrho}{a}\cos\varphi$ atque $\frac{\varrho}{a}\sin\varphi$ evolutiones earum in series in praecedentibus datas substituerimus, nanciscimur

$$\Gamma = L - \tfrac{3}{2}eG - \tfrac{1}{2}G\,\Sigma_{-\infty}^{+\infty}\frac{dR_{2}^{(\varkappa)}}{de}\cos\varkappa\gamma - \tfrac{1}{2}\frac{\sqrt{1-e^2}}{e}H\,\Sigma_{-\infty}^{+\infty}\varkappa R_{2}^{(\varkappa)}\sin\varkappa\gamma$$

ubi sub signis summationis valor $\varkappa = o$ excludendus est, quia terminum ad hunc spectantem separatim adscripsimus. Secundum hypothesin statutam ponere licet

$$-\tfrac{1}{2}G = \Sigma V\cos(\alpha t + \beta)\ ,\ \tfrac{1}{2}\frac{\sqrt{1-e^2}}{e}H = \Sigma W\sin(\alpha t + \beta)$$

unde

$$\Gamma = L - \tfrac{3}{2} eG + \Sigma\, \Sigma_{-\infty}^{+\infty} \alpha^{(\varkappa)} \cos\,[\varkappa\gamma + \alpha t + \beta]$$

ubi in summatione etiam valor $\varkappa = o$ excludendus est, quia termini omnes, in quibus $\varkappa = o$ est, extra summationis signum appositi sunt, et ubi

$$\alpha^{(\varkappa)} = V \frac{dR_2^{(\varkappa)}}{de} + W \varkappa R_2^{(\varkappa)}$$

Positis deinceps in hac aequatione et $\varkappa + 1$, et $-\varkappa$, et $-\varkappa - 1$ loco $\varkappa$, elicitur

$$\alpha^{(\varkappa+1)} = V \frac{dR_2^{(\varkappa+1)}}{de} + W(\varkappa + 1)\, R_2^{(\varkappa+1)}$$

$$\alpha^{(-\varkappa)} = V \frac{dR_2^{(\varkappa)}}{de} - W \varkappa R_2^{(\varkappa)}$$

$$\alpha^{(-\varkappa-1)} = V \frac{dR_2^{(\varkappa+1)}}{de} - W(\varkappa + 1)\, R_2^{(\varkappa+1)}$$

ubi V et W ubique eaedem quantitates sunt, quoniam e $\varkappa$ non pendent. Eliminatis deinceps V et W ope primae et tertiae harum quatuor aequationum, nanciscimur

$$V = \frac{\alpha^{(\varkappa)} + \alpha^{(-\varkappa)}}{2\left(\frac{dR_2^{(\varkappa)}}{de}\right)}$$

$$W = \frac{\alpha^{(\varkappa)} - \alpha^{(-\varkappa)}}{2 \varkappa R_2^{(\varkappa)}}$$

quae in secunda et quarta aequatione substitutae subministrant

$$(4)\ldots\ldots \begin{cases} \alpha^{(\varkappa+1)} = \eta^{(\varkappa)} \,.\, \alpha^{(\varkappa)} + \theta^{(\varkappa)} \,.\, \alpha^{(-\varkappa)} \\ \alpha^{(-\varkappa-1)} = \eta^{(\varkappa)} \,.\, \alpha^{(-\varkappa)} + \theta^{(\varkappa)} \,.\, \alpha^{(\varkappa)} \end{cases}$$

ubi $\varkappa$ absolute, hoc est sine respectu signi sui algebraici sumenda est, et ubi

$$\eta^{(\varkappa)} = \tfrac{1}{2} \left\{ \frac{\left(\frac{dR_2^{(\varkappa+1)}}{de}\right)}{\left(\frac{dR_2^{(\varkappa)}}{de}\right)} + \frac{(\varkappa + 1)\, R^{(\varkappa+1)}}{\varkappa R_2^{(\varkappa)}} \right\}$$

$$\theta^{(\varkappa)} = \tfrac{1}{2} \left\{ \frac{\left(\frac{dR_2^{(\varkappa+1)}}{de}\right)}{\left(\frac{dR_2^{(\varkappa)}}{de}\right)} - \frac{(\varkappa + 1)\, R^{(\varkappa+1)}}{\varkappa R_2^{(\varkappa)}} \right\}$$

Habemus igitur hoc

Theorema.

Quoties Γ est functio huius formae

$$\Gamma = L + G\,\frac{\varrho}{a}\cos\varphi + H\,\frac{\varrho}{a}\sin\varphi$$

ubi L, G atque H indolis supra designatae sunt: ope aequationum (4), quarum prior ad terminos ipsius Γ, in quibus $\varkappa$ positivus, posterior vero ad terminos, in quibus $\varkappa$ negativus, adhibenda est, termini, in quibus $\varkappa$ sine respectu signi sui algebraici maior est quam 1, ex iis, in quibus $\varkappa = 1$ et $\varkappa = -1$, facillima opera computantur, unde in evolutione ipsius Γ ceteroquin instituenda non nisi ad terminos, in quibus $\varkappa = o$, $\varkappa = 1$ et $\varkappa = -1$ est, respicere opus est.

Facile quoque reperitur theorema idem locum habere, quoties G formae $V\sin(\alpha t + \beta)$ et H formae $W\cos(\alpha t + \beta)$ sunt.

Iam facili computandi ratione expressio ipsius T in initio art. praec. data pro approximatione prima in hanc transfertur expressionem

$$\begin{aligned}\int T dt = \; & 2\,\frac{\varrho}{a}\cos\varphi\,\frac{an}{\sqrt{1-e^2}}\int\left\{\left(\frac{a}{r}\cos f + \frac{\cos f}{1-e^2} + \frac{e}{1-e^2}\right)\left(\frac{d\Omega}{dv_{\prime}}\right) + \frac{a}{r}\sin f\, r\left(\frac{d\Omega}{dr}\right)\right\} dt \\ & + 2\,\frac{\varrho}{a}\sin\varphi\,\frac{an}{\sqrt{1-e^2}}\int\left\{\left(\frac{a}{r}\sin f + \frac{\sin f}{1-e^2}\right)\left(\frac{d\Omega}{dv_{\prime}}\right) - \frac{a}{r}\cos f\, r\left(\frac{d\Omega}{dr}\right) - \frac{ey}{a\sqrt{1-e^2}}\right\} dt \\ & - 3\,\frac{an}{\sqrt{1-e^2}}\int\left(\frac{d\Omega}{dv_{\prime}}\right) dt\end{aligned}$$

unde manifestum est $\int T dt$ eiusdem formae esse ac Γ, itaque adiumento aequationis (3) terminos in quibus $\varkappa$ maior est quam ± 1 computari non oportere.

Infra monstrabitur, quousque theorema hoc in approximationibus subsequentibus locum habeat.

10.

Transcendentium $X_c^{\varkappa, k}$ et $X_s^{\varkappa, k}$ evolutiones in series infinitas in theoria Lunae commodissime adhibebuntur, quare per formulas supra traditas computavi

$$\frac{dW^{(0)}}{de} = -\frac{1}{2}e - \frac{1}{8}e^3 - \frac{1}{16}e^5 - \text{etc.}$$

$$W^{(1)} = \frac{1}{2}e - \frac{3}{16}e^3 - \frac{1}{128}e^5 - \frac{127}{18432}e^7 - \text{etc.}$$

$$W^{(2)} = \frac{3}{8}\,e^2 - \frac{11}{48}\,e^4 + \frac{3}{128}\,e^6 + \text{etc.}$$

$$W^{(3)} = \frac{17}{48}\,e^3 - \frac{77}{256}\,e^5 + \frac{743}{10240}\,e^7 + \text{etc.}$$

$$W^{(4)} = \frac{71}{192}\,e^4 - \frac{129}{320}\,e^6 + \text{etc.}$$

$$W^{(5)} = \frac{523}{1280}\,e^5 - \frac{10039}{18432}\,e^7 + \text{etc.}$$

$$W^{(6)} = \frac{899}{1920}\,e^6 - \text{etc.}$$

$$W^{(7)} = \frac{355081}{645120}\,e^7 - \text{etc.}$$

$$V^{(1)} = e - \frac{1}{8}\,e^3 + \frac{5}{192}\,e^5 + \frac{107}{9216}\,e^7 + \text{etc.}$$

$$V^{(2)} = \frac{5}{8}\,e^2 - \frac{11}{48}\,e^4 + \frac{17}{384}\,e^6 + \text{etc.}$$

$$V^{(3)} = \frac{13}{24}\,e^3 - \frac{43}{128}\,e^5 + \frac{95}{1024}\,e^7 + \text{etc.}$$

$$V^{(4)} = \frac{103}{192}\,e^4 - \frac{451}{960}\,e^6 + \text{etc.}$$

$$V^{(5)} = \frac{1097}{1920}\,e^5 - \frac{109433}{230400}\,e^7 + \text{etc.}$$

$$V^{(6)} = \frac{1223}{1920}\,e^6 - \text{etc.}$$

$$V^{(7)} = \frac{47273}{64512}\,e^7 - \text{etc.}$$

hinc emerserunt

$$X^{0,0}_c = -\frac{1}{\sqrt{1-e^2}}\left\{3 - \frac{3e^2}{1+\sqrt{1-e^2}}\right\}$$

$$X^{1,0}_c = -\frac{1}{2}\,e - \frac{3}{16}\,e^3 - \frac{71}{384}\,e^5 - \text{etc.}$$

$$X^{-1,1}_c = 2 - e^2 + \frac{9}{32}\,e^4 + \frac{35}{576}\,e^6 + \text{etc.}$$

$$X^{0,1}_c = -3e + \frac{3}{8}\,e^3 - \frac{5}{64}\,e^5 + \text{etc.}$$

$$X^{1,1}_c = -\frac{1}{12}\,e^4 - \frac{25}{384}\,e^6 - \text{etc.}$$

$$X^{-1,2}_c = \frac{5}{2}\,e - 2e^3 - \frac{87}{128}\,e^5 - \text{etc.}$$

$$X_c^{0,2} = -\frac{15}{4} e^2 + \frac{11}{8} e^4 - \frac{17}{64} e^6 + \text{etc.}$$

$$X_c^{1,2} = \frac{7}{48} e^3 - \frac{3}{32} e^5 - \text{etc.}$$

$$X_c^{-1,3} = \frac{13}{4} e^2 - \frac{7}{2} e^4 + \frac{401}{256} e^6 - \text{etc.}$$

$$X_c^{0,3} = -\frac{39}{8} e^3 + \frac{387}{128} e^5 - \text{etc.}$$

$$X_c^{1,3} = \frac{34}{128} e^4 - \frac{65}{512} e^6 + \text{etc.}$$

$$X_c^{-1,4} = \frac{103}{24} e^3 - \frac{277}{48} e^5 - \text{etc.}$$

$$X_c^{0,4} = -\frac{103}{16} e^4 + \frac{451}{80} e^6 + \text{etc.}$$

$$X_c^{1,4} = \frac{129}{320} e^5 - \text{etc.}$$

$$X_c^{-1,5} = \frac{1097}{192} e^4 - \frac{68429}{9600} e^6 + \text{etc.}$$

$$X_c^{0,5} = -\frac{1097}{128} e^5 + \text{etc.}$$

$$X_c^{1,5} = \frac{13579}{57600} e^6 - \text{etc.}$$

$$X_c^{-1,6} = \frac{1223}{160} e^5 - \text{etc.}$$

$$X_c^{0,6} = -\frac{3669}{320} e^6 + \text{etc.}$$

$$X_c^{1,7} = \frac{47273}{4608} e^6 + \text{etc.}$$

$$X_c^{-x,-k} = X_c^{x,k}$$

$$X_s^{0,0} = 0$$

$$X_s^{1,0} = \frac{1}{2} e + \frac{1}{16} e^3 + \frac{19}{384} e^5 + \text{etc.}$$

$$X_s^{-1,1} = 1 - e^2 + \frac{7}{64} e^4 - \frac{1}{18} e^6 + \text{etc.}$$

$$X_s^{0,1} = -\frac{3}{2} e + \frac{9}{16} e^3 + \frac{3}{128} e^5 + \text{etc.}$$

$$X_s^{1,1} = \frac{1}{4} e^2 + \frac{1}{24} e^4 + \frac{19}{512} e^6 + \text{etc.}$$

$$X_s^{-1,2} = \frac{3}{2} e - \frac{7}{4} e^3 + \frac{191}{384} e^5 + \text{etc.}$$

$$X^{0,2}_{s} = -\frac{9}{4}e^2 + \frac{11}{8}e^4 - \frac{9}{64}e^6 + \text{etc.}$$

$$X^{1,2}_{s} = \frac{13}{48}e^3 - \frac{1}{48}e^5 + \text{etc.}$$

$$X^{-1,3}_{s} = \frac{17}{8}e^2 - \frac{47}{16}e^4 + \frac{329}{256}e^6 + \text{etc.}$$

$$X^{0,3}_{s} = -\frac{51}{16}e^3 + \frac{693}{256}e^5 + \text{etc.}$$

$$X^{1,3}_{s} = \frac{43}{128}e^4 - \frac{113}{960}e^6 + \text{etc.}$$

$$X^{-1,4}_{s} = \frac{71}{24}e^3 - \frac{229}{48}e^5 + \text{etc.}$$

$$X^{0,4}_{s} = -\frac{71}{16}e^4 + \frac{387}{80}e^6 + \text{etc.}$$

$$X^{1,4}_{s} = \frac{419}{960}e^5 - \text{etc.}$$

$$X^{-1,5}_{s} = \frac{523}{128}e^4 - \frac{1451}{192}e^6 + \text{etc.}$$

$$X^{0,5}_{s} = -\frac{1569}{256}e^5 + \text{etc.}$$

$$X^{1,5}_{s} = \frac{1333}{2304}e^6 - \text{etc.}$$

$$X^{-1,6}_{s} = \frac{899}{160}e^5 - \text{etc.}$$

$$X^{0,6}_{s} = -\frac{2697}{320}e^6 + \text{etc.}$$

$$X^{-1,7}_{s} = \frac{355081}{46080}e^6 - \text{etc.}$$

$$X^{-n,-k}_{s} = -X^{n,k}_{s}$$

Praeterea computavi

$$\eta^{(1)} = \frac{1}{2}e - \frac{1}{8}e^3 \quad o e^5 + \text{etc.}$$

$$\eta^{(2)} = \frac{3}{4}e - \frac{3}{16}e^3 - \frac{1}{1536}e^5 - \text{etc.}$$

$$\eta^{(3)} = \frac{8}{9}e - \frac{2}{9}e^3 - \text{etc.}$$

$$\eta^{(4)} = \frac{125}{128}e - \frac{125}{512}e^3 - \text{etc.}$$

$$\eta^{(5)} = \frac{648}{625}e - \text{etc.}$$

$$\eta^{(6)} = \frac{16807}{15552}e - \text{etc.}$$

$$\theta^{(1)} = -\frac{1}{48}e^3 - \frac{1}{192}e^5 - \text{etc.}$$
$$\theta^{(2)} = -\frac{1}{64}e^3 - \frac{9}{1280}e^5 - \text{etc.}$$
$$\theta^{(3)} = -\frac{1}{90}e^3 - \text{etc.}$$
$$\theta^{(4)} = -\frac{25}{3072}e^3 - \text{etc.}$$

Hi termini appositi ad perturbationes Lunae usque ad millesimam minutae secundae partem computandas satis superque sufficiunt, ultimos enim terminos ubique fere abscindere licet. Substituto valore numerico excentricitatis Lunae in seriebus praecedentibus, quantitates $X_s^{x,k} h_x - X_c^{x,k} f_x$ pro diversis ipsius x valoribus facile computantur, et quatuor quidem casus secundum art. 7. aderunt, nam valor numericus $h_x = 2$ cum valoribus numericis $f_x = o$ et $f_x = 2$, nec non valor numericus $h_x = 3$ cum valoribus numericis $f_x = 1$ et $f_x = 3$ una existit.

11.

Computatis adiumento aequationis (3) omnibus ipsius T terminis in quibus $x = -1$, $x = o$ et $x = 1$, termini aderunt ipsi $x = 1$ respondentes hi

$$2n\,\Sigma_{-\infty}^{+\infty}\, X_s^{-1,k}\,[-k, o]_1 \sin(-\gamma) + n\frac{y_{\prime}}{\sqrt{1-e^2}}(-1)R_2^{(1)}\sin(-\gamma)$$
$$+2n\,\Sigma_{-\infty}^{+\infty}\, X_s^{1,k}\,[\,k,\ o\,]_1 \sin\gamma \qquad + n\frac{y_{\prime}}{\sqrt{1-e^2}}(+1)R_2^{(1)}\sin\gamma$$

Quodsi hi termini cifrae aequati fuerint, habebitur

$$y_{\prime} = -\frac{\Sigma_{-\infty}^{+\infty}\{X_s^{1,k} - X_s^{-1,k}\}[-k,\ o]_1}{R_2^{(1)}}\sqrt{1-e^2}$$

quae est expressio analytica ipsius $y_{\prime}$, qualis in approximatione prima elicitur, qua quidem expressione facile demonstratur $y_{\prime}$ semper esse debere quantitatem positivam. Concinnius vero computatio haec et subsequentes computationes ita explicantur.

Computatione terminorum ipsius T, in quibus $x = -1$, $x = o$ et $x = 1$ est, peracta, habetur

$$T = n\Sigma\{X^{\varkappa,k}_{i} h_x - X^{\varkappa,k}_{c} f_x\}[i,i']_x \sin[\varkappa\gamma + (k+i)g + i'g' + H_x] + n\frac{2y_{,}}{\sqrt{1-e^2}} R^{(1)}_2 \sin\gamma$$

sive

$$T = n\Sigma\{X^{\varkappa,k}_{i} h_x - X^{\varkappa,k}_{c} f_x\}[i-k,i']_x \sin[\varkappa\gamma + ig + i'g' + H_x] + n\frac{2y_{,}}{\sqrt{1-e^2}} R^{(1)}_2 \sin\gamma$$

quam quantitatem ita repraesentabo

$$(5)\dots\dots\quad T = n\Sigma\vartheta^{i,i',x}_{\varkappa} \sin(\varkappa\gamma + ig + i'g' + H_x) + n\,[\vartheta^{o,o,1}_{1} - \vartheta^{o,o,1}_{-1}]\sin\gamma + n\frac{2y_{,}}{\sqrt{1-e^2}} R^{(1)}_2 \sin\gamma$$

in qua expressione ipsi $\varkappa$ solummodo valores -1, 0, et 1 attributos esse supponitur, et

$$\vartheta^{i,i',x}_{\varkappa} = \Sigma\{X^{\varkappa,k}_{i} h_x - X^{\varkappa,k}_{c} f_x\}[i-k,\ i']_x$$

ubi tamen sub signo summationis casus in quo simul $\varkappa = 1$, $i = o$, $i' = o$, $x = 1$ et casus in quo simul $\varkappa = -1$, $i = o$, $i' = o$, $x = 1$ excludendus est, quia terminos ad hos casus pertinentes separatim adscripsi. Integrata autem haec expressio suppeditat

$$\int Tdt = -\Sigma\frac{\vartheta^{i,i',x}_{\varkappa}}{i + i'u + \nu_x}\cos[\varkappa\gamma + ig + i'g' + H_x] + nt[\vartheta^{o,o,1}_{1} - \vartheta^{o,o,1}_{-1}]\sin\gamma + nt\frac{2y_{,}}{\sqrt{1-e^2}} R^{(1)}_2 \sin\gamma$$

ubi ν_x coefficientem ipsius nt in H_x denotat, et ut supra u loco $\frac{n'}{n}$ scripta est. Itaque ad terminos per $nt \sin\gamma$ multiplicatos tollendos habemus aequationem hanc

$$(6)\dots\dots\quad y_{,} = -\frac{\vartheta^{o,o,1}_{1} - \vartheta^{o,o,1}_{-1}}{2R^{(1)}_2}\sqrt{1-e^2}$$

atque tum

$$(7)\dots\dots\quad \int Tdt = -\Sigma\frac{\vartheta^{i,i',x}_{\varkappa}}{i + i'u + \nu_x}\cos(\varkappa\gamma + ig + i'g' + H_x)$$

cuius valoris ipsius $y_{,}$ ope non modo termini formae $nt \sin\gamma$ sed omnes termini in $\int Tdt$ per t multiplicati sublati erunt. Omnes enim huius formae sunt

$$nt\,[\vartheta^{o,o,1}_{\varkappa} - \vartheta^{o,o,1}_{-\varkappa}] \sin\varkappa\gamma$$

ubi ipsi $\varkappa$ positivi tantum valores attribuendi sunt. Theorema vero in art. 9. demonstratum et ad hos terminos applicatum suppeditat, scripta $\vartheta^{o,o,1}_{\varkappa}$ loco $\alpha^{(\varkappa)}$,

$$\vartheta^{o,o,1}_{\varkappa+1} - \vartheta^{o,o,1}_{-\varkappa-1} = (\eta^{(\varkappa)} - \theta^{(\varkappa)})(\vartheta^{o,o,1}_{\varkappa} - \vartheta^{o,o,1}_{-\varkappa})$$

sive, substitutis valoribus ipsarum $\eta^{(\varkappa)}$ et $\theta^{(\varkappa)}$,

$$\vartheta^{0,0,1}_{\varkappa+1} - \vartheta^{0,0,1}_{-\varkappa-1} = \frac{(\varkappa+1)\, R_2^{(\varkappa+1)}}{\varkappa\, R_2^{(\varkappa)}} \left(\vartheta^{0,0,1}_{\varkappa} - \vartheta^{0,0,1}_{-\varkappa}\right)$$

unde

$$\frac{\vartheta^{0,0,1}_{\varkappa+1} - \vartheta^{0,0,1}_{-\varkappa-1}}{(\varkappa+1)\, R_2^{(\varkappa+1)}} = \frac{\vartheta^{0,0,1}_{\varkappa} - \vartheta^{0,0,1}_{-\varkappa}}{\varkappa\, R_2^{(\varkappa)}}$$

Ergo quum coefficiens ipsius $y_, \sin \varkappa\gamma$ in valore integro ipsius $\int Tdt$ sit ipsi $\varkappa R_2^{(\varkappa)}$ proportionalis, colligitur valore ipsius $y_,$ sub (6) dato terminos omnes in $\int Tdt$ per tempus ipsum multiplicatos sublatum iri; id quod semper fieri posse in art. 5. Sect. II. iam demonstravimus.

Aequationes igitur (5) et (7) monstrant, deletis terminis per $\sin \gamma$ multiplicatis, $\int Tdt$ ex T, multiplicata $\vartheta^{i,i',x}_{\varkappa}$ per factorem $\frac{1}{i+i'u+\nu_x}$, inverso signo algebraico producti et mutato sinu in cosinum, obtineri. Si producta haec simul per $\eta^{(1)}$, $\eta^{(2)}$, etc. $\theta^{(1)}$, etc. multiplicantur, termini ipsius $\int Tdt$ in quibus $\varkappa = 2$, $\varkappa = 3$, etc. $\varkappa = -2$, $\varkappa = -3$, etc. sunt, statim obtinentur et valor integer ipsius $\int Tdt$ huius formae erit

$$\begin{aligned}
&- \frac{\vartheta^{i,i',x}_{0}}{i+i'u+\nu_x} \cos (ig + i'g' + H_x) \\
&- \frac{\vartheta^{i-1,i',x}_{1}}{i-1+i'u+\nu^x} \cos [\gamma + (i-1) g + i'g' + H_x] - \frac{\vartheta^{i-2,i',x}_{2}}{i-2+i'u+\nu_x} \cos [2\gamma + (i-2) g + i'g' + H_x] - \text{etc.} \\
&- \frac{\vartheta^{i+1,i',x}_{-1}}{i+1+i'u+\nu_x} \cos [-\gamma + (i+1) g + i'g' + H_x] - \frac{\vartheta^{i+2,i',x}_{-2}}{i+2+i'u+\nu_x} \cos [-2\gamma + (i+2) g + i'g' + H_x] - \text{etc.}
\end{aligned}$$

quae forma generalis est, nec ullam patitur exceptionem.

12.

Secundum art. 2. habemus

$$W = -b + A_,(1-b)\xi + A_{,,}\xi^2 + \int \mathring{T}dt - \frac{y_,}{\sqrt{1-e^2}}(n) \int \frac{d \cdot \frac{(\rho)^2}{(a)^2}}{d\gamma}\, dt$$

sive

$$W = -b + A_,(1-b)\xi + A_{,,}\xi^2 + \int Tdt$$

quare ante omnia $A_,$ et $A_{,,}$ evolvendae sunt. Art. 6. Sect. III. suppeditat

$$A_{,} = 2\frac{\varrho}{a}\cos\varphi + 3e$$

$$A_{,,} = 3\frac{\varrho^2}{a^2}\cos^2\varphi + 12e\frac{\varrho}{a}\cos\varphi + 2\frac{\varrho}{a}\cos^2\varphi - \frac{5}{2} + 10e^2$$

quae ope formularum artt. 4. et 5., mutata t in τ, facile primum transformantur in

$$A_{,} = -\frac{d.\varrho^2}{a^2 de} + 3e$$

$$A_{,,} = \frac{3\varrho^2}{e^2a^2} - \frac{(4+6e^2)\varrho}{e^2a} + 2\frac{(1-e^2)^2a}{e^2\varrho} - \frac{1-\frac{15}{2}e^2-e^4}{e^2}$$

et deinde in

$$A_{,} = -2\Sigma_{\mathrm{I}}^{\infty}\frac{dR_2^{(i)}}{de}\cos i\gamma$$

$$A_{,,} = \Sigma_{\mathrm{I}}^{\infty}\left\{\frac{6-2i^2(1-e^2)^2}{e^2}R_2^{(i)} - \frac{4+6e^2}{e}\frac{dR_2^{(i)}}{de}\right\}\cos i\gamma$$

Ideo si ponitur

$$A_{,} = \Sigma_{\mathrm{I}}^{\infty} A_{,}^{(i)}\cos i\gamma$$
$$A_{,,} = \Sigma_{\mathrm{I}}^{\infty} A_{,,}^{(i)}\cos i\gamma$$

erit nobis

$$A_{,}^{(i)} = -2\frac{dR_2^{(i)}}{de}$$

$$A_{,,}^{(i)} = \frac{6-2i^2(1-e^2)^2}{e^2}R_2^{(i)} - \frac{4+6e^2}{e}\frac{dR_2^{(i)}}{de}$$

quae in series evolutae produnt

$$A_{,}^{(1)} = 2 - \frac{3}{4}e^2 - \frac{5}{96}e^4 - \text{etc.}$$

$$A_{,}^{(2)} = e - \frac{2}{3}e^3 + \text{etc.}$$

$$A_{,}^{(3)} = \frac{3}{4}e^2 - \frac{45}{64}e^4 + \text{etc.}$$

$$A_{,}^{(4)} = \frac{2}{3}e^3 - \text{etc.}$$

$$A_{,}^{(5)} = \frac{125}{192}e^4 - \text{etc.}$$

etc.

$$A_{\prime\prime}^{(1)} = e + \frac{37}{96} e^3 + \text{etc.}$$

$$A_{\prime\prime}^{(2)} = \frac{5}{2} - \frac{5}{2} e^2 + \text{etc.}$$

$$A_{\prime\prime}^{(3)} = 3e - \frac{9}{2} e^3 + \text{etc.}$$

$$A_{\prime\prime}^{(4)} = \frac{7}{2} e^2 + \text{etc.}$$

etc.

Substitutis his valoribus, nanciscimur

$$W = -b + \Sigma_1^\infty \{(1-b)\xi A_{\prime}^{(i)} + \xi^2 A_{\prime\prime}^{(i)}\} \cos i\gamma + \Sigma \alpha_x^{i,i',x} \cos(x\gamma + ig + i'g' + H_x)$$

ubi

$$\alpha_x^{i,i',x} = -\frac{\vartheta_x^{i,i',x}}{i + i'u + \nu_x}$$

et notandum est, propter valorem, quem ipsi y, attribuimus, esse

$$\alpha_x^{0,0,1} = 0$$

Hinc mutata τ in t elicitur

$$\overline{W} = -b + \Sigma_1^\infty \{(1-b)\xi A_{\prime}^{(i)} + \xi^2 A_{\prime\prime}^{(i)}\} \cos ig + \Sigma l^{i,i',x} \cos(ig + i'g' + H_x)$$

ubi

$$l^{i,i',x} = \alpha_0^{i,i',x} + \alpha_1^{i-1,i',x} + \text{etc.} + \alpha_{-1}^{i+1,i',x} + \text{etc.}$$

Ex art. 2. vero habetur

$$(n)z = g + n\int \overline{W}\, dt - \frac{y_{\prime}}{\sqrt{1-e^2}}\, n\int \frac{r^2}{a^2}\, dt$$

ex quo sicuti ex eo quod est

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2} e^2 + 2\, \Sigma_1^\infty R_2^{(i)} \cos ig$$

emergit

$$(n)z = g - b\,nt + \Sigma_1^\infty \left\{(1-b)\xi \frac{A_{\prime}^{(i)}}{i} + \xi^2 \frac{A_{\prime\prime}^{(i)}}{i}\right\} \sin ig - y_{\prime} \frac{2+3e^2}{2\sqrt{1-e^2}} nt - 2\Sigma \frac{y_{\prime} R_2^{(i)}}{i\sqrt{1-e^2}} \sin ig$$
$$+ l^{0,0,1} nt + [l^{1,0,1} - l^{-1,0,1}] \sin g + \Sigma \frac{l^{i,i',x}}{i + i'u + \nu_x} \sin(ig + i'g' + H_x)$$

ubi in ultimo termino sub signo Σ valores speciales indicum, pro quibus terminos separatim adscripsi, excipiendi sunt.

Quibus computationibus peractis, arbitrariae b et ξ ope aequationum determinandae sunt harum

$$o = -b + l^{0,0,1} - y_{,}\frac{2+3e^2}{2\sqrt{1-e^2}}$$

$$o = \xi^2 A^{(1)}_{,,} + (1-b)\xi A^{(1)} + l^{1,0,1} - l^{-1,0,1} - 2y_{,}\frac{R^{(1)}_{2}}{\sqrt{1-e^2}}$$

quo facto, in $(n)z$ praeter terminum nt qui in ipsa g continetur, nullus terminus neque tempori proportionalis, neque per sin g multiplicatus adest.

Substituto valore ipsius b ex priore praecedentium aequationum desumto in posteriore, emergit

$$o = \xi^2 B^{(1)} + \xi\left\{1 - l^{0,0,1} + y_{,}\frac{2+3e^2}{2\sqrt{1-e^2}}\right\} A^{(1)}_{,} + \left\{l^{1,0,1} - l^{-1,0,1} - 2y_{,}\frac{R^{(1)}_{2}}{\sqrt{1-e^2}}\right\}$$

quae aequatio quadratica valorem ipsius ξ suppeditat, pro quo ea aequationis huius radix habenda est, cuius valor approximatus est $=$

$$-\frac{l^{1,0,1} - l^{-1,0,1} - 2y_{,}\frac{R^{(1)}_{2}}{\sqrt{1-e^2}}}{1 - l^{0,0,1} + y_{,}\frac{2+3e^2}{2\sqrt{1-e^2}}}$$

Praeterea habetur

$$b = l^{0,0,1} - y_{,}\frac{2+3e^2}{2\sqrt{1-e^2}}$$

qui ipsarum b et ξ valores in coefficientibus ipsarum sin $2g$, sin $3g$, etc. expressionis praecedentis pro $(n)z$ substituendi sunt.

13.

In art. 2. allata est aequatio haec

$$w = C + \tfrac{1}{2}\varepsilon - \tfrac{1}{2}n\int\overline{\left(\frac{dW}{d\gamma}\right)}\,dt + \frac{y_{,}}{2\sqrt{1-e^2}}\,n\int\frac{d.\left(\frac{r}{a}\right)^2}{dg}\,dt$$

ubi

$$C = \text{term. const. in}\left\{-\frac{y_{,}}{2\sqrt{1-e^2}}\cdot\left(\frac{r}{a}\right)^2\right\}$$

et

$$\tfrac{1}{2}\varepsilon = \tfrac{1}{6}b + \tfrac{1}{12}b^2 - \tfrac{1}{2}e\xi - \tfrac{1}{4}(1+e^2)\xi^2$$

Valorem vero ipsius W in art. praec. datum ita quoque exhibere licet

$$W = \Sigma \alpha_{x}^{i,i',x} \cos(x\gamma + ig + i'g' + H_x)$$

ubi

$$\alpha_{x}^{i,i',x} = -\frac{\vartheta_{x}^{i,i',x}}{i + i'u + v_x}$$

si excipis

$$\alpha_{x}^{0,0,1} = (1-b)\xi A_{,}^{(x)} + \xi^2 A_{,,}^{(x)}$$

si denuo excipis

$$\alpha_{0}^{0,0,1} = -b$$

in quibus valores numericos quantitatum, quas continent, secundum regulas in praecedentibus traditas computatos substitutos esse supponitur.

Hinc habetur

$$\frac{dW}{d\gamma} = -\Sigma x \alpha_{x}^{i,i',x} \sin(x\gamma + ig + i'g' + H_x)$$

atque hinc

$$\left(\overline{\frac{dW}{d\gamma}}\right) = \Sigma \beta^{i,i',x} \sin(ig + i'g' + H_x)$$

si ponitur

$$\beta^{i,i',x} = -\alpha_{1}^{i-1,i',x} - 2\alpha_{2}^{i-2,i',x} - \text{etc.} + \alpha_{-1}^{i+1,i',x} + 2\alpha_{-2}^{i+2,i',x} + \text{etc.}$$

porro

$$n\int \frac{d.\left(\frac{r}{a}\right)^2}{dg} dt = 2\,\Sigma_1^\infty R_2^{(i)} \cos ig$$

et

$$\text{Term. const. in} \left\{-\left(\frac{r}{a}\right)^2\right\} = -1 - \tfrac{3}{2}e^2$$

Substitutis his valoribus in aequatione praecedenti pro w, nanciscimur

$$w = \tfrac{1}{6}b + \tfrac{1}{12}b^2 - \tfrac{1}{2}e\xi - \tfrac{1}{4}(1+e^2)\xi^2 - y_{,}\frac{2+3e^2}{4\sqrt{1-e^2}} + \frac{y_{,}}{\sqrt{1-e^2}}\Sigma_1^\infty R_2^{(i)}\cos ig + \tfrac{1}{2}\Sigma\frac{\beta^{i,i',x}}{i+i'u+v_x}\cos(ig+i'g'+H_x)$$

quae aequatio valorem ipsius w perfacile computandum suppeditat. Quibus absolutis, art. 11. Sect. III. suppeditat terminos primi ordinis ipsius $(S+\varepsilon)$, qui in perturbationum secundi ordinis computatione in usum vocandi sunt, per formulam hanc

$$S+\varepsilon = 2w + \delta\,\frac{dz}{dt} + y_{,}\frac{2+3e^2}{2\sqrt{1-e^2}} + y_{,}\frac{2}{\sqrt{1-e^2}}\Sigma_1^\infty R_2^{(i)}\cos ig$$

ubi

$$\delta\,\frac{dz}{dt} = \frac{dz}{dt} - 1$$

cuius quantitatis valor numericus ex computationibus praecedentibus iam praesto est. Eadem vero aequatio confirmando calculo numerico inserviet, si praeterea S calculo directo computatur, id quod fit per formulam hanc

$$S+\varepsilon = \varepsilon + \frac{1}{\sqrt{1-e^2}}\,\Sigma\,\frac{f_x[i,i']_x}{i+i'u+\nu_x}\cos(ig+i'g'+H_x)$$

Valoribus numericis ipsius $S+\varepsilon$ ex his duabus aequationibus, quae congruere debent, elicitis et comparatis, errores in calculo numerico perturbationum, si qui forte commissi sint, facile detegi et corrigi possunt.

14.

Formulae perturbationes primi ordinis ipsarum P, Q et K exhibentes secundum art. 12. Sect. III. sunt hae

$$\frac{dP}{dt} = -2(n)\alpha_{,}\sin\tfrac{1}{2}I - \frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dI}\right)\cos\tfrac{1}{2}I$$
$$+ n\,\frac{dp'}{n\cos i'dt}\cos\tfrac{1}{2}I\cos[\pi'-\nu+k-n(\alpha+\eta)t] - n\,\frac{dq'}{n\cos i'dt}\cos\tfrac{1}{2}I\sin[\pi'-\nu+k-n(\alpha+\eta)t]$$

$$\frac{dQ}{dt} = \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{d\nu}\right)\frac{\cos^2\frac{1}{2}I}{2\sin\frac{1}{2}I} - \tfrac{1}{2}\left(\frac{d\Omega}{dk}\right)\sin\tfrac{1}{2}I\right\}$$
$$- n\,\frac{dp'}{n\cos i'dt}\cos\tfrac{1}{2}I\sin[\pi'-\nu+k-n(\alpha+\eta)t] - n\,\frac{dq'}{n\cos i'dt}\cos\tfrac{1}{2}I\cos[\pi'-\nu+k-n(\alpha+\eta)t]$$

$$\frac{dK}{dt} = n\eta_{,} + \tfrac{1}{2}\,\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dI}\right)\operatorname{tg}\tfrac{1}{2}I$$
$$+ \tfrac{1}{2}n\,\frac{dp'}{n\cos i'dt}\operatorname{tg}\tfrac{1}{2}I\cos[\pi'-\nu+k-n(\alpha+\eta)t] - \tfrac{1}{2}n\,\frac{dq'}{n\cos i'dt}\operatorname{tg}\tfrac{1}{2}I\sin[\pi'-\nu+k-n(\alpha+\eta)t]$$

Pars ea ipsarum dp' et dq' cuius caussa attractio Lunae est, formam habet hanc

$$\frac{dp'}{n\cos i'dt} = \tfrac{1}{2}\,\frac{m}{M+m}\,\frac{\sin I}{\sqrt{1-e'^2}}\,\frac{a^3r'}{u\,a'^2r^3}\left\{\begin{array}{l}\cos[f+f'+n(y+y'+\alpha-\eta)t+\pi'+\nu+k]\\ -\cos[f-f'+n(y-y'+\alpha-\eta)t-\pi'+\nu+k]\end{array}\right\}$$

$$\frac{dq'}{n\cos i'dt} = -\tfrac{1}{2}\,\frac{m}{M+m}\,\frac{\sin I}{\sqrt{1-e'^2}}\,\frac{a^3r'}{u\,a'^2r^2}\left\{\begin{array}{l}\sin[f+f'+n(y+y'+\alpha-\eta)t+\pi'+\nu+k]\\ +\sin[f-f'+n(y-y'+\alpha-\eta)t-\pi'+\nu+k]\end{array}\right\}$$

et termini non-periodici attractione planetarum in motu orbitae terrae producti sunt hi

$$\frac{dp'}{\cos i' \, dt} = b + ct$$

$$\frac{dq'}{\cos i' \, dt} = b_{,} + c_{,}t$$

ubi b, $b_{,}$, c et $c_{,}$ quantitates numericae sunt ex noto Solis motu desumendae. Quas formulas, positis

$$\frac{b}{n} + \frac{c_{,}}{n^2(\alpha+\eta)} = -\beta \cos B$$

$$\frac{b_{,}}{n} - \frac{c}{n^2(\alpha+\eta)} = \beta \sin B$$

$$\frac{c}{n^2} = -\beta_{,} \cos B_{,}$$

$$\frac{c_{,}}{n^2} = \beta_{,} \sin B_{,}$$

sub hac redigere licet forma

$$\frac{dp'}{n \cos i' \, dt} = -\beta \cos B - \frac{\beta_{,} \sin B_{,}}{\alpha+\eta} - nt \, \beta_{,} \cos B_{,}$$

$$\frac{dq'}{n \cos i' \, dt} = \beta \sin B - \frac{\beta_{,} \cos B_{,}}{\alpha+\eta} + nt \, \beta_{,} \sin B_{,}$$

Substitutis his valoribus in expressionibus praecedentibus ipsarum dP et dQ, neglectisque excentricitatibus in valoribus ipsarum dp' et dq', id quod licitum est, nanciscimur in $\frac{dP}{dt}$ terminos hos

$$-n\beta \cos \tfrac{1}{2} I \cos [n(\alpha+\eta)t - \pi' + v - k + B] - \frac{n\beta_{,}}{\alpha+\eta} \cos \tfrac{1}{2} I \sin [n(\alpha+\eta)t - \pi' + v - k + B_{,}]$$
$$-n^2 t \beta_{,} \cos \tfrac{1}{2} I \cos [n(\alpha+\eta)t - \pi' + v - k + B_{,}]$$
$$+ \tfrac{1}{2} \frac{m}{M+m} \sin I \frac{na}{ua'} \left\{ \begin{array}{l} \cos [g + g' + n(y + y' + 2\alpha)t + 2v] \\ - \cos [g - g' + n(y - y' - 2\eta)t + 2k] \end{array} \right\}$$

et in $\frac{dQ}{dt}$ hos

$$-n\beta \cos \tfrac{1}{2} I \sin [n(\alpha+\eta)t - \pi' + v - k + B] + \frac{n\beta_{,}}{\alpha+\eta} \cos \tfrac{1}{2} I \cos [n(\alpha+\eta)t - \pi' + v - k + B_{,}]$$
$$-n^2 t \beta \cos \tfrac{1}{2} I \sin [n(\alpha+\eta)t - \pi' + v - k + B_{,}]$$
$$+ \tfrac{1}{2} \frac{m}{M+m} \sin I \frac{na}{ua'} \left\{ \begin{array}{l} \sin [g + g' + n(y + y' + 2\alpha)t + 2v] \\ + \sin [g - g' + n(y - y' - 2\eta)t + 2k] \end{array} \right\}$$

In his terminis ponere licet $-\pi'+v-k=-\Theta$, ubi, uti in art. 34 Sect. II. Θ denotat longitudinem nodi ascendentis orbitae Lunae cum ecliptica tempori $t=o$ respondentem.

Substituto hoc valore in expressionibus praecedentibus per dt multiplicatis, nanciscimur post integrationes peractas in P terminos hos

$$-\frac{\beta}{\alpha+\eta}\cos\tfrac{1}{2}I\sin[n(\alpha+\eta)t-\Theta+B]-\frac{nt\beta_{,}}{\alpha+\eta}\cos\tfrac{1}{2}I\sin[n(\alpha+\eta)t-\Theta+B_{,}]$$
$$+\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1+u+y+y'+2\alpha)}\sin[g+g'+n(y+y'+2\alpha)t+2v]$$
$$-\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1-u+y-y'-2\eta)}\sin[g-g'+n(y-y'-2\eta)t+2k]$$

et in Q hos

$$+\frac{\beta}{\alpha+\eta}\cos\tfrac{1}{2}I\cos[n(\alpha+\eta)t-\Theta+B]+\frac{nt\beta_{,}}{\alpha+\eta}\cos\tfrac{1}{2}I\cos[n(\alpha+\eta)t-\Theta+B_{,}]$$
$$-\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1+u+y+y'+2\alpha)}\cos[g+g'+n(y+y'+2\alpha)t+2v]$$
$$-\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1-u+y-y'-2\eta)}\cos[g-g'+n(y-y'-2\eta)t+2k]$$

Expressio ipsius Ω haec

$$a\Omega=[i,i']_x\cos(ig+i'g'+H_x)$$

in art. 7. definita suppeditat

$$(8)\ldots\ldots\left\{\begin{aligned}-a\left(\frac{d\Omega}{dI}\right)\cos\tfrac{1}{2}I&=P_x[i,i']_x\cos(ig+i'g'+H_x)\\ +a\left\{\left(\frac{d\Omega}{dv}\right)\frac{\cos^2\frac{1}{2}I}{2\sin\frac{1}{2}I}-\tfrac{1}{2}\left(\frac{d\Omega}{dk}\right)\sin\tfrac{1}{2}I\right\}&=Q_x[i,i']_x\sin(ig+i'g'+H_x)\end{aligned}\right.$$

ubi P_x et Q_x functiones ipsius I sunt, quae ex differentiatis expressionibus ipsarum $a\Omega_1$, $a\Omega_2$, etc. in art. 3. datis inveniuntur. Differentiationibus his institutis, emergit

$$P_1=\frac{\frac{3}{8}\sin\frac{1}{2}I-\frac{9}{2}\sin^3\frac{1}{2}I+3\sin^5\frac{1}{2}I}{\frac{1}{4}-\frac{3}{4}\sin^2\frac{1}{2}I+\frac{3}{4}\sin^4\frac{1}{2}I},\qquad Q_1=o$$
$$P_2=2\sin\tfrac{1}{2}I,\qquad Q_2=2\sin\tfrac{1}{2}I$$
$$P_3=-\frac{1-2\sin^2\frac{1}{2}I}{\sin\frac{1}{2}I},\qquad Q_3=-\frac{1-2\sin^2\frac{1}{2}I}{\sin\frac{1}{2}I}$$
$$P_4=P^3,\qquad Q_4=-\frac{1}{\sin\frac{1}{2}I}$$

$$P_5 = -2\,\frac{\cos^2\frac{1}{2}I}{\sin\frac{1}{2}I}\,, \qquad Q_5 = -2\,\frac{\cos^2\frac{1}{2}I}{\sin\frac{1}{2}I}$$

$$P_6 = \frac{\frac{33}{8}\sin\frac{1}{2}I - \frac{183}{8}\sin^3\frac{1}{2}I + \frac{75}{4}\sin^5\frac{1}{2}I}{\frac{3}{8} - \frac{33}{8}\sin^2\frac{1}{2}I + \frac{75}{8}\sin^4\frac{1}{2}I}\,, \qquad Q_6 = \sin\tfrac{1}{2}I$$

$$P_7 = \frac{\frac{15}{8}\sin\frac{1}{2}I - \frac{45}{8}\sin^3\frac{1}{2}I + \frac{15}{4}\sin^5\frac{1}{2}I}{\frac{5}{8} - \frac{15}{8}\sin^2\frac{1}{2}I + \frac{15}{8}\sin^4\frac{1}{2}I}\,, \qquad Q_7 = 3\sin\tfrac{1}{2}I$$

$$P_8 = -\frac{\frac{9}{4}\sin\frac{1}{2}I - \frac{69}{4}\sin^3\frac{1}{2}I + 15\sin^5\frac{1}{2}I}{\frac{9}{4}\sin^2\frac{1}{2}I - \frac{15}{2}\sin^4\frac{1}{2}I}\,, \qquad Q_8 = -\frac{\cos^2\frac{1}{2}I}{\sin\frac{1}{2}I}$$

$$P_9 = -\frac{\frac{15}{8}\sin\frac{1}{2}I - \frac{75}{8}\sin^3\frac{1}{2}I + \frac{15}{2}\sin^5\frac{1}{2}I}{\frac{15}{8}\sin^2\frac{1}{2}I - \frac{15}{4}\sin^4\frac{1}{2}I}\,, \qquad Q_9 = -\frac{1-3\sin^2\frac{1}{2}I}{\sin\frac{1}{2}I}$$

$$P_{10} = P_9\,, \qquad Q_{10} = \frac{1+\sin^2\frac{1}{2}I}{\sin\frac{1}{2}I}$$

Substitutis expressionibus praecedentibus in formulis pro $\frac{dP}{dt}$ et $\frac{dQ}{dt}$ in huius articuli initio allatis, emergunt post integrationem, si perpendimus esse

$$\begin{aligned} n(y+y'+2\alpha)t+2\nu &= H_8\,, & n(y-y'-2\eta)t+2k &= H_6 \\ y+y+2\alpha &= \nu_8\,, & y-y'-2\eta &= \nu_6 \end{aligned}$$

valores ipsarum P et Q hi

$$\begin{aligned} \delta P = &-\frac{\beta}{\alpha+\eta}\cos\tfrac{1}{2}I\sin[n(\alpha+\eta)t-\Theta+B] - \frac{nt\beta_{,}}{\alpha+\eta}\cos\tfrac{1}{2}I\sin[n(\alpha+\eta)t-\Theta+B_{,}] \\ &+\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1+u+\nu_8)}\sin(g+g'+H_8) \\ &-\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1-u+\nu_6)}\sin(g-g'+H_6) \\ &+\frac{1}{\sqrt{1-e^2}}\,\Sigma\,\frac{P_x[i,i']_x}{i+i'u+\nu_x}\sin(ig+i'g'+H_x) \end{aligned}$$

$$\begin{aligned} \delta Q = &\frac{\beta}{\alpha+\eta}\cos\tfrac{1}{2}I\cos[n(\alpha+\eta)t-\Theta+B] + \frac{nt\beta_{,}}{\alpha+\eta}\cos\tfrac{1}{2}I\cos[n(\alpha+\eta)t-\Theta+B_{,}] \\ &-\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1+u+\nu_8)}\cos(g+g'+H_8) \\ &-\frac{m}{M+m}\,\frac{a\sin I}{2a'u(1-u+\nu_6)}\cos(g-g'+H_6) \\ &-\frac{1}{\sqrt{1-e^2}}\,\Sigma\,\frac{Q_x[i,i']_x}{i+i'u+\nu_x}\cos(ig+i'g'+H_x) \end{aligned}$$

In hoc ipsius P valore terminus e $[o,o]_1$ pendens excludendus est, cuius loco habemus

$$\alpha = \frac{P_1[o,o]_1}{2\sin\frac{1}{2}I}\cdot\frac{1}{\sqrt{1-e^2}}$$

Quum valor numericus ipsius $\frac{\beta}{\alpha+\eta}\cos\frac{1}{2}I$ sit circiter 1,"5, et valores

numerici coefficientium expressionum praecedentium per m multiplicatorum circiter $3''$: sequitur in valore ipsius K terminos e dp' et dq' emersuros optimo iure negligi posse. Quibus positis, formulae in huius articuli initio allatae statim suppeditant pro approximatione prima

$$\delta K = -\delta P \frac{\sin \tfrac{1}{2} I}{2 \cos^2 \tfrac{1}{2} I}$$
$$\eta = \quad \alpha \operatorname{tg}^2 \tfrac{1}{2} I$$

His computationibus omnibus peractis, approximatio prima ad perturbationes Lunae obtinendas absoluta est, sed antequam computationes approximationis secundae explicabimus, necesse est ostendatur quomodo perturbationes longitudinis mediae logarithmique radii vectoris Solis ad formam hoc loco requisitam redigantur.

15.

Formulae praecedentes omnes supponunt, expressionem verae Solis longitudinis ad formam hanc

$$v'_{,} = \overline{f'} + ny't + \pi'$$

redactam esse, dum in theoria nostra planetarum ipsa y' non continetur. Perturbationibus Solis (sive terrae) ad formam praecedentem redigendis inserviunt, ut iam explicui, formulae art. 15. Sect. II. Suppono igitur datas

$$\lambda' = \overline{\varphi}'_{,} + \pi'$$
$$l\varrho' = l\overline{\varrho}'_{,} + \beta',$$

ubi $\overline{\varphi}'_{,}$ et $\overline{\varrho}'_{,}$ ex data quantitate $\zeta'_{,}$ et aequationibus ipsis (13) art. 13. Sect. II. plane similibus pendent, et inveniendas esse

$$(9)\ldots\ldots\left\{\begin{aligned} \lambda' &= \overline{\varphi}' + ny't + \pi' \\ l\varrho' &= l\overline{\varrho}' + \beta' \end{aligned}\right.$$

ubi $\overline{\varphi}'$ et β' ex incognita quantitate ζ' eodem modo quo illae ex $\zeta'_{,}$, pendent. Consideretur $ny't$ quasi incrementum ipsius $\pi'_{,}$: quod quum sit et in casu quem nunc tractamus, incrementum excentricitatis non existat, aequationes (22*) Sect. II., scriptis $\pi' + ny't$ loco π, π' loco π_0 et e' loco e_0 atque (e), suppeditant, si tertia altioresque ipsius $ny't$ potestates negliguntur,

$$\eta = \frac{e'}{1-e'^2}\, ny't\,, \quad \xi = -\frac{e'}{2(1-e'^2)}\, n^2 y'^2 t^2$$

Positis porro in aequationibus (26) et (27) Sect. II. n' loco (n), ζ' loco (ζ), $\zeta_{\prime}'$ loco τ, e' loco (e), a' loco (a), $\overline{\varphi}'$ loco $(\overline{\varphi})$, $\overline{\varrho}'$ loco $(\overline{\varrho})$, $\beta' - \beta_{\prime}'$ loco (β), nec non valoribus praecedentibus ipsarum η et ξ et praeterea $b = o$ et $(c) = o$, nanciscimur

$$\left.\begin{aligned} n'\zeta &= n'\zeta_{\prime}' - \frac{2e'}{1-e'^2} ny't \int \frac{\overline{\varrho}'}{a'} \sin\overline{\varphi}' \, n' d\zeta_{\prime}' + \frac{e'}{(1-e'^2)^2} n^2 y'^2 t^2 \int \left\{ e' \frac{\overline{\varrho}'^2}{a'^2} \sin^2\overline{\varphi}' - (1-e'^2) \frac{\overline{\varrho}'}{a'} \cos\overline{\varphi}' \right\} n' d\zeta_{\prime}' \\ \beta' &= \beta_{\prime}' + \frac{e'}{1-e'^2} ny't \frac{\overline{\varrho}'}{a'} \sin\overline{\varphi}' + \frac{e'}{2(1-e'^2)^2} n^2 y'^2 t^2 \left\{ e' \frac{\overline{\varrho}'^2}{a'^2} \sin^2\overline{\varphi}' + (1-e'^2) \frac{\overline{\varrho}'}{a'} \cos\overline{\varphi}' \right\} \end{aligned}\right\} \ldots\ldots (10)$$

quibus aequationibus problema nostrum solutum est *).

Neglecta primum y'^2, positisque $\overline{\varrho}_{\prime}'$ et $\overline{\varphi}_{\prime}'$ resp. loco $\overline{\varrho}'$ et $\overline{\varphi}'$, aequatio prior post integrationem peractam praebet

$$n'\zeta' = n'\zeta_{\prime}' - \frac{ny't}{\sqrt{1-e'^2}} \frac{\overline{\varrho}_{\prime}'^2}{a'^2}$$

Si in hac aequatione ponitur $\zeta_{\prime}' = o$, habetur

$$n'\zeta' = -\frac{ny't}{\sqrt{1-e'^2}} (1-e')^2$$

pro valore vero $\zeta_{\prime}' = o$ necessario esse debet $\overline{\varphi}_{\prime}' = o$, unde $\lambda' = \pi'$. Aequationes vero hae

$$\operatorname{tg}\overline{\varphi}' = \frac{\sqrt{1-e'^2} \, . \sin v'}{\cos v' - e'}$$

$$n'\zeta' = v' - e' \sin v' = -\frac{ny't}{\sqrt{1-e'^2}} (1-e')^2$$

ex aequationibus ipsis (17) Sect. II. analogis derivatae suppeditant usque ad quantitates tertii ordinis respectu ipsius y',

$$\varphi' = -ny't$$

quo ipsius $\overline{\varphi}'$ valore substituto, aequatio prior (9) etiam suppeditat $\lambda' = \pi'$. Hinc sequitur integralibus, quae in valore praecedenti ipsius ζ' continentur, constantem arbitrariam addendam non esse. Posita

$$n'\delta\zeta' = -\frac{ny't}{\sqrt{1-e'^2}} \frac{\overline{\varrho}_{\prime}'^2}{a'^2}$$

habetur

$$\frac{\overline{\varrho}'}{a'} \sin\overline{\varphi}' = \frac{\overline{\varrho}_{\prime}'}{a'} \sin\overline{\varphi}_{\prime}' + \frac{d \, . \frac{\overline{\varrho}_{\prime}'}{a'} \sin\overline{\varphi}_{\prime}'}{n' d\zeta_{\prime}'} \, n'\delta\zeta_{\prime}' = \frac{\overline{\varrho}_{\prime}'}{a'} \sin\overline{\varphi}_{\prime}' - \frac{ny't}{1-e'^2} \left\{ e' \frac{\overline{\varrho}_{\prime}'^2}{a'^2} \sin^2\overline{\varphi}_{\prime}' + (1-e'^2) \frac{\overline{\varrho}_{\prime}'}{a'} \cos\overline{\varphi}_{\prime}' \right\}$$

*) Vide aliam huius problematis solutionem in Astr. Nachr. No. 296. art. 76.

Substituta hac aequatione in (10), neglectaque tertia ipsius y' potestate, emergit post integrationes peractas

$$\begin{aligned}
n'\zeta' &= n'\zeta_{\prime}' - \frac{ny't}{\sqrt{1-e'^2}}\,\frac{\bar{\varrho}_{\prime}'^2}{a'^2} + \frac{n^2y'^2t^2}{(1-e'^4)^{\frac{3}{2}}}\,e'\,\frac{\bar{\varrho}_{\prime}'^3}{a'^3}\sin\bar{\varphi}_{\prime}' \\
\beta' &= \beta_{\prime}' + e'\frac{ny't}{1-e'^2}\,\frac{\bar{\varrho}_{\prime}'}{a'}\sin\bar{\varphi}_{\prime}' - e'\frac{n^2y'^2t^2}{2(1-e'^2)^2}\left\{e'\,\frac{\bar{\varrho}_{\prime}'^2}{a'^2}\sin^2\bar{\varphi}_{\prime}' + (1-e'^2)\,\frac{\bar{\varrho}_{\prime}'}{a'}\cos\bar{\varphi}_{\prime}'\right\}
\end{aligned}$$

Si hae aequationes formularum artt. 4. seqq. adminiculo evolutae, simul

$$y' = y_{\prime}' + \tfrac{1}{2}y'_{\prime\prime}nt$$

substituta, et τ in t mutata fuerit, habetur

$$\begin{aligned}
n'z' = n'z_{\prime}' &- \frac{1+\frac{3}{2}e'^2}{\sqrt{1-e'^2}}\,ny_{\prime}'t - 2\,\frac{ny_{\prime}'t}{\sqrt{1-e'^2}}\,\Sigma_{\mathrm{I}}^{\infty}R_2^{(i)}\cos i\,n'z_{\prime}' - \frac{1+\frac{3}{2}e'^2}{2\sqrt{1-e'^2}}\,y'_{\prime\prime}n^2t^2 - \frac{n^2y'_{\prime\prime}t^2}{\sqrt{1-e'^2}}\,\Sigma_{\mathrm{I}}^{\infty}R_2^{(i)}\cos i\,n'z_{\prime}' \\
&+ \frac{n^2y_{\prime}'^2t^2}{1-e'^2}\,\Sigma_{\mathrm{I}}^{\infty}\left\{\frac{5e'}{i}\,\frac{dR_2^{(i)}}{de} - \frac{6}{i}\,R_2^{(i)}\right\}\sin i\,n'z_{\prime}' \\
w' = w_{\prime}' &- \frac{ny_{\prime}'t}{\sqrt{1-e'^2}}\,\Sigma_{\mathrm{I}}^{\infty}\,i\,R_2^{(i)}\sin i\,n'z_{\prime}' + \frac{e'^2n^2y_{\prime}'^2t^2}{2(1-e'^2)} - \frac{n^2y'_{\prime\prime}t^2}{2\sqrt{1-e'^2}}\,\Sigma_{\mathrm{I}}^{\infty}\,i\,R_2^{(i)}\sin i\,n'z_{\prime}' \\
&+ \frac{n^2y_{\prime}'^2t^2}{1-e'^2}\,\Sigma_{\mathrm{I}}^{\infty}\left\{R_2^{(i)} - \frac{e'}{2}\,\frac{dR_2^{(i)}}{de}\right\}\cos i\,n'z_{\prime}'
\end{aligned}$$

Termini ipsarum $z_{\prime}'$ et $w_{\prime}'$, qui in motu Lunae vim habent, huius formae sunt

$$\begin{aligned}
n'z_{\prime}' = {} & n't + c' + t\,\Sigma_{\mathrm{I}}^{\infty}\,\alpha_s^{(i)}\sin i(n't+c') + t\,\Sigma_{\mathrm{I}}^{\infty}\,\alpha_c^{(i)}\cos i(n't+c') \\
& + t^2\,\Sigma_{\mathrm{I}}^{\infty}\,\beta_s^{(i)}\sin i(n't+c') + t^2\,\Sigma_{\circ}^{\infty}\,\beta_c^{(i)}\cos i(n't+c') \\
& + Z\sin[g - g' + n(y - y' - 2\eta)t + 2k] \\
w_{\prime}' = {} & t\;\Sigma_{\circ}^{\infty}\,\varepsilon_c^{(i)}\cos i(n't+c') + t\;\Sigma_{\mathrm{I}}^{\infty}\,\varepsilon_s^{(i)}\sin i(n't+c') \\
& + t^2\,\Sigma_{\circ}^{\infty}\,\theta_c^{(i)}\cos i(n't+c') + t^2\,\Sigma_{\mathrm{I}}^{\infty}\,\theta_s^{(i)}\sin i(n't+c') \\
& + W\cos(g - g' + n(y - y' - 2\eta)t + 2k)
\end{aligned}$$

ubi ultimi per coefficientes Z atque W multiplicati termini perturbationes a Luna ipsa productae sunt. Substitutis his valoribus in aequationibus praecedentibus, positisque

$$n' - \frac{1+\frac{3}{2}e'^2}{\sqrt{1-e'^2}}\,ny_{\prime}' = (n')$$

et ut antea

$$(n')t + c' = g'$$

emergunt hae

$$
\begin{aligned}
(n')z' = \;& g' + t\,\Sigma_1^\infty \alpha_s^{(i)} \sin ig' + t\,\Sigma_1^\infty \left\{\alpha_c^{(i)} - \frac{2ny_,'}{\sqrt{1-e'^2}} R_2^{(i)}\right\} \cos ig' \\
& + t^2\, \Sigma_1^\infty \left\{\beta_s^{(i)} + \frac{n^2 y_,'^2}{1-e'^2}\left\{\frac{5e'}{i}\frac{dR_2^{(i)}}{de'} - \frac{6}{i} R_2^{(i)}\right\}\right\} \sin ig' + t^2 \frac{2ny_,'}{\sqrt{1-e'^2}}\, n'\delta z'\, \Sigma_1^\infty\, i\, R_2^{(i)} \sin ig' \\
& + t^2\left\{\beta_c^{(o)} - \frac{1+\frac{3}{2}e'^2}{2\sqrt{1-e'^2}} y'_{,,} n^2\right\} + t^2\, \Sigma_1^\infty \left\{\beta_c^{(i)} + \frac{1+\frac{3}{2}e'^2}{\sqrt{1-e'^2}} ny_,' i\, \alpha_s^{(i)} - \frac{n^2 y'_{,,}}{\sqrt{1-e'^2}} R_2^{(i)}\right\} \cos ig' \\
& + Z \sin[g - g' + n(y - y' - 2\eta)t + 2k]
\end{aligned}
$$

$$
\begin{aligned}
w' = \;& t\,\Sigma_0^\infty \varepsilon_c^{(i)} \cos ig' + t\,\Sigma_1^\infty \left\{\varepsilon_s^{(i)} - \frac{ny_,'}{\sqrt{1-e'^2}}\, i\, R_2^{(i)}\right\} \sin ig' \\
& + t^2\left\{\theta_c^{(o)} + \frac{e'^2 n^2 y_,'^2}{2(1-e'^2)}\right\} + t^2\, \Sigma_1^\infty \left\{\theta_c^{(i)} + \frac{n^2 y_,'^2}{1-e'^2}\left\{R_2^{(i)} - \frac{e'}{2}\frac{dR_2^{(i)}}{de'}\right\}\right\} \cos ig' - t^2 \frac{ny_,'}{\sqrt{1-e'^2}}\, n'\delta z'\, \Sigma_1^\infty\, i^2 R_2^{(i)} \cos ig' \\
& + t^2\, \Sigma_1^\infty \left\{\theta_s^{(i)} - \frac{1+\frac{3}{2}e'^2}{\sqrt{1-e'^2}} ny_,' i\, \varepsilon_c^{(i)} - \frac{n^2 y'_{,,}}{2\sqrt{1-e'^2}}\, i\, R_2^{(i)}\right\} \sin ig' \\
& + W \cos[g - g' + n(y - y' - 2\eta)t + 2k]
\end{aligned}
$$

In his aequationibus quantitates $y_,'$ et $y'_{,,}$ omnino arbitrariae sunt, et ad libitum determinari possunt, rei tamen accommodatissimum est $y_,'$ ita determinare, ut terminus in $(n')z'$ per $t \cos g'$, et $y'_{,,}$ ita ut in $(n')z'$ terminus per $t^2 \cos g'$ multiplicatus evanescat. Ideo habetur

$$
\begin{aligned}
y_,' &= \frac{\alpha_c^{(1)}}{2nR_2^{(1)}}\sqrt{1-e'^2} \\
y'_{,,} &= \frac{\beta_c^{(1)}}{n^2 R_2^{(1)}}\sqrt{1-e'^2} - \frac{1+\frac{3}{2}e'^2}{2n^4 (R_2^{(1)})^2}\, \alpha_s^{(1)} \alpha_c^{(1)} \sqrt{1-e'^2}
\end{aligned}
$$

ubi non minus quam in formulis praecedentibus $R_2^{(i)}$ adiumento excentricitatis terrae computanda est. Quoties $y_,'$ et $y'_{,,}$ ita determinantur, non modo termini in $(n')z'$ per $t \cos g'$ et $t^2 \cos g'$ multiplicati evanescunt, sed etiam propter relationem inter terminos $t \cos ig'$ et $t \sin ig'$ alio loco demonstratam termini omnes in $(n')z'$ per $t \cos ig'$, termini omnes in w' per $t \sin ig'$ et terminus in w' per $t^2 \sin g'$ evanescere debent. Deletis igitur his terminis ex expressionibus praecedentibus, omissisque terminis reliquis qui vim non habent, emergunt denique

$$
\begin{aligned}
(n')z' = \;& g' + t\alpha_s^{(1)} \sin g' + t\alpha_s^{(2)} \sin 2g' + t^2 \{\beta_s^{(1)} + e' n^2 y_,'^2\} \sin g' \\
& + Z \sin[g - g' + (n)(y - y' - 2\eta)t + 2k] \\
w' = \;& t\varepsilon_c^{(o)} + t\varepsilon_c^{(1)} \cos g' + t\varepsilon_c^{(2)} \cos 2g' + t^2 \{\theta_c^{(1)} - \tfrac{1}{2} e' n^2 y_,'^2\} \cos g' \\
& + W \cos[g - g' + (n)(y - y' - 2\eta)t + 2k]
\end{aligned}
$$

qui sunt valores ipsarum $(n')z'$ et w' in hac Lunae theoria adhibendi.

SECTIO V.

EXPLICATIO COMPVTATIONVM APPROXIMATIONIS SECVNDAE ET SVBSEQVENTIVM.

1.

Si in formulis art. 7. Sect. III. termini tertii ordinis abscinduntur, habemus

$$(1)\ldots\ldots \quad (n)z = g + (n)\int\left\{\overline{W} + (n)\delta z\left[\overline{\left(\frac{dW}{dy}\right)} - \frac{y_{,}}{\sqrt{1-(e)^2}}\frac{d.\frac{(r)^2}{(a)^2}}{dg}\right] + w^2\right\}dt$$
$$- \frac{y_{,}}{\sqrt{1-(e)^2}}(n)\int\frac{(r)^2}{(a)^2}dt - \frac{y_{,,}}{\sqrt{1-(e)^2}}(n)\int\frac{(r)^2}{(a)^2}(n)t\,dt$$

ubi

$$W = -b + A_{,}(1-b)\xi + A_{,,}\xi^2 + Z$$

atque

$$Z = (n)\int\left\{\begin{array}{l}\frac{1}{(n)}(\mathring{T}) - \frac{y_{,}}{\sqrt{1-(e)^2}}\frac{d.\frac{(\varrho)^2}{(a)^2}}{d\gamma} - \frac{2}{(n)}U(S+\varepsilon) + \frac{1}{(n)}\frac{d(\mathring{T})}{dg}(n)\delta z + \frac{1}{(n)}r\frac{d(\mathring{T})}{dr}w + \frac{1}{(n)}\frac{d(\mathring{T})}{dg'}(n')dz' \\ + \frac{1}{(n)}r'\frac{d(\mathring{T})}{dr'}w' + \frac{1}{(n)}\frac{d(\mathring{T})}{dP}\delta P + \frac{1}{(n)}\frac{d(\mathring{T})}{dQ}\delta Q + \frac{1}{(n)}\frac{d(\mathring{T})}{dK}\delta K - \frac{y_{,,}}{\sqrt{1-(e)^2}}\frac{d.\frac{(\varrho)^2}{(a)^2}}{d\gamma}(n)t \\ + \frac{y_{,}}{\sqrt{1-(e)^2}}\left\{\frac{(\varrho)^2}{(a)^2}\frac{dW}{d\gamma} - \frac{1}{2}\frac{d.\frac{(\varrho)^2}{(a)^2}}{d\gamma}[W+S+\varepsilon]\right\}\end{array}\right\}dt$$

Hae igitur expressiones terminos omnes ipsius $(n)z$ tum primi tum

secundi ordinis continent, et quum in approximatione prima terminos $\int(\overset{\circ}{T})\,dt - \frac{ny_{,}}{\sqrt{1-e^2}}\,\frac{d.\frac{(\varrho)^2}{(a)^2}}{d\gamma}$ contemplati simus, in approximatione secunda termini reliqui expressionum praecedentium considerandi sunt.

2.

Ante omnia necesse est quantitates auxiliares, quas expressio ipsius Z requirit, explicentur. Sunt autem hae

$$U,\ \frac{d(\overset{\circ}{T})}{dg},\ r\frac{d(\overset{\circ}{T})}{dr},\ \frac{d(\overset{\circ}{T})}{dg'},\ r'\frac{d(\overset{\circ}{T})}{dr'},\ \frac{d(\overset{\circ}{T})}{dP},\ \frac{d(\overset{\circ}{T})}{dQ},\ \frac{d(\overset{\circ}{T})}{dK}$$

Ad $\frac{d(\overset{\circ}{T})}{dg}$ et $\frac{d(\overset{\circ}{T})}{dg'}$ obtinendas nihil fere negotii requiritur, nam approximatio prima ipsam $(\overset{\circ}{T})$ explicite functionem ipsarum g atque g' suppeditabat, quare ad hos quotientes differentiales obtinendos differentiationes et respectu g et respectu g' immediate institui possunt. Ad ipsam U explicandam moneo in art. 5. Sect. III. statutam esse

$$U = 2n\,\frac{\varrho}{a(1-e^2)^{\frac{1}{2}}}\,[\cos(v_{,}-\lambda)-1]\,a\left(\frac{d\Omega}{dv_{,}}\right) \qquad \text{........(2)}$$

in qua formula quantitatum, quas continet, valores pure elliptici et adiumento elementorum (n), (a), (e), etc. computati substituendi sunt.

Adiumento computationum Sectionis praecedentis facile invenitur

$$\frac{2\varrho}{a(1-e^2)^{\frac{1}{2}}}[\cos(v_{,}-\lambda)-1] = \frac{1}{2\sqrt{1-e^2}}\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}\left\{k^2\frac{dR_2^{(\varkappa)}}{de}\cdot\frac{R_2^{(k)}}{e} - \varkappa k\frac{R_2^{(\varkappa)}}{e}\cdot\frac{dR_2^{(k)}}{de}\right\}\cos(\varkappa\gamma+kg) - \frac{2}{\sqrt{1-e^2}}$$

Positis igitur

$$\left.\begin{aligned} E^{\varkappa,k} &= \frac{1}{2\sqrt{1-e^2}}\left\{\varkappa\,\frac{R_2^{(\varkappa)}}{e}.k\frac{dR_2^{(k)}}{de} - \frac{dR_2^{(\varkappa)}}{de}.k^2\,\frac{R_2^{(k)}}{e}\right\} \\ &\quad \text{si excipis} \\ E^{0,0} &= \frac{2}{\sqrt{1-e^2}} \end{aligned}\right\} \qquad \text{........(3)}$$

habetur

$$\frac{2\varrho}{a(1-e^2)^{\frac{1}{2}}}[\cos(v_{,}-\lambda)-1] = -\Sigma_{-\infty}^{+\infty}\Sigma_{-\infty}^{+\infty}E^{\varkappa,k}\cos(\varkappa\gamma+kg)$$

atque hinc

$$(4)\ldots\ldots \qquad U = n\Sigma E^{\varkappa,k} f_x[i, i']_x \sin[\varkappa\gamma + (i+k)g + i'g' + H_x]$$

quae formula, si forma eius mutata erit, cum illa in theoria mea Iovis atque Saturni data plane congruit. Formula (2) perfacile monstrat, U ita quoque exhiberi posse

$$U = 2\frac{\varrho}{a}\cos\varphi\left\{\frac{\cos f}{(1-e^2)^{\frac{1}{2}}} + \frac{e}{(1-e^2)^{\frac{1}{2}}}\right\}\left(\frac{d\Omega}{dv_{\prime}}\right) + 2\frac{\varrho}{a}\sin\varphi\frac{\sin f}{(1-e^2)^{\frac{1}{2}}}\left(\frac{d\Omega}{dv_{\prime}}\right) - \frac{2}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{\prime}}\right)$$

unde manifestum est, U functionem eiusdem formae esse ac functionem in art. 9. Sect. praec. Γ appellatam, itaque theorematì ibidem demonstrato subiectam, quare in evolutione computationeque numerica ipsius U ope formularum (3) et (4) absolvenda ad valores $\varkappa = o$, $\varkappa = 1$ et $\varkappa = -1$ solummodo respicere nos oportet, unde magnum calculi compendium adducimus.

Si vero, peracta ipsius U computatione modo descripta, adiumento theorematis huius, hoc est adiumento formularum (4) Sect. praec. termini, in quibus sine respectu signi eorum algebraici $\varkappa$ est maior quam 1, computantur, calculus numericus confirmari potest: nam restitutis his terminis mutataque τ in t, U cifrae aequalis fieri debet.

3.

Quantitates $E^{\varkappa,k}$ formulis (3) datae, in series infinitas evolutae ita se habent:

$$E^{0,0} = \frac{2}{\sqrt{1-e^2}}$$

$$E^{1,0} = o$$

$$E^{-1,1} = -1 - \frac{13}{64}e^2 - \frac{91}{576}e^4 - \text{etc.}$$

$$E^{0,1} = \frac{3}{2}e + \frac{9}{16}e^3 + \frac{61}{128}e^5 + \text{etc.}$$

$$E^{1,1} = o$$

$$E^{-1,2} = -e + \frac{1}{4}e^3 - \frac{13}{64}e^5 + \text{etc.}$$

$$E^{0,2} = \frac{3}{2}e^2 + \frac{1}{4}e^4 + \frac{3}{8}e^6 + \text{etc.}$$

$$E^{1,2} = -\frac{1}{24}e^3 - \frac{1}{96}e^5 - \text{etc.}$$

$$
\left.
\begin{aligned}
E^{-1,3} &= -\frac{9}{8}\,e^2 + \frac{9}{16}\,e^4 - \frac{153}{512}\,e^6 + \text{etc.} \\
E^{0,3} &= \frac{27}{16}\,e^4 - \frac{27}{256}\,e^6 + \text{etc.} \\
E^{1,3} &= -\frac{9}{128}\,e^4 - \frac{3}{640}\,e^6 + \text{etc.} \\
E^{-1,4} &= -\frac{4}{3}\,e^3 + e^5 - \text{etc.} \\
E^{0,4} &= 2e^4 - \frac{3}{5}\,e^6 + \text{etc.} \\
E^{1,4} &= -\frac{1}{10}\,e^6 + \text{etc.} \\
E^{-1,5} &= -\frac{625}{384}\,e^4 + \frac{625}{384}\,e^6 - \text{etc.} \\
E^{0,5} &= \frac{625}{256}\,e^6 - \text{etc.} \\
E^{1,5} &= -\frac{625}{4608}\,e^6 + \text{etc.} \\
E^{-1,6} &= -\frac{81}{40}\,e^5 + \text{etc.} \\
E^{0,6} &= \frac{243}{80}\,e^6 - \text{etc.} \\
E^{-1,7} &= -\frac{117649}{46080}\,e^6 + \text{etc.} \\
E^{-x,-k} &= E^{x,k}
\end{aligned}
\right\} \quad \ldots\ldots(6)
$$

quarum numerus in theoria Lunae satis superque sufficit. ......(8)

4.

Reliquae quantitates auxiliares perfacile ex $(\mathring{T})$ computantur. Statuta

$$
\left.
\begin{aligned}
V = {} & \frac{an}{\sqrt{1-e^2}}\left\{2\,\frac{\varrho}{r}\cos(v_{,}-\lambda) - 1 + 2\,\frac{\varrho}{a(1-e^2)}\left[\cos(v_{,}-\lambda) - 1\right]\right\} r\left(\frac{d^2\Omega}{dv_{,}\,.\,dr}\right) \\
& + 2\,\frac{\varrho}{r}\sin(v_{,}-\lambda)\,\frac{an}{\sqrt{1-e^2}}\left\{r^2\left(\frac{d^2\Omega}{dr^2}\right) + r\left(\frac{d\Omega}{dr}\right)\right\}
\end{aligned}
\right\} \ldots\ldots(5)
$$

ubi non minus quam in U et in $(\mathring{T})$ valores pure elliptici coordinatarum adiumento elementorum (a), (n), (e), etc. computandarum substituendi sunt, facile invenitur

$$(6)\ldots\ldots\qquad r\frac{d(\mathring{T})}{dr}=V-(\mathring{T})+U-\frac{d(S)}{dt}$$

Quum autem in art. 3. Sect. praec. ipsam Ω in partes decem Ω_1, Ω_2, etc. distribuerimus, quas generaliter Ω_x denotavimus, ita ut sit

$$\Omega=\Sigma_1^{10}\,\Omega_x$$

et ex art. 7. Sect. praec. habeamus

$$a\left(\frac{d\Omega}{dv_{,}}\right)=-\Sigma_1^{10}\,f_x\,[i,i']_x\sin(ig+i'g'+H_x)$$
$$ar\left(\frac{d\Omega}{dr}\right)=\quad\Sigma_1^{10}\,h_x\,[i,i']_x\cos(ig+i'g'+H_x)$$

ex his aequationibus et ex indole ipsarum Ω_x elicitur

$$(7)\ldots\ldots\left\{\begin{aligned}ar\left(\frac{d^2\Omega}{dv_{,}\,.\,dr}\right)&=-\Sigma_1^{10}\,h_x f_x\,[i,i']_x\sin(ig+i'g'+H_x)\\ ar^2\left(\frac{d^2\Omega}{dr^2}\right)+ar\left(\frac{d\Omega}{dr}\right)&=\quad\Sigma_1^{10}\,h_x^{\,2}\;[i,i']_x\cos(ig+i'g'+H_x)\end{aligned}\right.$$

Si igitur significationem analogam introducimus hanc

$$(\mathring{T})=\Sigma_1^{10}\,(\mathring{T})_x$$

aequationes (5) et (7) cum valore ipsius $(\mathring{T})$ comparatae suppeditant

$$V=\Sigma_1^{10}h_x(\mathring{T})_x$$

unde ex (6) emergit

$$(8)\ldots\ldots\qquad r\frac{d(\mathring{T})}{dr}=\Sigma_1^{10}(h_x-1)(\mathring{T})_x+U-\frac{d(S)}{dt}$$

Quum Ω sit functio homogenea ipsarum r et r' dimensionis — 1, habetur aequatio haec

$$-\Omega=r\left(\frac{d\Omega}{dr}\right)+r'\left(\frac{d\Omega}{dr'}\right)$$

unde

$$(9)\ldots\ldots\left\{\begin{aligned}-\left(\frac{d\Omega}{dv_{,}}\right)&=r\left(\frac{d^2\Omega}{dv_{,}\,.\,dr}\right)+r'\left(\frac{d^2\Omega}{dv_{,}\,.\,dr'}\right)\\ -r\left(\frac{d\Omega}{dr}\right)&=r^2\left(\frac{d^2\Omega}{dr^2}\right)+r\left(\frac{d\Omega}{dr}\right)+rr'\left(\frac{d^2\Omega}{dr\,.\,dr'}\right)\end{aligned}\right.$$

Quum vero sit

$$r'\frac{d(\mathring{T})}{dr'}=\frac{an}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+2\frac{\varrho}{a(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}r'\left(\frac{d^2\Omega}{dv_{,}\,.\,dr'}\right)$$
$$+\frac{an}{\sqrt{1-e^2}}\,2\frac{\varrho}{r}\sin(v_{,}-\lambda)\,rr'\left(\frac{d^2\Omega}{dr\,.\,dr'}\right)$$

aequationes praecedentes monstrant esse

$$-(\mathring{T})=V+r'\frac{d(\mathring{T})}{dr'}$$

quae, substitutis valoribus ipsarum $(\mathring{T})$ et V per $(\mathring{T})_x$ expressis, suppeditat

$$r'\frac{d(\mathring{T})}{dr'}=-\Sigma_1^{10}(h_x+1)(\mathring{T})_x \qquad \text{......(9)}$$

5.

Formulis aeque simplicibus reliquae quantitates auxiliares computantur. Propter aequationes

$$\left(\frac{d\Omega}{dP}\right)=\left(\frac{d\Omega}{dv}\right)\frac{1}{2\sin\frac{1}{2}I},\quad \left(\frac{d\Omega}{dQ}\right)=\left(\frac{d\Omega}{dI}\right)\frac{1}{\cos\frac{1}{2}I},\quad \left(\frac{d\Omega}{dK}\right)=\left(\frac{d\Omega}{dk}\right)$$

habetur

$$\frac{d(\mathring{T})}{dP}=\frac{an}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{r}\cos(v_,-\lambda)-1+2\frac{\varrho}{a(1-e^2)}[\cos(v_,-\lambda)-1]\right\}\left(\frac{d^2\Omega}{dv_,\,dv}\right)\frac{1}{2\sin\frac{1}{2}I}$$
$$+\frac{an}{\sqrt{1-e^2}}2\frac{\varrho}{r}\sin(v_,-\lambda)\,r\left(\frac{d^2\Omega}{dr\,dv}\right)\frac{1}{2\sin\frac{1}{2}I}$$
$$\frac{d(\mathring{T})}{dQ}=\frac{an}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{r}\cos(v_,-\lambda)-1+2\frac{\varrho}{a(1-e^2)}[\cos(v_,-\lambda)-1]\right\}\left(\frac{d^2\Omega}{dv_,\,dI}\right)\frac{1}{\cos\frac{1}{2}I}$$
$$+\frac{an}{\sqrt{1-e^2}}2\frac{\varrho}{r}\sin(v_,-\lambda)\,r\left(\frac{d^2\Omega}{dr\,dI}\right)\frac{1}{\cos\frac{1}{2}I}$$
$$\frac{d(\mathring{T})}{dK}=\frac{an}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{r}\cos(v_,-\lambda)-1+2\frac{\varrho}{a(1-e^2)}[\cos(v_,-\lambda)-1]\right\}\left(\frac{d^2\Omega}{dv_,\,dk}\right)$$
$$+\frac{an}{\sqrt{1-e^2}}2\frac{\varrho}{r}\sin(v_,-\lambda)\,r\left(\frac{d^2\Omega}{dr\,dk}\right) \qquad \text{...(10)}$$

Consideremus expressionem

$$a\left(\frac{d\Omega}{dP}\right)\delta P+a\left(\frac{d\Omega}{dQ}\right)\delta Q+a\left(\frac{d\Omega}{dK}\right)\delta K$$

quae, substitutis valoribus ipsarum $\left(\frac{d\Omega}{dP}\right)$, $\left(\frac{d\Omega}{dQ}\right)$ atque $\left(\frac{d\Omega}{dK}\right)$ modo datis nec non valore ipsius δK hoc

$$\delta K=-\delta P\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}$$

in art. 14. Sect. praec. invento, transit in hanc

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$$a\left\{\left(\frac{d\Omega}{dv}\right)\frac{1}{2\sin\frac{1}{2}I}-\left(\frac{d\Omega}{dk}\right)\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\right\}\delta P+a\left(\frac{d\Omega}{dI}\right)\frac{1}{\cos\frac{1}{2}I}\delta Q$$

quae, habita ratione aequationum (8) art. 14. Sect. praec., transfertur in hanc

$$(9)\ldots\ldots\quad \delta P\Sigma_1^{\infty}\frac{Q_x}{\cos^2\frac{1}{2}I}[i,i']_x\sin(ig+i'g'+H_x)-\delta Q\Sigma_1^{\infty}\frac{P_x}{\cos^2\frac{1}{2}I}[i,i']_x\cos(ig+i'g'+H_x)$$

Comparata hac expressione cum expressionibus ipsarum $\frac{d(\mathring{T})}{dP}$, $\frac{d(\mathring{T})}{dQ}$ atque $\frac{d(\mathring{T})}{dK}$ modo datis et cum expressione ipsius $(\mathring{T})$, elicitur

$$\frac{d(\mathring{T})}{dP}\delta P+\frac{d(\mathring{T})}{dQ}\delta Q+\frac{d(\mathring{T})}{dK}\delta K=$$

$$-\delta P\Sigma_1^{\infty}\frac{Q_x}{\cos^2\frac{1}{2}I}(\mathring{T})_x\operatorname{cotg}(x\gamma+ig+i'g'+H_x)-\delta Q\Sigma_1^{\infty}\frac{P_x}{\cos^2\frac{1}{2}I}(\mathring{T})_x$$

ubi $(\mathring{T})_x \operatorname{cotg}(x\gamma+ig+i'g'+H_x)$ est $(\mathring{T})_x$ ipsa, in qua post evolutionem in seriem infinitam sinus in cosinus mutandi sunt.

6.

Expressionibus artt. praecc. quantitates auxiliares perfacile computantur, quibus calculis peractis, ante omnia ipsa Z computanda est. Habetur vero secundum praecedentia pro approximatione secunda

$$(10)\ldots\ldots\left\{\begin{aligned}\frac{dZ}{dt}=&-2U(S+\varepsilon)+\frac{d(\mathring{T})}{dg}(n)\delta z+r\frac{d(\mathring{T})}{dr}w+\frac{d(\mathring{T})}{dg'}(n')\delta z'+r'\frac{d(\mathring{T})}{dr'}w'\\&+M\delta P+N\delta Q-\frac{(n)y_{,}}{\sqrt{1-e^2}}\frac{d.\varrho^2}{a^2d\gamma}-\frac{(n)y_{,,}}{\sqrt{1-e^2}}\frac{d.\varrho^2}{a^2d\gamma}(n)t\\&+\frac{(n)y_{,}}{\sqrt{1-e^2}}\left\{\frac{\varrho^2}{a^2}\frac{dW}{d\gamma}-\frac{1}{2}\frac{d.\varrho^2}{a^2d\gamma}[W+S+\varepsilon]\right\}\end{aligned}\right.$$

ubi

$$M=-\Sigma\frac{Q_x}{\cos^2\frac{1}{2}I}(\mathring{T})_x\operatorname{cotg}(x\gamma+ig+i'g'+H_x)$$

$$N=-\Sigma\frac{P_x}{\cos^2\frac{1}{2}I}(\mathring{T})_x$$

scripsi, et ubi loco U, $\frac{d(\mathring{T})}{dg}$, $r\frac{d(\mathring{T})}{dr}$, etc. valores earum in praecedentibus explicati substituendi sunt.

Iam explicationes praecedentes quantitatum U, $\frac{d(\overset{\circ}{T})}{dg}$, $r\frac{d(\overset{\circ}{T})}{dr}$, etc. monstrant, has functiones eiusdem formae esse, ac functionem in art. 9. Sect. IV. Γ appellatam; praeterea $\frac{d.\varrho^2}{a^2 dy}$, quum sit aequalis ipsi $2\frac{\varrho e \sin\varphi}{a\sqrt{1-e^2}}$, eidem formae se aggregat. Hinc sequitur, quantitatem $\frac{dZ}{dt}$, qualis supra pro approximatione secunda exhibita est, si terminum eius ultimum excipias, theoremati l. c. demonstrato esse subiectam. Consideremus nunc terminum timum.

Quum $y_{,}$ sit quantitas primi ordinis, ad terminos secundi ordinis obtinendos in factore ipsius $\frac{y_{,}}{\sqrt{1-e^2}}$, hoc est in quantitate

$$\frac{\varrho^2}{a^2}\frac{dW}{dy}-\frac{1}{2}\frac{d.\varrho^2}{a^2 dy}[W+S+\varepsilon]$$

termini primi tantum ordinis recipiendi sunt. Formulae vero pro approximatione prima suppeditant

$$W=\int(T)\,dt-b+A(1-b)\xi$$

ubi terminum $B\xi^2$ omisi, quia secundi ordinis est: qui quidem terminus, quum minutissimus sit, in theoria Lunae omnino negligi potest. Substitutis valoribus ipsarum (T) et A his

$$\begin{aligned}(T) = \;& 2\frac{\varrho}{a}\cos\varphi\left\{\left(\frac{a}{r}\cos f+\frac{\cos f}{1-e^2}+\frac{e}{1-e^2}\right)\left(\frac{d\Omega}{dv_{,}}\right)+\frac{a}{r}\sin f.r\left(\frac{d\Omega}{dr}\right)\right\}\frac{an}{\sqrt{1-e^2}}\\ &+2\frac{\varrho}{a}\sin\varphi\left\{\left(\frac{a}{r}\sin f+\frac{\sin f}{1-e^2}\right)\left(\frac{d\Omega}{dv_{,}}\right)-\frac{a}{r}\cos f.r\left(\frac{d\Omega}{dr}\right)-\frac{ey_{,}}{a\sqrt{1-e^2}}\right\}\frac{an}{\sqrt{1-e^2}}\\ &-3\frac{an}{\sqrt{1-e'^2}}\left(\frac{d\Omega}{dv_{,}}\right)\end{aligned}$$

$$A = 2\frac{\varrho}{a}\cos\varphi+3e$$

in praecedente ipsius W expressione, nanciscimur

$$W=2\frac{\varrho}{a}\cos\varphi\left\{G+(1-b)\xi\right\}+2\frac{\varrho}{a}\sin\varphi.H-3S-b$$

ubi brevitatis caussa feci

$$G=\frac{an}{\sqrt{1-e^2}}\int\left\{\left(\frac{a}{r}\cos f+\frac{\cos f+e}{1-e^2}\right)\left(\frac{d\Omega}{dv_{,}}\right)+\frac{a}{r}\sin f.r\left(\frac{d\Omega}{dr}\right)\right\}dt$$

$$H=\frac{an}{\sqrt{1-e^2}}\int\left\{\left(\frac{a}{r}\sin f+\frac{\sin f}{1-e^2}\right)\left(\frac{d\Omega}{dv_{,}}\right)-\frac{a}{r}\sin f.r\left(\frac{d\Omega}{dr}\right)-\frac{ey_{,}}{a\sqrt{1-e^2}}\right\}dt$$

Ex hac ipsius W expressione nanciscimur differentiatione

$$\frac{dW}{d\gamma} = -2\frac{\sin\varphi}{\sqrt{1-e^2}}[G+(1-b)\xi] + 2\frac{e+\cos\varphi}{\sqrt{1-e^2}} H$$

atque hinc

$$\frac{\varrho^2}{a^2}\left(\frac{dW}{d\gamma}\right) - \frac{d.\frac{\varrho^2}{a^2}}{2d\gamma}[W+S+\varepsilon] = -\frac{\varrho}{a}\sin\varphi\left\{2\sqrt{1-e^2}\,[G+(1-b)\xi] - \frac{e}{\sqrt{1-e^2}}(2S+b-\varepsilon)\right\}$$
$$+ 2\frac{\varrho}{a}\cos\varphi\,.\sqrt{1-e^2}\,.\,H$$

Quantitas igitur $\frac{\varrho^2}{a^2}\frac{dW}{d\gamma} - \frac{1}{2}\frac{d.\frac{\varrho^2}{a^2}}{d\gamma}[W+S+\varepsilon]$, qualis in approximatione secunda adhibenda est, non minus quam ceterae ipsius Z partes, functio eiusdem formae est ac functio in art. 9. Sect. IV. Γ appellata, ex quo sequitur, theorema illic demonstratum in approximatione secunda pro Z quoque, et quum sit

$$W = -b + (1-b)\xi\left(2\frac{\varrho}{a}\cos\varphi + 3e\right) + B\xi^2 + Z$$

etiam pro W locum habere, dummodo non nisi maximus terminus in B recipiatur, qui satis superque sufficit *).

Ex indole ipsius W in fine art. 7. Sect. III. usque ad quantitates quarti ordinis exhibitae, et ex iis quae in praecedentibus protulimus, statim elucet, terminos tertii ordinis, qui ex quotientibus differentialibus ipsarum U et $(\mathring{T})$ pendent, eidem theoremati subiectos esse; iam quum in termino

$$\frac{y_{\prime}}{\sqrt{1-(e)^2}}\left\{\frac{(\varrho)^2}{(a)^2}\frac{dW}{d\gamma} - \frac{1}{2}\frac{d.\frac{(\varrho)^2}{(a)^2}}{d\gamma}\left[W+(S+\varepsilon)+\tfrac{1}{2}(S+\varepsilon)^2\right]\right\}$$

pro approximatione tertia ii ipsarum $\frac{dW}{d\gamma}$, W et $S+\varepsilon$ valores substituendi sint, quos approximatio secunda protulerat, quumque modo demonstraverimus, hunc ipsius W valorem eidem theoremati subiectum esse, et denique reliqui ipsius W termini tertii ordinis secundum praecedentia eadem proprietate gaudeant: sequitur in tertia quoque approximatione quantitatem

*) Coefficiens maximi termini in $B\xi^2$ aequalis est 0,"059, reliqui igitur termini optimo iure negligi possunt.

W eidem theorematis subiectam esse. Quum igitur in approximatione prima, secunda et tertia W hac proprietate gaudeat; concludere licet eandem proprietatem in approximationibus omnibus subsequentibus locum habere*).

Hac proprietate magnum calculi compendium lucramur, sive evolutiones analyticae amplius producuntur, sive perturbationes secundi altiorumque ordinum per methodum computantur, quam nunc explicabo.

7.

Consideremus septem primos expressionis ipsius $\frac{dZ}{dt}$ in art. praec. exhibitae terminos, quorum quisque duobus constat factoribus, quorum alter ipsam τ non continet. Hi factores partim computationibus approximationis primae dati sunt, partim eo modo, quem in praecedentibus explicavi, obtinentur. Omnes vero factores series sunt, quae secundum sinus et cosinus angulorum $\varkappa\gamma + ig + i'g' + H_x$ et resp. $ig + i'g' + H_x$ procedunt, quarum serierum coefficientes rapidissime convergunt. Habetur

$$\text{aut}\ \frac{1}{(n)}U,\ \text{aut}\ \frac{1}{(n)}r\frac{d(\mathring{T})}{dr},\ \text{aut}\ \frac{1}{(n)}r'\frac{d(\mathring{T})}{dr'},\ \text{aut}\ \frac{1}{(n)}N = \Sigma\,\theta_{\varkappa}^{i,i',x}\sin(\varkappa\gamma + ig + i'g' + H_x)$$

$$\text{aut}\ \frac{1}{(n)}\frac{d(\mathring{T})}{dg},\ \text{aut}\ \frac{1}{(n)}\frac{d(\mathring{T})}{dg'},\ \text{aut}\ \frac{1}{(n)}M = \Sigma\,\theta_{\varkappa}^{i,i',x}\cos(\varkappa\gamma + ig + i'g' + H_x)$$

$$\text{aut}\ (S+\varepsilon),\ \text{aut}\ w,\ \text{aut}\ \delta Q = \Sigma\,\lambda^{i_,,i'_,,x_,}\cos(i_,g + i'_,g' + H_{x_,})$$

$$\text{aut}\ (n)\delta z,\ \text{aut}\ \delta P = \Sigma\,\lambda^{i_,,i'_,,x_,}\sin(i_,g + i'_,g' + H_{x_,})$$

$$(n')\delta z' = Z'\sin(g - g' + H_6) + (n)t\,\Sigma\,\alpha^{(i'_,)}\sin i'_,g'$$

$$w' = W'\cos(g - g' + H_6) + (n)t\,\Sigma\,\varepsilon^{(i'_,)}\cos i'_,g'$$

ubi $i_,$ et $i'_,$ numeri integri sunt, qui tum ab i et i' diversi, tum his aequales sunt, index $x_,$ tum ab indice x diversus, tum huic aequalis est, et in quaque quantitate coefficientes $\theta_{\varkappa}^{i,i',x}$ et $\lambda^{i_,,i'_,,x_,}$ a se invicem diversi sunt. Multiplicatis his quantitatibus emergit

$$\text{aut}\ \frac{1}{(n)}\frac{d(\mathring{T})}{dg}(n)\delta z,\ \text{aut}\ \frac{1}{(n)}M\delta P = \tfrac{1}{2}\Sigma\,\theta_{\varkappa}^{i,i',x}.\lambda^{i_,,i'_,,x_,}.\sin[\varkappa\gamma + (i+i_,)g + (i'+i'_,)g' + H_x + H_{x_,}]$$
$$-\tfrac{1}{2}\Sigma\,\theta_{\varkappa}^{i,i',x}.\lambda^{i_,,i'_,,x_,}.\sin[\varkappa\gamma + (i-i_,)g + (i'-i'_,)g' + H_x + H_{x_,}]$$

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*) Quod theorema infra rigorose demonstrabitur.

$$\text{aut}\,\frac{1}{(n)}U(S+\varepsilon),\ \text{aut}\,\frac{1}{(n)}r\frac{d(\mathring{T})}{dr}w,\ \text{aut}\,\frac{1}{(n)}N\delta Q =$$
$$\tfrac{1}{2}\Sigma\theta_{\varkappa}^{i,i',x}.\lambda^{i_{,},i'_{,},x_{,}}.\sin[\varkappa\gamma+(i+i_{,})g+(i'+i'_{,})g'+H_x+H_{x_{,}}]$$
$$+\tfrac{1}{2}\Sigma\theta_{\varkappa}^{i,i',x}.\lambda^{i_{,},i'_{,},x_{,}}.\sin[\varkappa\gamma+(i-i_{,})g+(i'-i'_{,})g'+H_x-H_{x_{,}}]$$

$$\frac{1}{(n)}\frac{d(\mathring{T})}{dg'}(n')\,\delta z' = \tfrac{1}{2}Z\Sigma\theta_{\varkappa}^{i,i',x}\sin[\varkappa\gamma+(i+1)g+(i'-1)g'+H_x+H_6]$$
$$-\tfrac{1}{2}Z\Sigma\theta_{\varkappa}^{i,i',x}\sin[\varkappa\gamma+(i-1)g+(i'+1)g'+H_x-H_6]$$
$$+\tfrac{1}{2}(n)t\,\Sigma\theta_{\varkappa}^{i,i',x}.\alpha^{(i_{,}')}\sin[\varkappa\gamma+ig+(i'+i'_{,})g'+H_x]$$
$$-\tfrac{1}{2}(n)t\,\Sigma\theta_{\varkappa}^{i,i',x}.\alpha^{(i_{,}')}\sin[\varkappa\gamma+ig+(i'-i'_{,})g'+H_x]$$

$$\frac{1}{(n)}r'\frac{d(\mathring{T})}{dr'}w' = \tfrac{1}{2}W'\Sigma\theta_{\varkappa}^{i,i',x}\sin[\varkappa\gamma+(i+1)g+(i'-1)g'+H_x+H_6]$$
$$+\tfrac{1}{2}W'\Sigma\theta_{\varkappa}^{i,i',x}\sin[\varkappa\gamma+(i-1)g+(i'+1)g'+H_x-H_6]$$
$$+\tfrac{1}{2}(n)t\,\Sigma\theta_{\varkappa}^{i,i',x}.\varepsilon^{(i_{,}')}\sin[\varkappa\gamma+ig+(i'+i'_{,})g'+H_x]$$
$$+\tfrac{1}{2}(n)t\,\Sigma\theta_{\varkappa}^{i,i',x}.\varepsilon^{(i_{,}')}\sin[\varkappa\gamma+ig+(i'-i'_{,})g'+H_x]$$

Summae et differentiae $H_x+H_{x_{,}}$ atque $H_x-H_{x_{,}}$ partim eosdem ipsius H_x valores decem denuo producunt, quos in approximatione prima evolutione quantitatis Ω invenimus et in artt. 3. et 7. Sect. praec. adscripsimus, partim novas quantitates analogas gignunt. E. g. quum sit $H_1=o$, erit semper pro valore $x_{,}=1$

$$H_x+H_1 = H_x-H_1 = H_x$$

atque pro valore $x_{,}=x$

$$H_x-H_x = H_1$$

Quum sint

$$H_2 = n(2y-2y'-4\eta)t+4k$$
$$H_6 = n(\ y-\ y'-2\eta)t+2k$$
$$H_7 = n(3y-3y'-6\eta)t+6k$$

erunt pro valoribus $x=2$, $x_{,}=6$

$$H_x+H_{x_{,}} = H_7$$
$$H_x-H_{x_{,}} = H_6$$

et sic porro, pro valoribus vero $x=2$, $x_{,}=2$ habetur

$$H_x+H_{x_{,}} = n(4y-4y_{,}-8\eta)t+8k$$

qui valor in decem ipsius H_x valoribus approximationis primae non inest,

et sic porro. Quamobrem, concinnius in his multiplicationibus introducuntur significationes hae

$$n(y+\alpha-\eta)t+v+k=\omega$$
$$n(y'+\alpha+\eta)t+v-k=\omega'$$

Hinc factum est, ut

$$\begin{array}{ll} H_1 = o & H_6 = \omega-\omega' \\ H_2 = 2\omega-2\omega' & H_7 = 3\omega-3\omega' \\ H_3 = 2\omega & H_8 = \omega+\omega' \\ H_4 = 2\omega' & H_9 = 3\omega-\omega' \\ H_5 = 2\omega+2\omega' & H_{10} = \omega-3\omega' \end{array}$$

et termini generales factorum supra exhibitorum sint $\frac{\sin}{\cos}(\varkappa\gamma+ig+i'g'+i''\omega+i'''\omega')$ et respective $\frac{\sin}{\cos}(i_,g+i_,'g'+i_,''\omega+i_,'''\omega')$. His introductis significationibus, anguli in productis supra exhibitis contenti has induunt formas:

$$\varkappa\gamma+(i+i_,)g+(i'+i_,')g'+(i''+i_,'')\omega+(i'''+i_,''')\omega'$$
$$\text{et } \varkappa\gamma+(i-i_,)g+(i'-i_,')g'+(i''-i_,'')\omega+(i'''-i_,''')\omega'$$

quae in simpliciore hac

$$\varkappa\gamma+ig+i'g'+i''\omega+i'''\omega'$$

continentur. Hinc sequitur terminum generalem omnium horum productorum esse

$$h\sin(\varkappa\gamma+ig+i'g'+i''\omega+i'''\omega')$$

et resp.

$$h'\sin(\varkappa\gamma+ig+i'g'+i''\omega+i'''\omega')+(n)t\,h''\sin(\varkappa\gamma+ig+i'g'+i''\omega+i'''\omega')$$

designantibus h, h' et h'' coefficientes constantes.

Inter valores ipsius $i''\omega+i'''\omega'$, qui ad terminos pertinent, quorum coefficientes in approximatione prima vim non habent, quatuor adsunt ad terminos spectantes, quorum coefficientes in approximatione secunda negligere non licet, qui quidem valores, computatione numerica indicati, sunt $4\omega-4\omega'$, $4\omega-2\omega'$, $2\omega-4\omega'$, et $5\omega-5\omega'$. Positis igitur

$$H_{11} = 4\omega-4\omega' = n(4y-4y'-8\eta)t+8k$$
$$H_{12} = 4\omega-2\omega' = n(4y-2y'+2\alpha-6\eta)t+2v+6k$$
$$H_{13} = 2\omega-4\omega' = n(2y-4y'-2\alpha-6\eta)t-2v+6k$$
$$H_{14} = 5\omega-5\omega' = n(5y-5y'-10\eta)t+10k$$

terminus generalis productorum illorum ita se habet

$$h \sin(\varkappa\gamma + ig + i'g' + H_x)$$

et resp.

$$h' \sin(\varkappa\gamma + ig + i'g' + H_x) + (n)t\, h'' \sin(\varkappa\gamma + ig + i'g' + H_x)$$

ubi index x quatuordecim valores accipit.

Perturbationes igitur approximatione secunda productae eiusdem formae sunt ac perturbationes approximationis primae, si excipis terminos praecedentes per $(n)t$ multiplicatos et quatuor valores 11, 12, 13 et 14 indicis x.

Ex approximatione prima praeterea notae sunt $S + \varepsilon + W$ et $\frac{dW}{d\gamma}$ sub forma hac

$$S + \varepsilon + W = \Sigma \zeta_x^{i,i',x} \cos(\varkappa\gamma + ig + i'g' + H_x)$$

$$\frac{dW}{d\gamma} = \Sigma \beta_x^{i,i',x} \sin(\varkappa\gamma + ig + i'g' + H_x)$$

ubi $\beta_x^{i,i',x} = -\varkappa \zeta_x^{i,i',x}$, porro

$$\left(\frac{\varrho}{a}\right)^2 = \Sigma R_2^{(\varkappa_1)} \cos \varkappa_1\gamma\,, \quad \frac{d.\varrho^2}{a^2 d\gamma} = \Sigma Q^{(\varkappa_1)} \sin \varkappa_1\gamma$$

ubi $Q^{(\varkappa_1)} = -\varkappa_1 R_2^{(\varkappa_1)}$; hinc sequitur

$$\left(\frac{\varrho}{a}\right)^2 \frac{dW}{d\gamma} = \tfrac{1}{2}\Sigma \beta_x^{i,i',x}.R_2^{(\varkappa_1)} \sin[(\varkappa+\varkappa_1)\gamma + ig + i'g' + H_x]$$
$$+ \tfrac{1}{2}\Sigma \beta_x^{i,i',x}.R_2^{(\varkappa_1)} \sin[(\varkappa-\varkappa_1)\gamma + ig + i'g' + H_x]$$

$$\frac{d.\left(\frac{\varrho}{a}\right)^2}{d\gamma}[W + S + \varepsilon] = \tfrac{1}{2}\Sigma \zeta_x^{i,i',x}.Q^{(\varkappa_1)} \sin[(\varkappa+\varkappa_1)\gamma + ig + i'g' + H_x]$$
$$- \tfrac{1}{2}\Sigma \zeta_x^{i,i',x}.Q^{(\varkappa_1)} \sin[(\varkappa-\varkappa_1)\gamma + ig + i'g' + H_x]$$

Quum quantitates

$$(\varkappa+\varkappa_1)\gamma + ig + i'g' + H_x \text{ atque } (\varkappa-\varkappa_1)\gamma + ig + i'g' + H_x$$

in forma hac $\varkappa\gamma + ig + i'g' + H_x$ contineantur, terminus generalis horum productorum est

$$h''' \sin(\varkappa\gamma + ig + i'g' + H_x)$$

idem atque in approximatione prima.

8.

Sicut in approximatione prima perturbationes Lunae aggregato duorum productorum constant, quorum factores habent terminos generales hos

$$A^{\sin}_{\cos}(\varkappa\gamma + kg) \text{ atque } B^{\cos}_{\sin}(ig + i'g' + H_x)$$

denotantibus A et B coefficientes, qui in praecedentibus explicati sunt, ita in approximatione secunda perturbationes aggregato productorum nonnullorum constant, quorum factores habent terminos generales hos

$$C^{\sin}_{\cos}(\varkappa\gamma + ig + i'g' + H_x) \text{ atque } D^{\cos}_{\sin}(i_{,}g + i'_{,}g' + H_x)$$

quos in art. praec. explicavi. In approximatione prima haec producta in series, quarum coefficientes secundum potestates excentricitatum inclinationisque mutuae progrediuntur, non evolvi, sed potius factorum coefficientium valores numericos, deinde horum valorum adiumento producta, denique perturbationes ipsas computavi. Hinc effeci, ut sine respectu semper dubio ordinis analytici terminorum, perturbationes approximationis primae usque ad fixum in initio calculi mihimet ipsi propositum limitem numericum nactus sim. Eadem ratione praestat in approximatione secunda et subsequentibus producta, quorum indolem in art. praec. exposui, nec non producta reliqua, quibus opus est, adiumento valorum numericorum coefficientium factorum eorum computare, quo facto secundae quoque approximationis et subsequentium perturbationes usque ad fixum determinatumque limitem numericum accuratae obtinentur. Haec methodus perturbationum computandarum tutissime ad finem propositum perducit, et in analyseos statu hodierno sola est, qua praecisio ad quemvis gradum evehi potest; evolutio enim ut dicitur analytica semper dubia et fallax est, quia coefficientes numerici quantitatum, quae in hoc evolutionis genere parvulae quantitates primi ordinis habentur, saepe permagni sunt, unde efficitur ut termini complures revera ad ordinem pertineant eo ordine inferiorem, cui pro compositione sua analytica adnumerandi sint. Fieri igitur potest ut termini, qui propter compositionem suam analyticam ad ordinem negligendum pertinent, ideoque in hoc evolutionis genere negliguntur, revera maiores sint quam ut iure negligi possint.

Quoties vero adiumento valorum numericorum coefficientium factorum, qui ex computationibus approximationis primae usque ad fixum limitem nu-

mericum accurati sunt, producta computantur, absoluta cuiusque termini magnitudo praesto est, et, nulla adhibita consideratione secundaria, termini limitem propositum superantes recipi, termini vero hoc limite minores omitti poterunt. Usque ad hoc tempus iam perturbationes longitudinis mediae Lunae approximationis primae et secundae hac methodo computavi, et celerrime hanc computationem absolvere potui, quia series, quae factores productorum repraesentant, rapidissime convergunt.

In hoc computationis genere multum interest, ut in serie argumentorum perturbationum ordo certus simplicissimusque conservetur, quare ordinem argumentorum simplicissimum et in methodo nostra se ipsum offerentem exponam. Primum valores indicis x distribuunt argumenta in tot sectiones quot x habet valores, et in quaque harum sectionum valores diversi indicis i tot subdivisiones efficiunt. Quibus statutis, in quaque subdivisione procedant argumenta secundum valores indicis i, ita ut e. g. primum locum teneat argumentum, in quo i maximum valorem negativum habet, et ultimum locum argumentum, in quo i maximum valorem positivum habet. Ut haec res clarius intelligatur, adiungam seriem argumentorum ex omnibus sectionibus, quarum coefficientes vim habent, electam et ut habeatur terminus comparationis, apponam argumenta aequipollentia secundum notationem Viri ill.[*] Damoiseau cum numeris qui ordinem ab hoc geometra adhibitum denotant. Designant igitur in tabellae subsequentis ultima columna t elongationem mediam Lunae a Sole, x anomaliam mediam Lunae, y argumentum latitudinis Lunae et z anomaliam mediam Solis.

Series argumentorum secundum nos.	Series argumentorum secundum ill. Damoiseau cum eorum numero.		Series argumentorum secundum nos.	Series argumentorum secundum ill. Damoiseau cum eorum numero.	
g	1	x	$-g-2g'$	23	$-(x+2z)$
$2g$	2	$2x$	$-2g'$	17	$-2z$
$3g$	4	$3x$	$g-2g'$	22	$x-2z$
etc.		etc.	etc.		etc.
$-2g-g'$	21	$-(2x+z)$	etc.		etc.
$-g+g'$	19	$-(x+z)$	H_2	vacat	$2t-2x+2z$
$-g'$	16	$-z$ (*	$g+H_2$	61	$2t-x+2z$
$g-g'$	18	$x-z$	$2g+H_2$	44	$2t+2z$
$2g-g'$	20	$2x-z$	etc.		etc.
etc.		etc.	$-g-g'+H_2$	vacat	$2t-3x+z$

Series argumentorum secundum nos.	Series argumentorum secundum ill. Damoiseau cum eorum numero.	
$-g'+H_2$	51	$2t-2x+z$
$g-g'+H_2$	39	$2t-x+z$
$2g-g'+H_2$	34	$2t+z$
etc.		etc.
$-g-2g'+H_2$	45	$2t-3x$
$-2g'+H_2$	38	$2t-2x$
$g-2g'+H_2$	31	$2t-x$ (**
$2g-2g'+H_2$	30	$2t$ (***
$3g-2g'+H_2$	32	$2t+x$
etc.		etc.
$-3g'+H_2$	53	$2t-z-2x$
$g-3g'+H_2$	41	$2t-z-x$
$2g-3g'+H_2$	33	$2t-z$
etc.		etc.
$g-4g'+H_2$	59	$2t-2z-x$
$2g-4g'+H_2$	43	$2t-2z$
etc.		etc.
etc.		etc.
$g'+H_3$	vacat	$2y+z-2x$
$g+g'+H_3$	vacat	$2y+z-x$
$2g+g'+H_3$	25	$2y+z$
etc.		etc.
$-g+H_3$	11	$-(3x-2y)$
H_3	7	$-(2x-2y)$
$g+H_3$	5	$-(x-2y)$
$2g+H_3$	3	$2y$
$3g+H_3$	6	$x+2y$
etc.		etc.
$-g'+H_3$	vacat	$2y-z-2x$
$g-g'+H_3$	vacat	$2y-z-x$
$2g-g'+H_3$	24	$2y-z$
etc. etc.		etc. etc.
$-g+3g'+H_4$	vacat	$-(2t-2y-z+x)$
$3g'+H_4$	57	$-(2t-2y-z)$
$g+3g'+H_4$	vacat	$-(2t-2y-z-x)$
etc.		etc.
$-g+2g'+H_4$	48	$-(2t+x-2y)$
$2g'+H_4$	37	$-(2t-2y)$
$g+2g'+H_4$	49	$-(2t-x-2y)$
$2g+2g'+H_4$	65	$-(2t-2x-2y)$
etc.		etc.
$-g+g'+H_4$	vacat	$-(2t-2y+z+x)$
$g'+H_4$	55	$-(2t-2y+z)$
$g+g'+H_4$	vacat	$-(2t-2y+z-x)$
etc.		etc.
etc.		etc.
$g+2g'+H_5$	vacat	$4y-2t-x$
$2g+2g'+H_5$	vacat	$4y-2t$
etc.		etc.
etc.		etc.
$-g+H_6$	vacat	$t-2x+z$
H_6	89	$t-x+z$
$g+H_6$	84	$t+z$
$2g+H_6$	92	$t+x+z$
etc.		etc.
$-g-g'+H_6$	85	$t-2x$
$-g'+H_6$	81	$t-x$
$g-g'+H_6$	80	t (†
$2g-g'+H_6$	82	$t+x$
etc.		etc.
$-2g'+H_6$	91	$t-x-z$
$g-2g'+H_6$	83	$t-z$
$2g-2g'+H_6$	vacat	$t+x-z$
etc.		etc.
etc.		etc.
$g-2g'+H_7$	vacat	$3t+z-2x$
$2g-2g'+H_7$	vacat	$3t+z-x$
$3g-2g'+H_7$	104	$3t+z$
etc.		etc.
$-3g'+H_7$	vacat	$3t-3x$
$g-3g'+H_7$	vacat	$3t-2x$
$2g-3g'+H_7$	101	$3t-x$
$3g-3g'+H_7$	100	$3t$
$4g-3g'+H_7$	102	$3t+x$
etc.		etc.
etc.		etc.
$g'+H_8$	vacant	$2y-t-x$
$g+g'+H_8$		$2y-t$
etc.		etc.
H_8		$2y-t-x-z$
$g+H_8$		$2y-t-z$
etc. etc.		etc. etc.

Series argumentorum secundum nos.	Series argumentorum secundum ill. Damoiseau cum eorum numero.	
$2g - g' + H_9$	vacant	$2y + t - x$
$3g - g' + H_9$		$2y + t$
etc. etc.		etc. etc.
H_{10}		$3t + 3z - 2y - x$
$g + H_{10}$		$3t + 3z - 2y$
etc.		etc.
$2g - 3g' + H_{11}$	141	$4t - 2x + z$
$3g - 3g' + H_{11}$	129	$4t - x + z$
$4g - 3g' + H_{11}$	124	$4t + z$
etc.		etc.
$g - 4g' + H_{11}$	135	$4t - 3x$
$2g - 4g' + H_{11}$	125	$4t - 2x$
$3g - 4g' + H_{11}$	121	$4t - x$
$4g - 4g' + H_{11}$	120	$4t$
$5g - 4g' + H_{11}$	122	$4t + x$
etc.		etc.
$2g - 5g' + H_{11}$	143	$4t - 2x - z$
$3g - 5g' + H_{11}$	131	$4t - x - z$
$4g - 5g' + H_{11}$	123	$4t - z$
etc. etc.		etc. etc.

Series argumentorum secundum nos.	Series argumentorum secundum ill. Damoiseau cum eorum numero.	
$2g - 2g' + H_{12}$	68	$2t + 2x + 2y$
$3g - 2g' + H_{12}$	47	$2t + x + 2y$
$4g - 2g' + H_{12}$	38	$2t + 2y$
etc.		etc.
$3g - 3g' + H_{12}$	vacat	$2t + 2y - x - z$
$4g - 3g' + H_{12}$	56	$2t + 2y - z$
etc. etc.		etc. etc.
$g + 4g' + H_{13}$	139	$4t - x - 2y$
$2g - 4g' + H_{13}$	127	$4t - 2y$
$3g - 4g' + H_{13}$	vacant	$4t + x - 2y$
etc.		etc.
$g - 5g' + H_{13}$		$4t - x - z - 2y$
$2g - 5g' + H_{13}$		$4t - x - 2y$
etc.		etc.
etc.		etc.
$2g - 5g' + H_{14}$		$5t - 3x$
$3g - 5g' + H_{14}$		$5t - 2x$
$4g - 5g' + H_{14}$		$5t - x$
etc.		etc.
etc.		etc.

*) Aequatio annua; **) Evectio; ***) Variatio; †) Aeq. parallactica.

Ex indole ipsius Ω in Sect. praec. exhibita facile reperitur, quamque subdivisionem seriem infinitam esse, cuius coefficientes a termino quodam maximo antrorsum et retrorsum convergunt. Quae quum in evoluta quantitate Ω ita sint, in perturbationibus ipsis etiam locum habere debent, et quidem non modo in approximatione prima sed etiam in approximationibus omnibus subsequentibus. Hac proprietate ducente, fieri non potest, ut termini limitem propositum superantes omittantur.

9.

Suppono igitur, valoribus numericis coefficientium ipsarum U, $\frac{d(T)}{dg}$ etc. secundum regulas in praecedentibus traditas computatis, productorum septem $-2U(S+\varepsilon)$, $\frac{d(T)}{dg}(n)dz$, etc. nec non productorum $\frac{y_1}{\sqrt{1-e^2}}\left(\frac{\lambda}{a}\right)^2\frac{dV}{dy}$

atque $\frac{1}{2}\frac{y_1}{\sqrt{1-e^2}}\frac{d\left(\frac{r}{a}\right)^2}{d\gamma}[H+S+s]$ coefficientes numericos immediate computari *). In hac computatione factorum U, $\frac{d(\mathring{T})}{dg}$, etc. termini tantum in quibus $x = o$, $x = -1$ et $x = 1$ recipiuntur, et in duobus ultimis productis termini solummodo adscribuntur, in quibus $x + x_{,}$ et $x - x_{,}$ aut $= o$, aut $= -1$, aut $= 1$ sunt. Praeterea in his productis omnibus termini distinguendi sunt, qui in integrationibus subsequentibus factorem permagnum vel divisorem perparvulum accipient, qui termini hanc ob rem pluribus notis decimalibus apponendi sunt quam reliqui. Termini vero qui parvulum divisorem accipient perfacile a ceteris distinguuntur, et in diversas classes distribui possunt. Primam classem constituunt termini, in quibus et $i = o$ et $i' = o$, itaque ii quorum argumenta sunt H_x; divisores autem horum terminorum omnium minimi sunt; secunda classis terminis constat in quibus $i = o$ et $i' = 1$ sive $i' = -1$, itaque iis quorum argumenta sunt $\pm g' + H_x$, factores autem quos hi termini in integrationibus obtinebunt rationi motus medii Lunae ad motum medium Solis fere aequantur, quae ratio circiter numero 13 aequatur; tertia classis terminis constat quorum argumenta sunt $\pm 2g' + H_x$, quorum igitur factor numeris 6 .. 7 fere aequatur; quartam classem, quae terminis constaret, quorum argumenta $\pm 3g' + H_x$ sunt, a reliquis terminis distingui necesse non est. Omnes igitur hi termini in forma generali $i'g' + H_x$ continentur, quibus insuper adnumerandi sunt termini in quibus $x = 1$, $x = -1$, $x + i = o$, qui omnes divisorem parvulum in integrationibus obtinebunt. Sunt igitur termini hi

$$
\begin{array}{c}
i'g' + H_x \\
-\gamma + g + i'g' + H_x \\
\gamma - g + i'g' + H_x \\
\gamma \quad + i'g' + H_x \\
-\gamma \quad + i'g' + H_x
\end{array}
$$

qui in integrationibus subsequentibus divisorem parvulum obtinebunt, quo-

*) Regulas ulteriores ad multiplicationes has instituendas in theoriae Iovis atque Saturni [illegible]

rum quidem terminus $i'g' + H_x$ divisorem hunc bis, reliqui vero semel tantum obtinebunt. Termini praecipue classis primae, qui sunt

$$H_x$$
$$-\gamma + g + H_x$$
$$\gamma - g + H_x$$
$$\gamma \quad + H_x$$
$$-\gamma \quad + H_x$$

et inter hos praecipue terminus H_x maxima cum cura computandi sunt, quia divisorem perparvulum obtinebunt. Minimum omnium divisorem obtinebit terminus H_{10}, qui est terminus notus, quem observationes indicavisse videntur, cuius vero coefficientem geometrae, qui ei operam navaverunt, insensibilem invenerunt. Hunc pro magnitudine appositi sequuntur termini H_4, H_6, H_{12}, H_8, H_2, H_3, etc. quorum tamen nonnulli coefficientes insensibiles habent.

Quae omnia non modo ad approximationem secundam, sed etiam ad omnes approximationes subsequentes referenda sunt. Quod autem attinet ad approximationem primam, restrictiones nonnullae locum habent. In hac enim approximatione coefficientes terminorum ubi et $x = o$ et $i = o$, hoc est terminorum formae $i'g' + H_x$ omnes cifrae aequales sunt, in approximatione igitur prima quadrata divisorum parvulorum non inveniuntur. Porro propter aequationes $G_0^{(o)} = o$ et $D_0^{(o)} = o$ in art. 5. Sect. praec. inventas approximatio prima terminis huius formae

$$x\gamma + ig + H_2$$
$$x\gamma + ig + H_4$$
$$x\gamma + ig + H_7$$
$$x\gamma + ig + H_{10}$$

omnino caret. Perturbationes igitur, quae divisores primae classis obtinerent, pro valoribus $x = 2$, $x = 4$, $x = 7$ et $x = 10$ in approximatione prima non adsunt.

10.

Computatis productis omnibus, quibus formula (10) art. 6. constat, in quorum ultimo is ipsius y valor, quem approximatio prima prodiderat, adhibendus est, ne in ultimo producto quantitates tertii ordinis recipiantur,

dum in reliquis productis earum ratio non habeatur, nanciscimur post additos coefficientes eorundem argumentorum

$$\frac{dZ}{dt} = (n)\,\Sigma\,\lambda_x^{i,i',x}\sin(x\gamma+ig+i'g'+H_x)+(n)^2t\,\Sigma\,\theta_x^{i,i',x}\sin(x\gamma+ig+i'g'+H_x)$$
$$+\,2\,\frac{(n)\overset{2}{y}_{,}}{\sqrt{1-e^2}}\,R_2^{(1)}\sin\gamma+2\,\frac{(n)^2t\,y_{,,}}{\sqrt{1-e^2}}\,R_2^{(1)}\sin\gamma$$

ubi tamen secundum praecedentia ii solummodo recepti sunt termini, in quibus $x=0$, $x=1$ et $x=-1$, et in duobus ultimis terminis $\overset{2}{y}_{,}$ et $y_{,,}$ adhuc indeterminatae sunt. $\overset{2}{y}_{,}$ vero scripsi ut hoc superposito numero 2 indicetur ea ipsius $y_{,}$ pars, quam approximatio secunda suppeditabit. Inter speciales huius expressionis terminorum valores sub signo summationis adsunt hi

$$(n)\,\lambda_1^{0,0,1}\sin\gamma+(n)\,\lambda_{-1}^{0,0,1}\sin(-\gamma)+(n)^2t\,\theta_1^{0,0,1}\sin\gamma+(n)^2t\,\theta_{-1}^{0,0,1}\sin(-\gamma)$$

qui simplicius sub forma hac exhiberi possunt

$$(n)\,\lambda_1^{0,0,1}\sin\gamma+(n)^2t\,\theta_1^{0,0,1}\sin\gamma$$

quia $\lambda_{-1}^{0,0,1}$ in $\lambda_1^{0,0,1}$ et $\theta_{-1}^{0,0,1}$ in $\theta_1^{0,0,1}$ contentam esse censeri potest. Quibus terminis ad terminos per $\overset{2}{y}_{,}$ et $y_{,,}$ multiplicatos tollendos adhibitis, emergit

$$\overset{2}{y}_{,}=-\frac{\lambda_1^{0,0,1}}{2R_2^{(1)}}\sqrt{1-e^2}\,;\quad y_{,,}=-\frac{\theta_1^{0,0,1}}{2R_2^{(1)}}\sqrt{1-e^2}\qquad\ldots\ldots(11)$$

Haec aequatio ipsius $y_{,,}$ ponit valorem, quatenus ex approximatione secunda emergit, et illa ipsius $y_{,}$ praebet valorem, qui ad illum eiusdem quantitatis in approximatione prima inventum additus accuratiorem ipsius $y_{,}$ valorem suppeditat. Quocirca denotato valore ipsius $y_{,}$, qui ex approximatione prima emersit, per $\overset{1}{y}_{,}$, accuratior ipsius $y_{,}$ valor erit aggregato $\overset{1}{y}_{,}+\overset{2}{y}_{,}$ aequalis.

Eodem modo quo in approximatione prima (v. art. 11. Sect. praec.) demonstratur, his ipsarum $\overset{2}{y}_{,}$ et $y_{,,}$ valoribus non modo terminos formae $\lambda_1^{0,0,1}\sin\gamma$ et $nt\,\theta_1^{0,0,1}\sin\gamma$, sed etiam terminos omnes formae $\lambda_x^{0,0,1}\sin x\gamma$ et $nt\,\theta_x^{0,0,1}\sin x\gamma$ tolli.

Quibus statutis, habetur

$$\frac{dZ}{dt}=(n)\,\Sigma\,\lambda_x^{i,i',x}\sin(x\gamma+ig+i'g'+H_x)+(n)^2t\,\Sigma\,\theta_x^{i,i',x}\sin(x\gamma+ig+i'g'+H_x)$$

ubi tamen termini excludendi sunt, in quibus simul $i=0$, $i'=0$ et $x=1$,

et termini, in quibus sine respectu signi eorum algebraici $\varkappa$ maior est quam 1, adhuc desunt. Integrata hac expressione, nanciscimur

$$Z = -\Sigma \frac{\lambda_\varkappa^{i,i',x}}{i+i'u+\nu_x} \cos(\varkappa\gamma + ig + i'g' + H_x) - (n)t\,\Sigma \frac{\theta_\varkappa^{i,i',x}}{i+i'u+\nu_x} \cos(\varkappa\gamma + ig + i'g' + H_x)$$
$$+ \Sigma \frac{\theta_\varkappa^{i,i',x}}{(i+i'u+\nu_x)^2} \sin(\varkappa\gamma + ig + i'g' + H_x)$$

simulque adiumento theorematis art. 9. Sect. praec. computentur termini in quibus sine respectu signi algebraici $\varkappa$ maior est quam 1, id quod facillima opera efficitur, quia logarithmis quantitatum $\lambda_\varkappa^{i,i',x}$ et $\theta_\varkappa^{i,i',x}$ ad divisiones per $i + i'u + \nu_x$ instituendas ceteroquin opus est. Itaque expressio praecedens integrum ipsius Z valorem repraesentat.

Quum sit

$$W = -b + (1-b)\xi A_{,} + \xi^2 A_{,,} + Z$$

sive secundum computationes art. 12. Sect. praec.

$$W = -b + \Sigma_1^\infty \{(1-b)\xi A_{,}^{(i)} + \xi^2 A_{,,}^{(i)}\} \cos i\gamma + Z$$

emergit, substituto valore praecedenti ipsius Z,

$$(11)\ldots\ldots \begin{cases} W = -b + \Sigma_1^\infty \{(1-b)\xi A_{,}^{(i)} + \xi^2 A_{,,}^{(i)}\} \cos i\gamma \\ \quad + \Sigma f_\varkappa^{i,i',x} \cos(\varkappa\gamma + ig + i'g' + H_x) + (n)t\,\Sigma h_\varkappa^{i,i',x} \cos(\varkappa\gamma + ig + i'g' + H_x) \\ \quad + \Sigma k_\varkappa^{i,i',x} \sin(\varkappa\gamma + ig + i'g' + H_x) \end{cases}$$

ubi

$$f_\varkappa^{i,i',x} = -\frac{\lambda_\varkappa^{i,i',x}}{i+i'u+\nu_x},\quad h_\varkappa^{i,i',x} = -\frac{\theta_\varkappa^{i,i',x}}{i+i'u+\nu_x},\quad k_\varkappa^{i,i',x} = \frac{\theta_\varkappa^{i,i',x}}{(i+i'u+\nu_x)^2},$$

et ubi propter valores, quos ipsis $\mathfrak{y}_{,}$ et $\mathfrak{y}_{,,}$ attribuimus, sunt

$$f_\varkappa^{0,0,1} = 0,\quad h_\varkappa^{0,0,1} = 0,\quad k_\varkappa^{0,0,1} = 0$$

Hinc mutata τ in t elicimus

$$\overline{W} = -b + \Sigma_1^\infty \{(1-b)\xi A_{,}^{(i)} + \xi^2 A_{,,}^{(i)}\} \cos ig + \Sigma L^{i,i',x} \cos(ig + i'g' + H_x)$$
$$+ (n)t\,\Sigma M^{i,i',x} \cos(ig + i'g' + H_x) \quad + \Sigma N^{i,i',x} \sin(ig + i'g' + H_x)$$

ubi

$$\begin{aligned} L^{i,i',x} &= f_0^{i,i',x} + f_1^{i-1,i',x} + f_2^{i-2,i',x} + \text{etc.} + f_{-1}^{i+1,i',x} + f_{-2}^{i+2,i',x} + \text{etc.} \\ M^{i,i',x} &= h_0^{i,i',x} + h_1^{i-1,i',x} + \text{etc.} + h_{-1}^{i+1,i',x} + \text{etc.} \\ N^{i,i',x} &= k_0^{i,i',x} + k_1^{i-1,i',x} + \text{etc.} + k_{-1}^{i+1,i',x} + \text{etc.} \end{aligned}$$

11.

Aequatio (1) praeter producta in praecedentibus explicata requirit producta duo haec

$$(n)\,\delta z\left[\left(\overline{\frac{dW}{d\gamma}}\right)-\frac{\dot{y}_{,}}{\sqrt{1-e^2}}\,\frac{d\,.\,\frac{(r)^2}{(a)^2}}{dg}\right] \text{ et } w^2$$

quae eodem modo computanda sunt quo producta reliqua, quorum computatio in praecedentibus explicata est. Quum vero in horum productorum factoribus angulus γ non adsit, invenietur

$$(n)\,\delta z\left[\left(\overline{\frac{dW}{d\gamma}}\right)-\frac{\dot{y}_{,}}{\sqrt{1-(e)^2}}\,\frac{d\,.\,\frac{(r)^2}{(a)^2}}{dg}\right]+w^2=\Sigma d^{i,i',x}\cos\,(ig+i'g'+H_x)$$

ubi $d^{i,i',x}$ non minus quam coefficientes in praecedentibus explicati quantitas numerica est.

Horum productorum termini huius formae $i'g'+H_x$ divisores parvulos, de quibus in art. 9. locuti sumus, in integratione subsequenti semel obtinebunt, quare in his terminis plures notae decimales quam in reliquis apponantur necesse est.

His absolutis, et quum sit

$$\frac{(r)^2}{(a)^2}=1+\tfrac{3}{2}\,e^2+2\Sigma_1^\infty R_2^{(i)}\cos ig$$

omnia praesto sunt, quae ad perturbationes secundi ordinis ipsius $(n)z$ computandas requiruntur. Ex computationibus vero primi ordinis Sect. praec. vidimus, aequationes, quibus b et ξ determinandae sunt, has incognitas mixtas involvere, quare in computatione perturbationum secundi ordinis quantitatum primi ordinis, e quibus b et ξ pendent, ratio habeatur necesse est. Quae quantitates, facta $l^{-1,0,1}=0$, id quod licitum est, secundum art. 12. Sect. praec. sunt in ipsa $\frac{dz}{dt}$

$$(n)\,l^{0,0,1}-(n)\frac{\dot{y}_{,}}{\sqrt{1-e^2}}\,(1+\tfrac{3}{2}\,e^2)+(n)\,l^{1,0,1}\cos g-(n)\,\frac{2\dot{y}_{,}}{\sqrt{1-e^2}}\,R_2^{(1)}\cos g$$

Substitutis igitur quantitatibus et in praecedentibus evolutis et modo memoratis primi ordinis in aequatione (1), nanciscimur ante peractam integrationem

$$(n)\frac{dz}{dt} = (n)-(n)(\overset{1}{b}+\overset{2}{b})+(n)\{d^{0,0,1}+L^{0,0,1}+l^{0,0,1}\}-(n)\frac{\overset{1}{y}_{,}+\overset{2}{y}_{,}}{\sqrt{1-e^2}}(1+\tfrac{3}{2}e^2)$$
$$+(n)^2 t M^{0,0,1}-(n)^2 t\frac{y_{,,}}{\sqrt{1-e^2}}(1+\tfrac{3}{2}e^2)$$
$$+(n)\{(1-\overset{1}{b}-\overset{2}{b})(\overset{1}{\xi}+\overset{2}{\xi})A_{,}^{(1)}+(\overset{1}{\xi}+\overset{2}{\xi})^2 A_{,,}^{(1)}\}\cos g+(n)\{d^{1,0,1}+L^{1,0,1}+l^{1,0,1}\}\cos g-(n)\frac{2(\overset{1}{y}_{,}+\overset{2}{y}_{,})}{\sqrt{1-e^2}}R_2^{(1)}\cos g$$
$$+(n)\Sigma_2^\infty\{[(1-\overset{1}{b}-\overset{2}{b})\overset{2}{\xi}-\overset{2}{b}\overset{1}{\xi}]A_{,}^{(i)}+\xi^2 A_{,,}^{(i)}\}\cos ig+(n)\Sigma\{d^{i,i',x}+L^{i,i',x}\}\cos(ig+i'g'+H_x)-(n)\frac{2\overset{2}{y}_{,}}{\sqrt{1-e^2}}\Sigma_2^\infty R_2^{(i)}\cos$$
$$+(n)^2 t\,\Sigma M^{i,i',x}\cos(ig+i'g'+H_x)-(n)^2 t\frac{2y_{,,}}{\sqrt{1-e^2}}\Sigma_1^\infty R_2^{(i)}\cos ig$$
$$+(n)\Sigma N^{i,i',x}\sin(ig+i'g'+H_x)$$

ubi in terminis sub signo Σ termini speciales, quos separatim adscripsimus, excludendi sunt. In praecedentibus iam supposuimus

$$y = y_{,} + y_{,,}(n)t + y_{,,,}(n)^2 t^2 + \text{etc.}$$
$$\alpha = \alpha_{,} + \alpha_{,,}(n)t + \alpha_{,,,}(n)^2 t^2 + \text{etc.}$$
$$\eta = \eta_{,} + \eta_{,,}(n)t + \eta_{,,,}(n)^2 t^2 + \text{etc.}$$

et quum in formula praecedenti adsint termini tempori ipsi proportionales, suppono praeterea

$$(n_0) = (n) + \mu(n)^2 t + \mu_{,}(n)^3 t^2 + \text{etc.}$$

unde $(n)\frac{dz}{dt}$ suppeditat

$$\mu = M^{0,0,1} - y_{,,}\frac{1+\frac{3}{2}e^2}{\sqrt{1-e^2}}$$

$\mu_{,}$ vero et $y_{,,,}$ in approximatione tertia eodem modo determinabuntur. Hinc efficitur, ut primi ipsius $(n)z$ termini non sint $(n)t+(c)$ sed $\int(n_0)dt+(c)$, quare, ut terminorum $\mu(n)^2 t$ et $\mu_{,}(n)^3 t^2+$ etc. recta habeatur ratio, in approximatione tertia ubique $\int(n_0)dt+(c)$ loco g et $\int(n_0)d\tau+(c)$ loco γ substituendae sunt. Simplicitatis caussa retinebo quidem signa g et γ, sed exinde sunt

$$g = \int(n_0)dt + (c)$$
$$\gamma = \int(n_0)d\tau + (c)$$

Eadem ratione in ipsis H_x ubique $\int y dt$ loco yt, $\int \alpha dt$ loco αt, et $\int \eta dt$ loco ηt substituendae sunt, id quod iam ex ratiocinationibus in Sectione secunda expositis patet.

Positis terminis constantibus aequationis praecedentis pro $(n)\frac{dz}{dt}$, nec non terminis per $\cos g$ multiplicatis cifrae aequalibus, emergunt ad ipsas $\overset{1}{b}+\overset{2}{b}$ et $\overset{1}{\xi}+\overset{2}{\xi}$ determinandas aequationes hae

$$o = -(\overset{1}{b}+\overset{2}{b}) + l^{0,0,1} + d^{0,0,1} + L^{0,0,1} - (\overset{1}{y}_{,}+\overset{2}{y}_{,})\frac{1+\frac{3}{2}e^2}{\sqrt{1-e^2}}$$

$$o = (1-\overset{1}{b}-\overset{2}{b})(\overset{1}{\xi}+\overset{2}{\xi})A_{,}^{(1)} + (\overset{1}{\xi}+\overset{2}{\xi})^2 A_{,,}^{(1)} + l^{1,0,1} + d^{1,0,1} + L^{1,0,1} - 2(\overset{1}{y}_{,}+\overset{2}{y}_{,})\frac{R_2^{(1)}}{\sqrt{1-e^2}}$$

et integrata eadem aequatione, dum ipsarum $y_{,,}$, $y_{,,,}$, etc., $\alpha_{,,}$, $\alpha_{,,,}$, etc., $\eta_{,,}$, $\eta_{,,,}$, etc. ubique et ipsarum μ, $\mu_{,}$, etc. si terminum primum excipis, ratio non habeatur, elicitur

$$\begin{aligned}(n)z = \; & g + \Sigma_2^\infty\Big\{\big[(1-\overset{1}{b}-\overset{2}{b})\overset{2}{\xi}-\overset{2}{b}\overset{1}{\xi}\big]\frac{A_{,}^{(i)}}{i} + \overset{2}{\xi}{}^2\frac{A_{,,}^{(i)}}{i}\Big\}\sin ig + \Sigma\frac{d^{i,i',x}+L^{i,i',x}}{i+i'u+\nu_x}\sin(ig+i'g'+H_x) \\ & -2\frac{\overset{2}{y}_{,}}{\sqrt{1-e^2}}\Sigma_2^\infty\frac{R_2^{(i)}}{i}\sin ig + (n)t\,\Sigma\frac{M^{i,i',x}}{i+i'u+\nu_x}\sin(ig+i'g'+H_x) - (n)t\frac{2y_{,,}}{\sqrt{1-e^2}}\Sigma_1^\infty\frac{R_2^{(i)}}{i}\sin ig \\ & + \Sigma\Big\{\frac{M^{i,i',x}}{(i+i'u+\nu_x)^2} - \frac{N^{i,i',x}}{i+i'u+\nu_x}\Big\}\cos(ig+i'g'+H_x) - \frac{2y_{,,}}{\sqrt{1-e^2}}\Sigma_1^\infty\frac{R_2^{(i)}}{i^2}\cos ig\end{aligned}$$

expressio in qua, praeter terminos quos g cohibet, neque termini tempori aut temporis potestatibus proportionales, neque termini per $\sin g$ multiplicati continentur.

Terminos per $\cos(ig+i'g'+H_x)$ multiplicatos in aequatione praecedenti integritatis caussa tantum computavi, qui omnes negligendi sunt, quoniam maximus eorum ad 0",003 tantum ascendit.

12.

Omissis terminis tertii ordinis in expressione (13) Sect. III., emergit

$$\left.\begin{aligned} w = C + \tfrac{1}{2}\varepsilon - \tfrac{1}{2}(n)\int\Big\{\Big(\overline{\frac{dW}{d\gamma}}\Big) + (n)dz\Big[\Big(\overline{\frac{d^2W}{d\gamma^2}}\Big) - \frac{\overset{1}{y}_{,}}{\sqrt{1-e^2}}\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2} - w\Big(\overline{\frac{dW}{d\gamma}}\Big)\Big\}dt \\ + \tfrac{1}{2}\frac{y_{,}}{\sqrt{1-e^2}}(n)\int\frac{d.\frac{(r)^2}{(a)^2}}{dg}dt + \tfrac{1}{2}\frac{y_{,,}'}{\sqrt{1-e^2}}(n)\int\frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)t\,dt\end{aligned}\right\}\text{......}(12)$$

Duo igitur producta nova haec

$$(n)dz\Big[\Big(\overline{\frac{d^2W}{d\gamma^2}}\Big) - \frac{\overset{1}{y}_{,}}{\sqrt{1-e^2}}\frac{d^2.\frac{(r)^2}{(a)^2}}{dg^2}\Big] \text{ et } w\Big(\overline{\frac{dW}{d\gamma}}\Big)$$

requiruntur, ubi pro $\overline{\left(\frac{d^2 W}{d\gamma^2}\right)}$, $\overline{\left(\frac{dW}{d\gamma}\right)}$ et w ii harum quantitatum valores, quos approximatio prima suppeditavit, adhibendi sunt. Quibus productis methodo in praecedentibus satis explicata computatis, suppono nos invenisse

$$(n)\delta z\left[\overline{\left(\frac{d^2W}{d\gamma^2}\right)}-\frac{\overset{1}{y}_{,}}{\sqrt{1-e^2}}\,\frac{d^2\cdot\frac{(r)^2}{(a)^2}}{dg^2}\right]-w\overline{\left(\frac{dW}{d\gamma}\right)}=\Sigma\delta^{i,i',x}\sin(ig+i'g'+H_x)$$

Ex computatione perturbationum secundi ordinis ipsius $(n)z$ habemus secundum aequationem (11)

$$W=\Sigma F_{\varkappa}^{i,i',x}\cos(\varkappa\gamma+ig+i'g'+H_x)+(n)t\,\Sigma h_{\varkappa}^{i,i',x}\cos(\varkappa\gamma+ig+i'g'+H_x)$$
$$+\Sigma k_{\varkappa}^{i,i',x}\sin(\varkappa\gamma+ig+i'g'+H_x)$$

ubi

$$F_{\varkappa}^{i,i',x}=f_{\varkappa}^{i,i',x}$$

si excipis $F_{\varkappa}^{0,0,1}=[(1-\overset{1}{b}-\overset{2}{b})\overset{2}{\xi}-\overset{2}{b}\overset{1}{\xi}]\,A_{,}^{(i)}+\xi^2 A_{,,}^{(i)}$

et si demo excipis $F_{0}^{0,0,1}=-\overset{2}{b}$

qua tamen ultima quantitate opus non est. Hinc habetur

$$\frac{dW}{d\gamma}=-\Sigma\varkappa F_{\varkappa}^{i,i',x}\sin(\varkappa\gamma+ig+i'g'+H_x)-(n)t\,\Sigma\varkappa h_{\varkappa}^{i,i',x}\sin(\varkappa\gamma+ig+i'g'+H_x)$$
$$+\Sigma\varkappa k_{\varkappa}^{i,i',x}\cos(\varkappa\gamma+ig+i'g'+H_x)$$

unde

$$\overline{\left(\frac{dW}{d\gamma}\right)}=\Sigma P^{i,i',x}\sin(ig+i'g'+H_x)+(n)t\,\Sigma Q^{i,i',x}\sin(ig+i'g'+H_x)$$
$$+\Sigma R^{i,i',x}\cos(ig+i'g'+H_x)$$

ubi

$$P^{i,i',x}=-F_{1}^{i-1,i',x}-2F_{2}^{i-2,i',x}-\text{etc.}+F_{-1}^{i+1,i',x}+2F_{-2}^{i+2,i',x}+\text{etc.}$$
$$Q^{i,i',x}=-h_{1}^{i-1,i',x}-2h_{2}^{i-2,i',x}-\text{etc.}+h_{-1}^{i+1,i',x}+2h_{-2}^{i+2,i',x}+\text{etc.}$$
$$R^{i,i',x}=k_{1}^{i-1,i',x}+2k_{2}^{i-2,i',x}+\text{etc.}-k_{-1}^{i+1,i',x}-2k_{-2}^{i+2,i',x}-\text{etc.}$$

Porro habemus

$$(n)\int\frac{d\cdot\frac{(r)^2}{(a)^2}}{dg}\,dt=2\Sigma_1^\infty R_2^{(i)}\cos ig$$

$$(n)\int\frac{d\cdot\frac{(r)^2}{(a)^2}}{dg}\,(n)t\,dt=2(n)t\,\Sigma_1^\infty R_2^{(i)}\cos ig-2\Sigma_1^\infty\frac{R_2^{(i)}}{i}\sin ig$$

Congestis his valoribus, aequatio (12) suppeditat

$$\begin{aligned} w = C + \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\Sigma \frac{P^{i,i',x} + \delta^{i,i',x}}{i + i'u + v_x} \cos(ig + i'g' + H_x) + \frac{\overset{2}{y}_{,}}{\sqrt{1-e^2}} \Sigma_1^\infty R_2^{(i)} \cos ig \\ + \tfrac{1}{2}(n)t\,\Sigma \frac{Q^{i,i',x}}{i + i'u + v_x} \cos(ig + i'g' + H_x) + (n)t\, \frac{y_{,,}}{\sqrt{1-e^2}} \Sigma_1^\infty R_2^{(i)} \cos ig \\ - \tfrac{1}{2}\Sigma \left\{ \frac{Q^{i,i',x}}{(i + i'u + v_x)^2} + \frac{R^{i,i',x}}{i + i'u + v_x} \right\} \sin(ig + i'g' + H_x) - \frac{y_{,,}}{\sqrt{1-e^2}} \Sigma_1^\infty \frac{R_2^{(i)}}{i} \sin ig \end{aligned}$$

ubi tamen termini per sin $(ig + i'g' + H_x)$ multiplicati non minus quam analogi in $(n)z$ vim non habent.

Art. 10. Sect. III. suppeditabat valorem ipsius C, et art. 17. Sect. II. valorem ipsius $\frac{1}{2}\varepsilon$, quos hoc loco ita transcribo

$$C = \text{term. const. in} \left\{ -\tfrac{1}{2} \frac{\overset{1}{y}_{,} + \overset{2}{y}_{,}}{\sqrt{1-e^2}} \left(\frac{r}{a}\right)^2 + \tfrac{1}{4} \frac{(\overset{1}{y}_{,} + \overset{2}{y}_{,})^2}{1-e^2} \left(\frac{r}{a}\right)^4 + \tfrac{1}{4}\left(\delta \frac{dz}{dt}\right)^2 \right\}$$

$$\tfrac{1}{2}\varepsilon = \tfrac{1}{6}(\overset{1}{b} + \overset{2}{b}) + \tfrac{1}{12}(\overset{1}{b} + \overset{2}{b})^2 - \tfrac{1}{2}e(\overset{1}{\xi} + \overset{2}{\xi}) - \tfrac{1}{4}(1 + e^2)(\overset{1}{\xi} + \overset{2}{\xi})^2$$

qui harum constantium integri valores sunt. Ex Sect. praec. habetur

$$\text{term. const. in} \left(\frac{r}{a}\right)^2 = 1 + \tfrac{3}{2}e^2$$

$$\text{term. const. in} \left(\frac{r}{a}\right)^4 = 1 + 5e^2 + \tfrac{15}{8}e^4$$

et terminus constans in $2\left(\delta \frac{dz}{dt}\right)^2$ est aggregatum quadratorum cuiusque termini ipsius $\delta \frac{dz}{dt}$. Quum vero ad calculum confirmandum quadrato integro ipsius $\delta \frac{dz}{dt}$ utemur, suppono methodo in praecedentibus explicata hoc quadratum computatum esse, et nos invenisse

$$\left(\delta \frac{dz}{dt}\right)^2 = \Sigma \Delta^{i,i',x} \cos(ig + i'g' + H_x)$$

hinc habetur

$$\begin{aligned} C + \tfrac{1}{2}\varepsilon = \; & \tfrac{1}{6}(\overset{1}{b} + \overset{2}{b}) + \tfrac{1}{12}(\overset{1}{b} + \overset{2}{b})^2 - \tfrac{1}{2}e(\overset{1}{\xi} + \overset{2}{\xi}) - \tfrac{1}{4}(1 + e^2)(\overset{1}{\xi} + \overset{2}{\xi})^2 \\ & - \frac{\overset{1}{y}_{,} + \overset{2}{y}_{,}}{\sqrt{1-e^2}}\left(\tfrac{1}{2} + \tfrac{3}{4}e^2\right) + \left(\frac{\overset{1}{y}_{,} + \overset{2}{y}_{,}}{\sqrt{1-e^2}}\right)^2 \left(\tfrac{1}{4} + \tfrac{5}{4}e^2 + \tfrac{15}{32}e^4\right) + \tfrac{1}{4}\Delta^{0,0,1} \end{aligned}$$

quae expressio integrum huius constantis valorem suppeditat; ad eam vero partem eius obtinendam, quae ad approximationem secundam proprie pertinet, subtrahatur is ipsius $C + \frac{1}{2}\varepsilon$ valor, quem approximatio prima suppeditavit.

13.

Si et termini tertii ordinis, quos aequatio (14) Sect. III. implicite cohibet, et termini primi ordinis omittuntur, erit

$$S+\varepsilon = 2\overset{2}{w} + \delta\frac{d\overset{2}{z}}{dt} + \frac{\overset{2}{y}_{\prime}}{\sqrt{1-e^2}}\frac{(r)^2}{(a)^2} - \frac{1}{2}\left(\delta\frac{d\overset{1}{z}}{dt}\right)^2 - \frac{\overset{1}{y}_{\prime}}{\sqrt{1-e^2}}\left[\frac{(r)^2}{(a)^2}\delta\frac{d\overset{1}{z}}{dt} - \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta\overset{1}{z}\right]$$
$$-\frac{1}{2}\frac{\overset{1}{y}_{\prime}{}^2}{1-e^2}\frac{(r)^4}{(a)^4} + (n)t\frac{y_{\prime\prime}}{\sqrt{1-e^2}}\frac{(r)^2}{(a)^2}$$

ubi ut supra numerus litteris superscriptus ordinem indicat. Haec igitur expressio terminos omnes secundi ordinis ipsius $S+\varepsilon$ continet. Quantitates $\overset{2}{w}$, $\delta\frac{d\overset{2}{z}}{dt}$ et $\left(\delta\frac{d\overset{1}{z}}{dt}\right)^2$ in praecedentibus eruimus, item

$$\frac{(r)^2}{(a)^2} = 1 + \tfrac{3}{2}e^2 + 2\Sigma_1^\infty R_2^{(i)}\cos ig$$
$$\frac{(r)^4}{(a)^4} = 1 + 5e^2 + \tfrac{15}{8}e^4 + 2\Sigma_1^\infty\left\{\frac{12}{i^2}R_2^{(i)} - e\frac{10}{i^2}\frac{dR_2^{(i)}}{de}\right\}\cos ig$$

Sit porro

$$\frac{\overset{1}{y}_{\prime}}{\sqrt{1-e^2}}\left[\frac{(r)^2}{(a)^2}\delta\frac{d\overset{1}{z}}{dt} - \frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)\delta\overset{1}{z}\right] = \Sigma\eta^{i,i',x}\cos(ig+i'g'+H_x)$$

et coefficiens ipsius $\cos(ig+i'g'+H_x)$ in $\overset{2}{w}$ per $(\overset{2}{w})^{i,i',x}$ et in $\delta\frac{d\overset{2}{z}}{dt}$ per $\left(\delta\frac{d\overset{2}{z}}{dt}\right)^{i,i',x}$ sit denotatus: tum, substitutis his valoribus, habetur

$$S+\varepsilon = \frac{\overset{2}{y}_{\prime}}{\sqrt{1-e^2}}(1+\tfrac{3}{2}e^2) - \frac{1}{2}\frac{\overset{1}{y}_{\prime}{}^2}{1-e^2}(1+5e^2+\tfrac{15}{8}e^4) + (n)t\frac{y_{\prime\prime}}{\sqrt{1-e^2}}(1+\tfrac{3}{2}e^2)$$
$$+\Sigma\left\{2(\overset{2}{w})^{i,i',x} + \left(\delta\frac{d\overset{2}{z}}{dt}\right)^{i,i',x} - \tfrac{1}{2}\Delta^{i,i',x} - \eta^{i,i',x}\right\}\cos(ig+i'g'+H_x)$$
$$+\Sigma_1^\infty\left[\frac{2\overset{2}{y}_{\prime}}{\sqrt{1-e^2}}R_2^{(i)} + \frac{\overset{1}{y}_{\prime}{}^2}{1-e^2}\left(\frac{10e}{i^2}\frac{dR_2^{(i)}}{de} - \frac{12}{i^2}R_2^{(i)}\right)\right]\cos ig + 2(n)t\frac{y_{\prime\prime}}{\sqrt{1-e^2}}\Sigma_1^\infty R_2^{(i)}\cos ig$$

Si praeterea termini secundi ordinis ipsius $S+\varepsilon$ adiumento formulae (15) Sect. III., quae propter ea quae in praecedentibus exposuimus nulla indiget explicatione, computantur, calculus totus confirmari eiusque errores, si qui adsint, detegi et corrigi possunt.

14.

Perturbationes secundi ordinis ipsarum P, Q et K, quas $\delta\overset{2}{P}$, $\delta\overset{2}{Q}$ et $\delta\overset{2}{K}$ appellabo, suppeditant aequationes art. 12. Sect. III. tales

$$\left.\begin{aligned}
\delta\overset{2}{P} &= -2\sin\tfrac{1}{2}I\overset{2}{\alpha}_{,}(n)t - \sin\tfrac{1}{2}I\alpha_{,,}(n)^2t^2 \\
&\quad + (n)\int\left\{\begin{aligned}&-A(S+\varepsilon)+\frac{dA}{dg}(n)\delta z+r\frac{dA}{dr}w+\frac{dA}{dg'}(n')\delta z'+r'\frac{dA}{dr'}w' \\ &+D\delta P+(E-\overset{1}{\alpha}_{,})\delta Q+\frac{dA}{dk}\delta K\end{aligned}\right\}dt \\
\delta\overset{2}{Q} &= \quad (n)\int\left\{\begin{aligned}&-B(S+\varepsilon)+\frac{dB}{dg}(n)\delta z+r\frac{dB}{dr}w+\frac{dB}{dg'}(n')\delta z'+r'\frac{dB}{dr'}w' \\ &+(F+\overset{1}{\alpha}_{,})\delta P+G\delta Q+\frac{dB}{dk}\delta K\end{aligned}\right\}dt \\
\delta\overset{2}{K} &= \quad \overset{2}{\eta}_{,}(n)t+\tfrac{1}{2}\eta_{,,}(n)^2t^2-\mathrm{tg}^2\tfrac{1}{2}I\overset{2}{\alpha}_{,}(n)t-\tfrac{1}{2}\mathrm{tg}^2\tfrac{1}{2}I\alpha_{,,}(n)^2t^2 \\
&\quad -\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\delta\overset{2}{P}+(n)\int\left\{\frac{B}{4\cos^2\frac{1}{2}I}\delta P-\left(\overset{1}{\alpha}_{,}\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}+\frac{1+\sin^2\frac{1}{2}I}{4\cos^4\frac{1}{2}I}A\right)\delta Q\right\}dt
\end{aligned}\right\}\ldots\ldots(13)$$

et indoles quantitatum A, B, D, E, F et G ex eodem articulo cognoscitur.

In aequationibus perturbationes secundi ordinis ipsarum $(n)z$ et w exhibentibus nonnullos quidem tertii ordinis terminos recipere potuissemus, sed quum omnes huius generis terminos in approximatione secunda in calculum vocare nobis non liceret, omnes omittere debuimus. In formulis vero praecedentibus pro ipsis $\delta\overset{2}{P}$, $\delta\overset{2}{Q}$ et $\delta\overset{2}{K}$ terminos tertii ordinis, quorum statim ratio haberi potest, recipere licet, quoniam harum expressionum termini, qui explicite tertii ordinis sunt, vim non habent. Itaque quum, peractis computationibus in praecedentibus explicatis, termini et primi et secundi ordinis ipsarum $(n)z$, w et $(S+\varepsilon)$ comperti sint, aggregata harum perturbationum statim pro $(n)z$, w et $S+\varepsilon$ substituere possumus, unde in approximatione secunda terminos quosdam tertii ordinis iam cum terminis secundi ordinis iunctos obtinebimus. Quum vero adhuc ipsarum P, Q et K perturbationes primi tantum ordinis compertae sint, loco δP, δQ et δK non nisi hos terminos substituere nobis licet. Quibus positis adiumento relationis inter δP atque δK in art. 14. Sect. praec. datae inveniuntur

$$D\delta P+\frac{dA}{dk}\delta K=\left(D-\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\frac{dA}{dk}\right)\delta P$$

$$(F+\overset{\mathrm{I}}{\alpha})\delta P+\frac{dB}{dk}\delta K=\left(F+\overset{\mathrm{I}}{\alpha}-\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\frac{dB}{dk}\right)\delta P$$

Sint

$$D-\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\frac{dB}{dk}=\frac{R_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)$$

$$F-\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\frac{dB}{dk}=\frac{X_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)$$

tum differentiationibus, quae in expressionibus ipsarum A, B, D et F indicatae sunt, institutis, facili opera inveniuntur

$$R_1=o,\qquad X_1=-\frac{3-9\sin^2\frac{1}{2}I+6\sin^4\frac{1}{2}I}{1-6\sin^2\frac{1}{2}I+6\sin^4\frac{1}{2}I}$$

$$R_2=\frac{1+3\sin^2\frac{1}{2}I}{\cos^2\frac{1}{2}I},\qquad X_2=-\frac{1+3\sin^2\frac{1}{2}I}{\cos^2\frac{1}{2}I}$$

$$R_3=\frac{\frac{1}{2}-\frac{5}{2}\sin^2\frac{1}{2}I+3\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I},\qquad X_3=-\frac{\frac{1}{2}-\frac{5}{2}\sin^2\frac{1}{2}I+3\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I}$$

$$R_4=\frac{\frac{1}{2}-\frac{3}{2}\sin^2\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I},\qquad X_4=-\frac{\frac{1}{2}+\frac{3}{2}\sin^2\frac{1}{2}I-\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I}$$

$$R_5=3\frac{\cos^2\frac{1}{2}I}{\sin^2\frac{1}{2}I},\qquad X_5=-3\frac{\cos^2\frac{1}{2}I}{\sin^2\frac{1}{2}I}$$

$$R_6=\frac{\frac{1}{2}+\frac{11}{2}\sin^2\frac{1}{2}I-\frac{75}{2}\sin^4\frac{1}{2}I}{1-11\sin^2\frac{1}{2}I+25\sin^4\frac{1}{2}I},\qquad X_6=-\frac{\frac{11}{2}-35\sin^2\frac{1}{2}I+\frac{89}{2}\sin^4\frac{1}{2}I}{(1-11\sin^2\frac{1}{2}I+25\sin^4\frac{1}{2}I)\cos^2\frac{1}{2}I}$$

$$R_7=\frac{\frac{3}{2}+\frac{9}{2}\sin^2\frac{1}{2}I-\frac{27}{2}\sin^4\frac{1}{2}I}{1-3\sin^2\frac{1}{2}I+3\sin^4\frac{1}{2}I},\qquad X_7=-\frac{\frac{3}{2}+3\sin^2\frac{1}{2}I-\frac{39}{2}\sin^4\frac{1}{2}I+24\sin^6\frac{1}{2}I}{(1-3\sin^2\frac{1}{2}I+3\sin^4\frac{1}{2}I)\cos^2\frac{1}{2}I}$$

$$R_8=\frac{\frac{1}{2}-\frac{11}{2}\sin^2\frac{1}{2}I+\frac{15}{3}\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-\frac{10}{3}\sin^4\frac{1}{2}I},\qquad X_8=-\frac{\frac{1}{2}\cos^2\frac{1}{2}I}{\sin^2\frac{1}{2}I-\frac{10}{3}\sin^4\frac{1}{2}I}$$

$$R_9=\frac{\frac{1}{2}-\frac{9}{2}\sin^2\frac{1}{2}I+9\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-2\sin^4\frac{1}{2}I},\qquad X_9=-\frac{\frac{1}{2}-5\sin^2\frac{1}{2}I+\frac{53}{2}\sin^4\frac{1}{2}I-16\sin^6\frac{1}{2}I}{(\sin^2\frac{1}{2}I-\frac{10}{3}\sin^4\frac{1}{2}I)\cos^2\frac{1}{2}I}$$

$$R_{10}=-\frac{\frac{1}{2}-\frac{5}{2}\sin^2\frac{1}{2}I-3\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-2\sin^4\frac{1}{2}I},\qquad X_{10}=-\frac{\frac{1}{2}+3\sin^2\frac{1}{2}I-\frac{15}{2}\sin^4\frac{1}{2}I}{(\sin^2\frac{1}{2}I-\frac{10}{3}\sin^4\frac{1}{2}I)\cos^2\frac{1}{2}I}$$

Positis denique

$$E=\frac{E_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)$$

$$G=\frac{G_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)$$

inveniuntur

$$E_1 = \frac{3-27\sin^2\frac{1}{2}I+30\sin^4\frac{1}{2}I}{1-6\sin^2\frac{1}{2}I+6\sin^4\frac{1}{2}I}, \qquad G_1 = 0,$$

$$E_2 = \frac{1-5\sin^2\frac{1}{2}I}{\cos^2\frac{1}{2}I}, \qquad G_2 = \frac{1-5\sin^2\frac{1}{2}I}{\cos^2\frac{1}{2}I}$$

$$E_3 = -\frac{\frac{3}{2}-\frac{9}{2}\sin^2\frac{1}{2}I+5\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I}, \qquad G_3 = -\frac{\frac{1}{2}-\frac{9}{2}\sin^2\frac{1}{2}I+5\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I},$$

$$E_4 = -\frac{\frac{1}{2}-\frac{9}{2}\sin^2\frac{1}{2}I+5\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I}, \qquad G_4 = -\frac{\frac{1}{2}-\frac{3}{2}\sin^2\frac{1}{2}I}{\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I}$$

$$E_5 = -\frac{3-5\sin^2\frac{1}{2}I}{\sin^2\frac{1}{2}I}, \qquad G_5 = -\frac{3-5\sin^2\frac{1}{2}I}{\sin^2\frac{1}{2}I}$$

$$E_6 = \frac{\frac{11}{2}-\frac{183}{2}\sin^2\frac{1}{2}I+125\sin^4\frac{1}{2}I}{1-11\sin^2\frac{1}{2}I+25\sin^4\frac{1}{2}I}, \qquad G_6 = \frac{\frac{1}{2}-\frac{33}{2}\sin^2\frac{1}{2}I+\frac{125}{2}\sin^4\frac{1}{2}I}{1-11\sin^2\frac{1}{2}I+25\sin^4\frac{1}{2}I}$$

$$E_7 = \frac{\frac{3}{2}-\frac{27}{2}\sin^2\frac{1}{2}I+15\sin^4\frac{1}{2}I}{1-3\sin^2\frac{1}{2}I+3\sin^4\frac{1}{2}I}, \qquad G_7 = \frac{\frac{3}{2}-\frac{27}{2}\sin^2\frac{1}{2}I+\frac{45}{2}\sin^4\frac{1}{2}I}{1-3\sin^2\frac{1}{2}I+3\sin^4\frac{1}{2}I}$$

$$E_8 = -\frac{\frac{1}{2}-\frac{23}{2}\sin^2\frac{1}{2}I+\frac{10}{3}\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-\frac{10}{3}\sin^4\frac{1}{2}I}, \qquad G_8 = -\frac{\frac{1}{2}-\frac{13}{2}\sin^2\frac{1}{2}I+\frac{25}{3}\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-\frac{10}{3}\sin^4\frac{1}{2}I}$$

$$E_9 = -\frac{\frac{1}{2}-\frac{15}{2}\sin^2\frac{1}{2}I+10\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-2\sin^4\frac{1}{2}I}, \qquad G_9 = -\frac{\frac{7}{2}-\frac{15}{2}\sin^2\frac{1}{2}I+15\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-2\sin^4\frac{1}{2}I}$$

$$E_{10} = -\frac{\frac{1}{2}-\frac{15}{2}\sin^2\frac{1}{2}I+10\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-2\sin^4\frac{1}{2}I}, \qquad G_{10} = \frac{\frac{1}{2}-\frac{3}{2}\sin^2\frac{1}{2}I-5\sin^4\frac{1}{2}I}{\sin^2\frac{1}{2}I-2\sin^4\frac{1}{2}I}$$

Substitutis valoribus praecedentibus ipsarum $D-\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\frac{dA}{dk}$, etc., nec non valoribus ipsarum A atque B in Sect. praec. datis, aequationes (13) abeunt in has

$$\delta \overset{2}{P} = -2\sin\tfrac{1}{2}I\,\overset{2}{\alpha}_{,}(n)t-\sin\tfrac{1}{2}I\,\alpha_{,,}(n)^2t^2$$

$$+(n)\int\left\{\begin{aligned}&-(S+\varepsilon)\Sigma\frac{P_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)-(n)\delta z\Sigma\frac{iP_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)\\&+w\Sigma\frac{h_xP_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)-(n)'\delta z'\Sigma\frac{i'P_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)\\&-w'\Sigma\frac{(h_x+1)P_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)+\delta P\Sigma\frac{R_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)\\&+\delta Q\left\{-\alpha_{,}+\Sigma\frac{E_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)\right\}\end{aligned}\right\}dt$$

$$\delta Q = (n)\int\left\{\begin{aligned}&-(S+\varepsilon)\Sigma\frac{Q_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)-(n)\delta z\Sigma\frac{iQ_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)\\&+w\Sigma\frac{h_xQ_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)-(n)'\delta z'\Sigma\frac{i'Q_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)\\&-w'\Sigma\frac{(h_x+1)Q_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)+\delta P\left\{\alpha_{,}+\Sigma\frac{X_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)\right\}\\&+\delta Q\,\Sigma\frac{G_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)\end{aligned}\right\}dt$$

$$\delta\overset{2}{K} = \overset{2}{\eta}_{,}(n)t + \tfrac{1}{2}\eta_{,,}(n)^2t^2 - \operatorname{tg}^2\tfrac{1}{2}I\overset{2}{\alpha}_{,}(n)t - \tfrac{1}{2}\operatorname{tg}^2\tfrac{1}{2}I\alpha_{,,}(n)^2t^2 - \frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\delta\overset{2}{P}$$
$$+(n)\int\Big\{\frac{\delta P}{4\cos^2\frac{1}{2}I}\Sigma\frac{Q_x}{\sqrt{1-e^2}}[i,i']_x\sin(ig+i'g'+H_x)$$
$$-\delta Q\Big\{\overset{2}{\alpha}_{,}\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I} + \frac{1+\sin^2\frac{1}{2}I}{4\cos^4\frac{1}{2}I}\Sigma\frac{P_x}{\sqrt{1-e^2}}[i,i']_x\cos(ig+i'g'+H_x)\Big\}\Big\}dt$$

Producta haec eodem modo computanda sunt quo reliqua de quibus iam disseruimus. Quo facto, aggregatum nanciscimur productorum quae in $\delta\overset{2}{P}$ continentur sub forma hac

$$P^{i,i',x}\cos(ig+i'g'+H_x)+(n)tp^{i,i',x}\cos(ig+i'g'+H_x)$$

aggregatum productorum quae in $\delta\overset{2}{Q}$ continentur sub forma hac

$$Q^{i,i',x}\sin(ig+i'g'+H_x)+(n)t\,q^{i,i',x}\sin(ig+i'g'+H_x)$$

et aggregatum productorum in $\delta\overset{2}{K}$ sub forma hac

$$K^{i,i',x}\cos(ig+i'g'+H_x)$$

designantibus $P^{i,i',x}$, $p^{i,i',x}$, $Q^{i,i',x}$, $q^{i,i',x}$ atque $K^{i,i',x}$ coefficientes numericos, Hinc emergunt

$$\delta\overset{2}{P} = \frac{P^{i,i',x}}{i+i'u+\nu_x}\sin(ig+i'g'+H_x)+(n)t\frac{p^{i,i',x}}{i+i'u+\nu_x}\sin(ig+i'g'+H_x)$$
$$+\frac{p^{i,i',x}}{(i+i'u+\nu_x)^2}\cos(ig+i'g'+H_x)$$

ubi casus specialis, in quo simul $i=o$, $i'=o$, $x=1$ est, excipitur, cuius loco habetur

$$\overset{2}{\alpha}_{,} = \frac{P^{o,o,1}}{2\sin\frac{1}{2}I}\;;\;\alpha_{,,} = \frac{p^{o,o,1}}{2\sin\frac{1}{2}I}$$

porro

$$\delta\overset{2}{Q} = -\frac{Q^{i,i',x}}{i+i'u+\nu_x}\cos(ig+i'g'+H_x) - (n)t\frac{q^{i,i',x}}{i+i'u+\nu_x}\cos(ig+i'g'+H_x)$$
$$+\frac{q^{i,i',x}}{(i+i'u+\nu_x)^2}\sin(ig+i'g'+H_x)$$

quae expressio exceptionem non patitur, quia $Q^{o,o,1}$ et $q^{o,o,1}$ in expressione ipsius $\frac{dQ}{dt}$ per $\sin o$ multiplicatae sunt; denique

$$\delta\overset{2}{K} = -\frac{\sin\frac{1}{2}I}{2\cos^2\frac{1}{2}I}\delta\overset{2}{P} + \frac{K^{i,i',x}}{i+i'u+\nu_x}\sin(ig+i'g'+H_x)$$

ubi in ultimo termino etiam casus $i = o$, $i' = o$, $x = 1$ excipiendus est, cuius loco habetur

$$\mathring{\eta}_{,} = \mathring{\alpha}\,\mathrm{tg}^2\tfrac{1}{2}I + K^{o,o,1};\quad \eta_{,,} = \alpha_{,,}\,\mathrm{tg}^2\tfrac{1}{2}I$$

15.

Quum in praecedentibus computationes approximationis secundae copiose explicaverimus, de computationibus approximationis tertiae, quae eodem modo quo illae perficiuntur, pauca tantum dicenda sunt.

In approximatione tertia termini omnes et quantitatis in art. 7. Sect. III. Z appellatae et aequationum (11) et (13) Sect. III., quae $(n)z$ et w suppeditant, in calculum vocandi sunt. Quum ex approximationibus prima et secunda valores numerici omnium coefficientium, quibus in approximatione tertia utemur, noti sint, computationis modus, quem descripsimus, ipse monstrat et eos terminos, qui in approximatione tertia vim habent, et eos, quos negligere licet, ita ut nullus calculatorem attentum fugere possit.

Termini tertii ordinis in aequationibus (11) et (13) Sect. III. atque in expressione ipsius Z duplici continentur modo, tum explicito tum implicito. Explicite enim adsunt in iis qui tribus, implicite autem in iis qui duobus factoribus constant, quorum ultimi generaliter maximos terminos tertii ordinis suppeditabunt. In terminis quidem $\frac{d(\mathring{T})}{dg}(n)\,\delta z$, $r\frac{d(\mathring{T})}{dr}w$, etc., in quibus in approximatione secunda loco $(n)\delta z$, w etc. harum quantitatum perturbationes primi ordinis substitutae erant, in approximatione tertia perturbationes secundi ordinis earundem quantitatum substituendae sunt, in terminis autem $\frac{d^2(\mathring{T})}{dg^2}(n)^2\,\delta z^2$, $r\frac{d^2(\mathring{T})}{dg\,.\,dr}(n)\,\delta z\,.\,w$, etc. termini primi ordinis ipsarum $(n)\delta z$, w, etc. ad terminos tertii ordinis obtinendos sufficiunt; sed quum termini, qui explicite quarti ordinis sunt, vim non habeant, in terminis modo allatis aggregata perturbationum primi et secundi ordinis ipsarum $(n)z$, w, etc. substitui possunt, unde statim praecisionem maiorem adipiscimur. Eadem ratione substituantur in utroque et resp. quoque factore terminorum

$$(n)\delta z\left\{\left(\overline{\frac{dW}{dy}}\right) - \frac{y_{,}}{\sqrt{1-(e)^2}}\,\frac{d.\frac{(r)^2}{(a)^2}}{dg} - \frac{y_{,,}}{\sqrt{1-(e)^2}}\,\frac{d.\frac{(r)^2}{(a)^2}}{dg}(n)t\right\},\ \text{etc.}$$

aggregata perturbationum primi et secundi ordinis. Quamquam perturbationes tertii ordinis in motu Lunae nullo modo negligendae sunt, tamen pauci admodum earum termini existunt, qui vim habent, quo fit, ut computatio haec brevior evadat quam pro specie, quam prae se fert, iudicaveris. Quum vero inter terminos tertii ordinis nonnulli haud parvuli adsint, approximatio quarta, in qua hi termini in iisdem formulis substituantur, instituatur, qua approximationes finitae erunt. Itaque perturbationes inventae, si in formulis nostris denuo substitutae fuissent, easdem perturbationes reproducerent, et formulas plane identicas redderent, id quod aliter non esse debet.

Restant singula quaedam explicanda. Approximatio tertia quantitates nonnullas requirit, quae ex prioribus approximationibus immediate datae non sunt. Hae sequentibus facili negotio computantur formulis, quarum demonstratio ex indole ipsius Ω facile peti potest.

Si U in partes decem ipsis Ω_x correspondentes distribuitur, quas partes signo U_x generaliter designabo, ita ut sit

$$U = \Sigma_1^{10} U_x \cos(x\gamma + ig + i'g' + H_x)$$

habetur

$$\frac{dU}{dg} = -\Sigma_1^{10}\, i\, U_x \sin(x\gamma + ig + i'g' + H_x)$$

$$\frac{dU}{dg'} = -\Sigma_1^{10}\, i'\, U_x \sin(x\gamma + ig + i'g' + H_x)$$

$$r\frac{dU}{dr} = \Sigma_1^{10} h_x U_x \cos(x\gamma + ig + i'g' + H_x)$$

$$r'\frac{dU}{dr'} = -\Sigma_1^{10} (h_x + 1) U_x \cos(x\gamma + ig + i'g' + H_x)$$

etc.

ubi h_x eadem est atque in art. 7. Sect. IV. Porro posita

$$(\mathring{T}) = \Sigma_1^{10} (\mathring{T})_x \cos(x\gamma + ig + i'g' + H_x)$$

habetur

$$\frac{d^2(\mathring{T})}{dg^2} = -\Sigma_1^{10}\, i^2 (\mathring{T})_x \cos(x\gamma + ig + i'g' + H_x)$$

$$\frac{d^2(\mathring{T})}{dg.dg'} = -\Sigma_1^{10}\, i i' (\mathring{T})_x \cos(x\gamma + ig + i'g' + H_x)$$

$$r\frac{d^2(\mathring{T})}{dr.dg} = -\Sigma_1^{10}\, i h_x (\mathring{T})_x \sin(x\gamma + ig + i'g' + H_x)$$

etc. etc.

Quum per aequationem (6) habeatur in terminis finitis

$$r^2\frac{d^2(\mathring{T})}{dr^2}+r\frac{d(\mathring{T})}{dr}=r\frac{dV}{dr}-r\frac{d(\mathring{T})}{dr}+r\frac{dU}{dr}-r\frac{d\left(\frac{d(S)}{dt}\right)}{dr}$$

et quum posita

$$V_{,}=\frac{an}{\sqrt{1-e^2}}\left\{2\frac{\varrho}{r}\cos(v_{,}-\lambda)-1+2\frac{\varrho}{a(1-e^2)}[\cos(v_{,}-\lambda)-1]\right\}\left\{r^2\left(\frac{d^2\Omega}{dv_{,}.\,dr^2}\right)+r\left(\frac{d^2\Omega}{dv_{,}.\,dr}\right)\right\}$$
$$+\frac{an}{\sqrt{1-e^2}}\,2\frac{\varrho}{r}\sin(v_{,}-\lambda)\left\{r^3\left(\frac{d^3\Omega}{dr^3}\right)+3r^2\left(\frac{d^2\Omega}{dr^2}\right)+r\left(\frac{d\Omega}{dr}\right)\right\}$$

ubi valores pure elliptici quantitatum, quas continet, solummodo substituendi sunt, aequatio (5) suppeditet

$$r\frac{dV}{dr}=V_{,}-V+r\frac{dU}{dr}-r\frac{d\left(\frac{d(S)}{dt}\right)}{dr}$$

invenitur

$$r^2\frac{d^2(\mathring{T})}{dr^2}+r\frac{d(\mathring{T})}{dr}=V_{,}-2V+(\mathring{T})-U+2r\frac{dU}{dr}+\frac{d(S)}{dt}-2r\frac{d\left(\frac{d(S)}{dt}\right)}{dr}$$

quae, introducta significatione illis analoga hac,

$$\frac{d(S)}{dt}=\Sigma_1^{10}\left(\frac{d(S)}{dt}\right)_x\cos(x\gamma+ig+i'g'+H_x)$$

ubi notandum est esse semper $x=o$, facili opera transformatur in hanc

$$r^2\frac{d^2(\mathring{T})}{dr^2}+r\frac{d(\mathring{T})}{dr}=\Sigma_1^{10}\left\{(h_x-1)^2(\mathring{T})_x+(2h_x-1)\left[U_x-\left(\frac{d(S)}{dt}\right)_x\right]\right\}\cos(x\gamma+ig+i'g'+H_x)$$

Denique habetur

$$rr'\frac{d^2(\mathring{T})}{dr.\,dr'}=-\Sigma_1^{10}\left\{(h_x^2-1)(\mathring{T})_x+(h_x-1)\left[U_x-\left(\frac{d(S)}{dt}\right)_x\right]\right\}\cos(x\gamma+ig+i'g'+H_x)$$

etc. etc.

In art. 11. introduximus significationes

$$g=\int(n)_\circ dt+(c)$$
$$\gamma=\int(n)_\circ d\tau+(c)$$

ubi

$$(n)^\circ=(n)+\mu(n)^2t+\mu_{,}(n)^3t^2+\text{etc.}$$

et animadvertimus in H_x ubique $\int y\,dt$ loco yt, $\int\alpha\,dt$ loco αt et $\int\eta\,dt$ loco ηt substituendas esse. In integratione igitur aequationum differentialium,

quibus $(n)z$ et w elicitur, in approximatione tertia harum quantitatum ratio habenda est.

Habetur

$$
\begin{aligned}
\int y\,dt &= y_{,}t + \tfrac{1}{2}y_{,,}(n)t^2 + \tfrac{1}{3}y_{,,,}(n)^2t^3 + \text{etc.}\\
\int \alpha\,dt &= \alpha_{,}t + \tfrac{1}{2}\alpha_{,,}(n)t^2 + \tfrac{1}{3}\alpha_{,,,}(n)^2t^3 + \text{etc.}\\
\int \eta\,dt &= \eta_{,}t + \tfrac{1}{2}\eta_{,,}(n)t^2 + \tfrac{1}{3}\eta_{,,,}(n)^2t^3 + \text{etc.}\\
\int (n)_0\,dt &= (n)t + \tfrac{1}{2}\mu(n)^2t^2 + \tfrac{1}{3}\mu_{,}(n)^3t^3 + \text{etc.}\\
\int (n)_0\,d\tau &= N + (n)\tau + \mu(n)^2t\tau + \mu_{,}(n)^3t^2\tau + \text{etc.}
\end{aligned}
$$

ubi N constans integrationi adiecta est, quae functio ipsius t esse potest, et ita determinanda est, ut mutata τ in t integrale $\int(n)_0 d\tau$ cum integrali $\int(n)_0 dt$ coincidat. Quae conditio suppeditat

$$N = -\tfrac{1}{2}\mu(n)^2t^2 - \tfrac{2}{3}\mu_{,}(n)^3t^3 - \text{etc.}$$

unde

$$\int(n)_0 d\tau = (n)\tau + \mu(n)^2(t\tau - \tfrac{1}{2}t^2) + \mu_{,}(n)^3(t^2\tau - \tfrac{2}{3}t^3) + \text{etc.}$$

Iam quum, nulla terminorum per $(n)t$ multiplicatorum habita ratione, generaliter sit

$$(14)\ldots\ldots \qquad \frac{d^2\zeta}{d\tau\,dt} = (n)\Sigma M \sin(\varkappa\gamma + ig + i'g' + H_x)$$

ubi M coefficientem numericum pro quovis argumento diversum repraesentat, et quum H_x sub forma hac exhiberi possit

$$H_x = i''y_{,}'(n)t + i''y_{,,}'(n)^2t^2 + i'''\int y\,dt + i^{iv}\int\alpha\,dt + i^{v}\int\eta\,dt + C_x$$

ubi C_x arcus constans est, argumenta ipsius $\frac{d^2\zeta}{d\tau\,dt}$ sub forma sua generali exhibita ita se habent

$$
\begin{aligned}
\varkappa\gamma + ig + i'g' + H_x = {} & (\varkappa+i)(c) + i'(c') + C_x + \varkappa(n)\tau + i(n)t + i'(n')t + i''y_{,}'(n)t + i'''y_{,}(n)t + i^{iv}\alpha_{,}t + i^{v}\eta_{,}t\\
& + \varkappa\mu(n)\tau^2t + \tfrac{1}{2}\{(i-\varkappa)\mu + i''y_{,,}' + i'''y_{,,} + i^{iv}\alpha_{,,} + i^{v}\eta_{,,}\}(n)^2t^2\\
& + \varkappa\mu_{,}(n)^3\tau t^2 + \tfrac{1}{3}\{(i-2\varkappa)\mu_{,} + i'''y_{,,,} + i^{iv}\alpha_{,,,} + i^{v}\eta_{,,,}\}(n)^3t^3 + \text{etc.}
\end{aligned}
$$

Iam, neglecto quadrato et potestatibus temporis superioribus in coefficientibus, expressio (14) integrata suppeditat, quia $u = \frac{(n)'}{(n)}$ et v_x ipsi $i''y_{,}' + i'''y_{,} + i^{iv}\alpha_{,} + i^{v}\eta_{,}$ aequatur,

$$
\begin{aligned}
\frac{d\zeta}{dt} = {} & -\Sigma\frac{M}{i + i'u + v_x}\cos(\varkappa\gamma + ig + i'g' + H_x)\\
& + \Sigma\frac{\varkappa\mu(n)\tau + [(i-\varkappa)\mu + i''y_{,,}' + i'''y_{,,} + i^{iv}\alpha_{,,} + i^{v}\eta_{,,}](n)t}{(i + i'u + v_x)^2}M\cos(\varkappa\gamma + ig + i'g' + H_x)\\
& - \Sigma\frac{(i-\varkappa)\mu + i''y_{,,}' + i'''y_{,,} + i^{iv}\alpha_{,,} + i^{v}\eta_{,,}}{(i + i'u + v_x)^2}M\sin(\varkappa\gamma + ig + i'g' + H_x)
\end{aligned}
$$

Mutata τ in t, ex terminis praecedentibus evadunt hi

$$-\Sigma \frac{M}{i+i'u+\nu_x} \cos\left[(\varkappa+i)g+i'g'+H_x\right]$$
$$+(n)t\,\Sigma \frac{i\mu+i''y'_{,,}+i'''y_{,,}+i^{IV}\alpha_{,,}+i^{V}\eta_{,,}}{(i+i'u+\nu_x)^2} M\cos\left[(\varkappa+i)g+i'g'+H_x\right]$$
$$-\Sigma \frac{(i-\varkappa)\mu+i''y'_{,,}+i'''y_{,,}+i^{IV}\alpha_{,,}+i^{V}\eta_{,,}}{(i+i'u+\nu_x)^3} M\sin\left[(\varkappa+i)g+i'g'+H_x\right]$$

unde manifestum est formam generalem ipsius $\frac{dz}{dt}$ esse hanc

$$\frac{dz}{dt} = \Sigma N\cos(ig+i'g'+H_x)+(n)t\,\Sigma O\cos(ig+i'g'+H_x)+\Sigma P\sin(ig+i'g'+H_x)$$

Hi igitur ipsius $\frac{dz}{dt}$ termini eiusdem formae sunt atque ii, quos in approximatione secunda, ubi in argumentis quantitates $\mu(n)^2t^2$, $\alpha_{,,}(n)^2t^2$, etc. pro constantibus habitae erant, invenimus. Primus expressionis praecedentis terminus idem est, qui, ratione ipsarum $\mu(n)^2t^2$, $\alpha_{,,}(n)^2t^2$, etc. non habita, existit, termini igitur $\mu(n)^2t^2$, $\alpha_{,,}(n)^2t^2$, etc. in argumentis existentes ad terminos per $(n)t\cos(ig+i'g'+H_x)$ et ad terminos per $\sin(ig+i'g'+H_x)$ multiplicatos, qui in ipsa $\frac{dz}{dt}$ aliunde iam adsunt, novos eiusdem formae addunt, ad quos tamen, quum in motu Lunae illis multo minores sint, opus non erit respicere.

SECTIO VI.

EXPLICATIO COMPVTATIONVM AD IPSAS $p_,$ ATQVE $q_,$, LATITVDINEM ET REDVCTIONEM LONGITVDINIS OBTINENDAS INSTITVENDARVM.

1.

Quantitatibus $(n)z$, w, P, Q et K secundum regulas in praecedentibus explicatas computatis, $p_,$ atque $q_,$ computandae sunt, e quibus latitudo versus planum proiectionis et reductio longitudinis in orbita ad idem planum, sive declinatio et differentia inter longitudinem $v_,$ appellatam atque ascensionem rectam Lunae pendent. Termini fere omnes ipsarum $p_,$ atque $q_,$ ex praecedentibus ipsarum P et Q computationibus iam noti sunt, uti ex formulis art. 12. Sect. III. patet. Quantitatem

$$A-A(S+\varepsilon)+\frac{dA}{dg}(n)\delta z+r\frac{dA}{dr}w+\frac{dA}{dg'}(n')\delta z'+r'\frac{dA}{dr'}w'+D\delta P+E\delta Q+\frac{dA}{dk}dK$$

quam per $P^{i,i',x}\cos(ig+i'g'+H_x)$ repraesentabo, ex computatione ipsius P, et quantitatem

$$B-B(S+\varepsilon)+\frac{dB}{dg}(n)\delta z+r\frac{dB}{dr}w+\frac{dB}{dg'}(n')\delta z'+r'\frac{dB}{dr'}w'+F\delta P+G\delta Q+\frac{dB}{dk}\delta K$$

quam per $Q^{i,i',x}\sin(ig+i'g'+H_x)$ repraesentabo, ex computatione ipsius Q novimus; restant igitur producta $A\delta Q$, $B\delta K$, $B\delta Q$ et $A\delta K$ per methodum supra descriptam computanda. Propter aequationem inter δK atque δP saepius memoratam

ex productis $A\delta Q$ atque $B\delta K$ in ft evadunt haec $\frac{\sin\frac{1}{2}I}{2\cos^3\frac{1}{2}I}\{A\delta Q - B\delta P\}$

et ex productis $B\delta Q$ atque $A\delta K$ in $-\varphi t$ haec $\frac{\sin\frac{1}{2}I}{2\cos^3\frac{1}{2}I}\{B\delta Q + A\delta P\}$

Positis igitur post multiplicationes institutas

$$\frac{\sin\frac{1}{2}I}{2\cos^3\frac{1}{2}I}\{A\delta Q - B\delta P\} = G^{i,i',x}\cos(ig+i'g'+H_x)$$

$$\frac{\sin\frac{1}{2}I}{2\cos^3\frac{1}{2}I}\{B\delta Q + A\delta P\} = \Gamma^{i,i',x}\sin(ig+i'g'+H_x)$$

et tum

$$\sec\tfrac{1}{2}I\,P^{i,i',x} + G^{i,i',x} = F^{i,i',x};\quad \sec\tfrac{1}{2}I\,Q^{i,i',x} + \Gamma^{i,i',x} = \Phi^{i,i',x}$$

habetur

$$ft = n\Sigma F^{i,i',x}\cos(ig+i'g'+H_x);\quad \varphi t = -n\Sigma\Phi^{i,i',x}\sin(ig+i'g'+H_x) \quad\ldots\ldots(1)$$

Coefficientes $G^{i,i',x}$ atque $\Gamma^{i,i',x}$ omnes minutissimi sunt et fere omnes negligendi. Coefficientes $P^{i,i',x}$ atque $Q^{i,i',x}$ huius formae sunt $P_, + ntP_{,,}$, $Q_, + ntQ_{,,}$, sed coefficientes $ntP_{,,}$ et $ntQ_{,,}$ in integrationibus tanquam constantes tractare licet, quia termini ex variabilitate horum coefficientium orientes vim non habent, sicut iam supra respectu ipsarum $(n)z$ et w annotavimus.

2.

Approximatio prima ad veros ipsarum $p_,$ et $q_,$ valores obtinendos in eo consistit, quod valores ipsarum $p_,$, $q_,$ et $u_,$ ex integralibus approximatis (60) Sect. II. petendi in formulis (62) Sect. II., et tum valores ipsarum H, L atque M hoc modo orti in integralibus rigorosis (64) Sect. II. substituuntur.

Formulae (1) in expressionibus (62) Sect. II. substitutae primum subministrant

$$\left.\begin{aligned} H &= u_,n\Sigma F^{i,i',x}\cos(ig+i'g'+H_x)\\ L &= -u_,n\Sigma\Phi^{i,i',x}\sin(ig+i'g'+H_x)\\ M &= p_,n\Sigma F^{i,i',x}\cos(ig+i'g'+H_x) - q_,n\Sigma\Phi^{i,i',x}\sin(ig+i'g'+H_x)\end{aligned}\right\}\ldots\ldots(2)$$

ubi casus $i=o$, $i'=o$, $x=1$ excipiendus est, qui valorem ipsius c suppeditat hunc

$$c = F^{o,o,1}$$

Substitutis porro in his expressionibus valoribus ipsarum $p_{,}$, $q_{,}$ atque u ex integralibus (60) Sect. II. petendis, nanciscimur

$$H = n\frac{\alpha-\eta}{\mathrm{v}}\cos\Gamma\Sigma F\cos(\mathrm{w}nt+W) - n\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma F\cos[n(\mathrm{w}+\mathrm{v})t+W-\Theta] - n\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma F\cos[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$

$$L = -n\frac{\alpha-\eta}{\mathrm{v}}\cos\Gamma\Sigma\Phi\sin(\mathrm{w}nt+W) + n\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma\Phi\sin[n(\mathrm{w}+\mathrm{v})t+W-\Theta] + n\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma\Phi\sin[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$

$$M = -n\frac{c}{\mathrm{v}}\cos\Gamma\Sigma\Phi\sin(\mathrm{w}nt+W) - n\sin\Gamma\Sigma\left\{\frac{\alpha-\eta}{2\mathrm{v}}\Phi-\tfrac{1}{2}F\right\}\sin[n(\mathrm{w}+\mathrm{v})t+W-\Theta] - n\sin\Gamma\Sigma\left\{\frac{\alpha-\eta}{2\mathrm{v}}\Phi+\tfrac{1}{2}F\right\}\sin[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$

ubi $ig+i'g'+H_x$ brevitatis caussa per $\mathrm{w}t+W$, et $\sqrt{(\alpha-\eta)^2+c^2}$, quia littera x, per quam haec quantitas in Sectione secunda denotata est, hoc loco ceteroquin utimur, per v reddidi, nec non indices i, i', x, qui in omnibus terminis iidem sunt, brevitatis caussa omisi. Quibus ipsarum H, L, M valoribus in integralibus (64) Sect. II. substitutis, emergunt

$$(3)\ldots\ldots\left\{\begin{aligned} p_{,} &= \sqrt{1-\lambda}\,.\,\sin\Gamma\sin(\mathrm{v}nt-\Theta)+\delta p_{,} \\ q_{,} &= \frac{\alpha-\eta}{\mathrm{v}}\sqrt{1-\lambda}\,.\,\sin\Gamma\cos(\mathrm{v}nt-\Theta)+\frac{c}{\mathrm{v}}\sqrt{1-\lambda}\,.\,\cos\Gamma+\delta q_{,} \\ u &= -\frac{c}{\mathrm{v}}\sqrt{1-\lambda}\,.\,\sin\Gamma\cos(\mathrm{v}nt-\Theta)+\frac{\alpha-\eta}{\mathrm{v}}\sqrt{1-\lambda}\,.\,\cos\Gamma+\delta u \end{aligned}\right.$$

ubi

$$\delta p_{,} = -\cos\Gamma\Sigma\left\{\frac{\alpha-\eta}{\mathrm{vw}}F+\frac{(\alpha-\eta)\mathrm{v}F+\mathrm{wv}\Phi}{\mathrm{w}(\mathrm{w}^2-\mathrm{v}^2)}\right\}\sin(\mathrm{w}nt+W)+\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma\frac{F}{\mathrm{w}}\sin[n(\mathrm{w}+\mathrm{v})t+W-\Theta]+\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma\frac{F}{\mathrm{w}}\sin[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$

$$\begin{aligned}\delta q_{,} = -\frac{\alpha-\eta}{\mathrm{v}}\cos\Gamma\Sigma\left\{\frac{\Phi}{\mathrm{w}}+\frac{(\alpha-\eta)\mathrm{w}F+\mathrm{v}^2\Phi}{\mathrm{w}(\mathrm{w}^2-\mathrm{v}^2)}\right\}\cos(\mathrm{w}nt+W)&+\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma\left\{\frac{\Phi}{\mathrm{w}+\mathrm{v}}+\frac{(\alpha-\eta)F}{\mathrm{w}(\mathrm{w}+\mathrm{v})}\right\}\cos[n(\mathrm{w}+\mathrm{v})t+W-\Theta] \\ &+\frac{c}{2\mathrm{v}}\sin\Gamma\Sigma\left\{\frac{\Phi}{\mathrm{w}-\mathrm{v}}+\frac{(\alpha-\eta)F}{\mathrm{w}(\mathrm{w}-\mathrm{v})}\right\}\cos[n(\mathrm{w}-\mathrm{v})t+W+\Theta]\end{aligned}$$

$$\begin{aligned}\delta u = \frac{c}{\mathrm{v}}\cos\Gamma\Sigma\left\{\frac{\Phi}{\mathrm{w}}+\frac{(\alpha-\eta)\mathrm{w}F+\mathrm{v}^2\Phi}{\mathrm{w}(\mathrm{w}^2-\mathrm{v}^2)}\right\}\cos(\mathrm{w}nt+W)&+\sin\Gamma\Sigma\left\{\frac{(\alpha-\eta)\Phi}{2\mathrm{v}(\mathrm{w}+\mathrm{v})}-\frac{\mathrm{wv}+c^2}{2(\mathrm{w}+\mathrm{v})\mathrm{wv}}F\right\}\cos[n(\mathrm{w}+\mathrm{v})t+W-\Theta] \\ &+\sin\Gamma\Sigma\left\{\frac{(\alpha-\eta)\Phi}{2\mathrm{v}(\mathrm{w}-\mathrm{v})}+\frac{\mathrm{wv}-c^2}{2(\mathrm{w}-\mathrm{v})\mathrm{wv}}F\right\}\cos[n(\mathrm{w}-\mathrm{v})t+W+\Theta]\end{aligned}$$

et secundum aequationem (64*) Sect. II. 2λ aggregatum ex quadratis omnium ipsarum $\delta p_{,}$, $\delta q_{,}$ atque δu coefficientium formatum denotat.

Quae quidem formulae terminos omnes approximationis primae continent, sed in hoc statu integro ad terminos omnes computandos adhibeantur, necesse non est. Maiores solummodo formulis praecedentibus, reliqui autem minores formulis computentur simplicioribus, quae, neglectis η et c^2 respectu ipsius v nec non parvula inter $F^{i,i',x}$ et $\Phi^{i,i',x}$ differentia, ex illis facili opera emergunt.

3.

Quantitatibus $p_,$, $q_,$ et u adiumento harum formularum computatis, accuratiores harum quantitatum valores innotescunt, et quum in calculis modo peractis valores ipsarum $p_,$, $q_,$ et u ex aequationibus (60) Sect. II. petiti in expressionibus (2) substituti sint, in approximatione secunda termini ipsarum $p_,$, $q_,$ et u substituendi sunt, quos approximatio prima sive formulae praecedentes ad illos addiderunt. Quamobrem habetur in approximatione secunda

$$\left.\begin{aligned} \delta H &= \delta u\, n\, \Sigma F^{i,i',x} \cos(ig+i'g'+H_x) \\ \delta L &= -\delta u\, n\, \Sigma \Phi^{i,i',x} \sin(ig+i'g'+H_x) \\ \delta M &= \delta p_, n\, \Sigma F^{i,i',x} \cos(ig+i'g'+H_x) - \delta q_, n\, \Sigma \Phi^{i,i',x} \sin(ig+i'g'+H_x) \end{aligned}\right\} \ldots\ldots (4)$$

Coefficientium ipsarum $\delta p_,$ $\delta q_,$ et δu valores numerici ex calculis approximationis primae noti sunt, multiplicationes igitur, quas formulae (4) requirunt, eodem modo perficiendae sunt, quem supra descripsi. Ex forma ipsarum $\delta p_,$, $\delta q_,$, δu in art. praec. datarum sequitur, multiplicationibus institutis aequationes (4) suppeditaturas esse

$$\left.\begin{aligned} \delta H = {} & n\,\Sigma G^{i,i',x} \cos(\mathrm{w}nt+W) + n\,\Sigma J^{i,i',x} \cos[n(\mathrm{w}+\mathrm{v})t+W-\Theta] \\ & + n\,\Sigma K^{i,i',x} \cos[n(\mathrm{w}-\mathrm{v})t+W+\Theta] \\ \frac{\alpha-\eta}{\mathrm{v}}\delta M - \frac{c}{\mathrm{v}}\delta L = {} & n\,\Sigma L^{i,i',x} \sin(\mathrm{w}nt+W) + n\,\Sigma M^{i,i',x} \sin[n(\mathrm{w}+\mathrm{v})t+W-\Theta] \\ & + n\,\Sigma N^{i,i',x} \sin[n(\mathrm{w}-\mathrm{v})t+W+\Theta] \\ \frac{\alpha-\eta}{\mathrm{v}}\delta L + \frac{c}{\mathrm{v}}\delta M = {} & n\,\Sigma O^{i,i',x} \sin(n\mathrm{w}t+W) + n\,\Sigma P^{i,i',x} \sin[n(\mathrm{w}+\mathrm{v})t+W-\Theta] \\ & + n\Sigma Q^{i,i',x} \sin[n(\mathrm{w}-\mathrm{v})t+W+\Theta] \end{aligned}\right\} \ldots\ldots (5)$$

ubi $G^{i,i',x}$, $J^{i,i',x}$, etc. coefficientes numerici sunt. Substitutis his expressionibus in integralibus (64) Sect. II. resp. loco H, $\frac{\alpha-\eta}{\mathrm{v}}M - \frac{c}{\mathrm{v}}L$ atque $\frac{\alpha-\eta}{\mathrm{v}}L + \frac{c}{\mathrm{v}}M$, emergit in approximatione secunda

$$\delta p_, = -\Sigma\left\{\frac{\mathrm{w}}{\mathrm{w}^2-\mathrm{v}^2}G - \frac{\mathrm{v}}{\mathrm{w}^2-\mathrm{v}^2}O\right\}\sin(n\mathrm{w}t+W) - \Sigma\left\{\frac{\mathrm{w}+\mathrm{v}}{(\mathrm{w}+\mathrm{v})^2-\mathrm{v}^2}J - \frac{\mathrm{v}}{(\mathrm{w}+\mathrm{v})^2-\mathrm{v}^2}P\right\}\sin[n(\mathrm{w}+\mathrm{v})t+W-\Theta]$$
$$-\Sigma\left\{\frac{\mathrm{w}-\mathrm{v}}{(\mathrm{w}-\mathrm{v})^2-\mathrm{v}^2}K - \frac{\mathrm{v}}{(\mathrm{w}-\mathrm{v})^2-\mathrm{v}^2}Q\right\}\sin[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$

$$\delta \overset{2}{q}_{,} = -\Sigma\left\{\frac{\alpha-\eta}{\mathrm{v}}\left[\frac{\mathrm{v}}{\mathrm{w}^2-\mathrm{v}^2}G - \frac{\mathrm{w}}{\mathrm{w}^2-\mathrm{v}^2}O\right] + \frac{c}{\mathrm{v}}\frac{L}{\mathrm{w}}\right\}\cos(n\mathrm{w}t+W)$$
$$-\Sigma\left\{\frac{\alpha-\eta}{\mathrm{v}}\left[\frac{\mathrm{v}}{(\mathrm{w}+\mathrm{v})^2-\mathrm{v}^2}J - \frac{\mathrm{w}+\mathrm{v}}{(\mathrm{w}+\mathrm{v})^2-\mathrm{v}^2}P\right] + \frac{c}{\mathrm{v}}\frac{M}{\mathrm{w}+\mathrm{v}}\right\}\cos[n(\mathrm{w}+\mathrm{v})t+W-\Theta]$$
$$-\Sigma\left\{\frac{\alpha-\eta}{\mathrm{v}}\left[\frac{\mathrm{v}}{(\mathrm{w}-\mathrm{v})^2-\mathrm{v}^2}K - \frac{\mathrm{w}-\mathrm{v}}{(\mathrm{w}-\mathrm{v})^2-\mathrm{v}^2}Q\right] + \frac{c}{\mathrm{v}}\frac{N}{\mathrm{w}-\mathrm{v}}\right\}\cos[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$
$$\delta \overset{2}{u} = \Sigma\left\{\frac{c}{\mathrm{v}}\left[\frac{\mathrm{v}}{\mathrm{w}^2-\mathrm{v}^2}G - \frac{\mathrm{w}}{\mathrm{w}^2-\mathrm{v}^2}O\right] - \frac{\alpha-\eta}{\mathrm{v}}\frac{L}{\mathrm{w}}\right\}\cos(n\mathrm{w}t+W)$$
$$+\Sigma\left\{\frac{c}{\mathrm{v}}\left[\frac{\mathrm{v}}{(\mathrm{w}+\mathrm{v})^2-\mathrm{v}^2}J - \frac{\mathrm{w}+\mathrm{v}}{(\mathrm{w}+\mathrm{v})^2-\mathrm{v}^2}P\right] - \frac{\alpha-\eta}{\mathrm{v}}\frac{M}{\mathrm{w}+\mathrm{v}}\right\}\cos[n(\mathrm{w}+\mathrm{v})t+W-\Theta]$$
$$+\Sigma\left\{\frac{c}{\mathrm{v}}\left[\frac{\mathrm{v}}{(\mathrm{w}-\mathrm{v})^2-\mathrm{v}^2}K - \frac{\mathrm{w}-\mathrm{v}}{(\mathrm{w}-\mathrm{v})^2-\mathrm{v}^2}Q\right] - \frac{\alpha-\eta}{\mathrm{v}}\frac{N}{\mathrm{w}-\mathrm{v}}\right\}\cos[n(\mathrm{w}-\mathrm{v})t+W+\Theta]$$

ubi etiam brevitatis caussa indices $i, i', \varkappa$ ipsarum G, J, K, L, etc., qui in omnibus terminis iidem sunt, omisi, et sicut in praecedentibus formulis est

$$\mathrm{w}t + W = ig + i'g' + H_{\varkappa} \text{ atque } \mathrm{v} = \sqrt{(\alpha-\eta)^2 + c^2}$$

quas formulas praeterea eodem modo quo formulas approximationis primae pro terminis minoribus abbreviare licet.

Additis his ipsarum $\delta\overset{2}{p}_{,}$, $\delta\overset{2}{q}_{,}$ atque $\delta\overset{2}{u}$ valoribus ad valores ipsarum $\delta p_{,}$, $\delta q_{,}$ atque δu, quos approximatio prima prodiderat, accuratiores harum quantitatum valores nanciscimur. Eaedem formulae praecedentes tertiae quoque approximationi et subsequentibus perficiendis inservient, si his opus sit.

Quum in formulis (5) indices i, i' omnes valores integros positivos et negativos inclusa cifra, et $\varkappa$ valores omnes positivos quos supra explicavimus, inclusa unitate, comprehendant, existunt in his expressionibus termini hi

$$(6)\ldots\ldots\begin{cases} \delta H = nG^{0,0,1} + n\{J^{0,0,1} + K^{0,0,1}\}\cos[\mathrm{v}nt - \Theta] \\ \frac{\alpha-\eta}{\mathrm{v}}\delta M - \frac{c}{\mathrm{v}}\delta L = n\{M^{0,0,1} - N^{0,0,1}\}\sin[\mathrm{v}nt - \Theta] \\ \frac{\alpha-\eta}{\mathrm{v}}\delta L + \frac{c}{\mathrm{v}}\delta M = n\{P^{0,0,1} - Q^{0,0,1}\}\sin[\mathrm{v}nt - \Theta] \end{cases}$$

Nisi hi termini insensibiles sint, artificium in art. 30. Sect. II. expositum in usum vocatur, quo termini per tempus ipsum multiplicati, qui ex terminis praecedentibus in valoribus ipsarum $p_{,}$, $q_{,}$ atque $u_{,}$ nascerentur, tollantur. Quem in finem facili opera reperitur, quantitates in aequationi-

bus (65) Sect. II. C atque D nominatas cum quantitatibus praecedentibus iunctas esse aequationibus his

$$C = \operatorname{cosec}\Gamma\left\{\frac{\alpha-\eta}{\mathrm{v}}(P^{0,0,1}-Q^{0,0,1})-\frac{c}{\mathrm{v}}(M^{0,0,1}-N^{0,0,1})\right\}$$

$$D = \operatorname{cosec}\Gamma\left\{\frac{\alpha-\eta}{\mathrm{v}}(M^{0,0,1}-N^{0,0,1})+\frac{c}{\mathrm{v}}(P^{0,0,1}-Q^{0,0,1})\right\}$$

Si igitur hi termini vim habent, ubique in valoribus ipsarum $p_{,}$, $q_{,}$ atque u modo exhibitis substituatur

$$\alpha-\eta+\operatorname{cosec}\Gamma\left\{\frac{\alpha-\eta}{\mathrm{v}}(P^{0,0,1}-Q^{0,0,1})-\frac{c}{\mathrm{v}}(M^{0,0,1}-N^{0,0,1})\right\}$$

loco $\alpha-\eta$, et

$$c+\operatorname{cosec}\Gamma\left\{\frac{\alpha-\eta}{\mathrm{v}}(M^{0,0,1}-N^{0,0,1})+\frac{c}{\mathrm{v}}(P^{0,0,1}-Q^{0,0,1})\right\}$$

loco c, et termini (6) in valoribus ipsarum δH, $\frac{\alpha-\eta}{\mathrm{v}}\delta M-\frac{c}{\mathrm{v}}\delta L$ atque $\frac{\alpha-\eta}{\mathrm{v}}\delta L+\frac{c}{\mathrm{v}}\delta M$ deleantur.

4.

Quantitatibus $p_{,}$ et $q_{,}$ ope formularum praecedentium computatis, latitudo Lunae supra planum ad quod Γ spectat et reductio longitudinis ad idem planum computandae sunt. Denotante s sinum huius latitudinis, habetur secundum art. 33. Sect. II.

$$s = q_{,}\sin V - p_{,}\cos V$$

ubi

$$V = \bar{f}+(y+\alpha-\eta)nt+v+k$$

ubi $\bar{f}$ est anomalia vera ope veri ipsius $(n)z$ valoris computanda. Positis

$$p_{,} = \sin\Gamma\sin[\mathrm{v}nt-\Theta]+\delta_{,}p_{,}$$
$$q_{,} = \sin\Gamma\cos[\mathrm{v}nt-\Theta]+\frac{c}{\mathrm{v}}\cos\Gamma+\delta_{,}q_{,}$$

ubi igitur

$$\delta_{,}p_{,} = \delta p_{,}+\delta\overset{2}{p}_{,}+\left\{\sqrt{1-\lambda}-1\right\}\sin\Gamma\sin(\mathrm{v}nt-\Theta),$$
$$\delta_{,}q_{,} = \delta q_{,}+\delta\overset{2}{q}_{,}+\left\{\frac{\alpha-\eta}{\mathrm{v}}\sqrt{1-\lambda}-1\right\}\sin\Gamma\cos(\mathrm{v}nt-\Theta)$$

denotantibus $\delta p_{,}$, $\delta \overset{2}{p}_{,}$, $\delta q_{,}$, $\delta \overset{2}{q}_{,}$ easdem in artt. praec. explicatas quantitates, propter aequationem quam proximam hanc $\mathrm{v} = \alpha + \eta$, habetur

$$s = \sin\Gamma \sin[\bar{f} + (y + \alpha - \eta - \mathrm{v})nt + v + k + \Theta]$$
$$+ \frac{c}{\mathrm{v}} \cos\Gamma \sin[\bar{f} + (y + \alpha - \eta)nt + v + k]$$
$$+ \delta_{,}q_{,} \sin[\bar{f} + (y + \alpha - \eta)nt + v + k) - \delta_{,}p_{,} \cos[\bar{f} + (y + \alpha - \eta)nt + v + k]$$

Praestat in primis duobus ipsius s terminis, qui omnium maximi sunt, ipsam $\bar{f}$ conservare, in reliquis vero loco $\bar{f}$ evolutionem eius per g expressam introducere.

Ex formulis Sect. IV. sequitur esse

$$\cos f = -e - \frac{1-e^2}{2} h^2 \frac{R_2^{(h)}}{e} \cos hg$$
$$\sin f = -\frac{\sqrt{1-e^2}}{2} h \left(\frac{dR_2^{(h)}}{de}\right) \sin hg$$

ubi h est index a valore $-\infty$ usque ad $+\infty$ per omnes numeros integros extendendus. Quamobrem rigorosae sunt aequationes hae

$$(7)\ldots\ldots \left\{\begin{aligned} \sin(\bar{f}+\omega) &= -e\sin\omega - \frac{\sqrt{1-e^2}}{2}\cos\omega\, \Sigma_{-\infty}^{+\infty} h \left(\frac{dR_2^{(h)}}{de}\right) \sin h[g + (n)\delta z] \\ &\quad - \frac{1-e^2}{2} \sin\omega\, \Sigma_{-\infty}^{+\infty} h^2 \frac{R_2^{(h)}}{e} \cos h[g + (n)\delta z] \\ \cos(\bar{f}+\omega) &= -e\cos\omega - \frac{1-e^2}{2}\cos\omega\, \Sigma_{-\infty}^{+\infty} h^2 \frac{R_2^{(h)}}{e} \cos h[g + (n)\delta z] \\ &\quad + \frac{\sqrt{1-e^2}}{2} \sin\omega\, \Sigma_{-\infty}^{+\infty} h \left(\frac{dR_2^{(h)}}{de}\right) \sin h[g + (n)\delta z] \end{aligned}\right.$$

ubi

$$\omega = (y + \alpha - \eta)nt + v + k$$

Quantitates $\delta_{,}p_{,}$ et $\delta_{,}q_{,}$ ita exhiberi possunt

$$\delta_{,}p_{,} = \Sigma P_{\beta}^{i,i',x} \sin(ig + i'g' + H_{x,\beta})$$
$$\delta_{,}q_{,} = \Sigma Q_{\beta}^{i,i',x} \cos(ig + i'g' + H_{x,\beta})$$

ubi

$$H_{x,\beta} = H_x + \beta[\mathrm{v}nt - \Theta]$$

denotante β indicem, qui valores 0, 1 et -1 tantum accipit. Quibus positis, invenitur per calculum, quem in theoria Iovis atque Saturni explicavi, formula rigorosa haec

$$s = \sin\Gamma \sin[\bar{f}+(y+\alpha-\eta-\mathrm{v})\,nt+v+k+\Theta]+\frac{c}{\mathrm{v}}\cos\Gamma\sin[\bar{f}+(y+\alpha-\eta)\,nt+v+k]$$
$$+\Sigma M^{(h)}\{P_\beta^{i,i',x}-Q_\beta^{i,i',x}\}\sin[h(n)\delta z+(h+i)g+i'g'+H_{x,\beta}+(y+\alpha-\eta)\,nt+v+k]$$
$$+\Sigma N^{(h)}\{P_\beta^{i,i',x}+Q_\beta^{i,i',x}\}\sin[h(n)\delta z+(h+i)g+i'g'+H_{x,\beta}-(y+\alpha-\eta)\,nt-v-k]$$

ubi

$$M^{(h)} = \frac{h^2}{4}\left\{(1-e^2)\frac{R^{(h)}_2}{e}+\sqrt{1-e^2}\cdot\left(\frac{dR^{(h)}_2}{hde}\right)\right\}$$
$$N^{(h)} = \frac{h^2}{4}\left\{(1-e^2)\frac{R^{(h)}_2}{e}-\sqrt{1-e^2}\cdot\left(\frac{dR^{(h)}_2}{hde}\right)\right\}$$

si excipis

$$M^{(0)} = N^{(0)} = \tfrac{1}{2}e$$

Si in formula praecedenti sinus et cosinus secundum formulam notam hanc

$$\sin[h(n)\delta z+(h+i)g+\text{etc.}] = \sin[(h+i)g+\text{etc.}]+h(n)\delta z\cos[(h+i)g+\text{etc.}]-\text{etc.}$$

evolvuntur, emergit

$$s = \sin\Gamma\sin[\bar{f}+(y+\alpha-\eta-\mathrm{v})\,nt+v+k+\Theta]+\frac{c}{\mathrm{v}}\cos\Gamma\sin[\bar{f}+(y+\alpha-\eta)\,nt+v+k]$$
$$+\Sigma M^{(h)}\{P_\beta^{i,i',x}-Q_\beta^{i,i',x}\}\sin[(h+i)g+i'g'+H_{x,\beta}+(y+\alpha-\eta)\,nt+v+k]$$
$$+\Sigma N^{(h)}\{P_\beta^{i,i',x}+Q_\beta^{i,i',x}\}\sin[(h+i)g+i'g'+H_{x,\beta}-(y+\alpha-\eta)\,nt-v-k]$$
$$+(n)\delta z\,\Sigma h\,M^{(h)}\{P_\beta^{i,i',x}-Q_\beta^{i,i',x}\}\cos[(h+i)g+i'g'+H_{x,\beta}+(y+\alpha-\eta)\,nt+v+k]$$
$$+(n)\delta z\,\Sigma h\,N^{(h)}\{P_\beta^{i,i',x}+Q_\beta^{i,i',x}\}\cos[(h+i)g+i'g'+H_{x,\beta}-(y+\alpha-\eta)\,nt-v-k]$$

in qua argumenta perturbationum aeque atque in reliquis tempori proportionalia sunt. Quum vero etiam sit

$$\sin[h(n)\delta z+(h+i)g+\text{etc.}] =$$
$$\sin[(n)\delta z+(h+i)g+\text{etc.}]+(h-1)(n)\delta z\cos[(n)\delta z+(h+i)g+\text{etc.}]-\text{etc.}$$

atque

$$\sin[h(n)\delta z+(h+i)g+\text{etc.}] =$$
$$\sin[-(n)\delta z+(h+i)g+\text{etc.}]+(h+1)(n)\delta z\cos[-(n)\delta z+(h+i)g+\text{etc.}]-\text{etc.}$$

formula ipsam s exhibens ita quoque exponi potest

$$s = \sin\Gamma\sin[\bar{f}+(y+\alpha-\eta-\mathrm{v})\,nt+v+k+\Theta]+\frac{c}{\mathrm{v}}\cos\Gamma\sin[\bar{f}+(y+\alpha-\eta)\,nt+v+k]$$
$$+\Sigma M^{(h)}\{P_\beta^{i,i',x}-Q_\beta^{i,i',x}\}\sin[(h+i)g+i'g'+H_{x,\beta}+(n)\delta z+(y+\alpha-\eta)\,nt+v+k]$$
$$+\Sigma N^{(h)}\{P_\beta^{i,i',x}+Q_\beta^{i,i',x}\}\sin[(h+i)g+i'g'+H_{x,\beta}-(n)\delta z-(y+\alpha-\eta)\,nt-v-k]$$
$$+(n)\delta z\,\Sigma(h-1)M^{(h)}\{P_\beta^{i,i',x}-Q_\beta^{i,i',x}\}\cos[(h+i)g+i'g'+H_{x,\beta}+(n)\delta z+(y+\alpha-\eta)\,nt+v+k]$$
$$+(n)\delta z\,\Sigma(h+1)N^{(h)}\{P_\beta^{i,i',x}+Q_\beta^{i,i',x}\}\cos[(h+i)g+i'g'+H_{x,\beta}-(n)\delta z-(y+\alpha-\eta)\,nt-v-k]$$

In hac igitur formula ad argumentorum tempori proportionales partes perturbationes longitudinis mediae adduntur et resp. ab iis subtrahuntur. Quum omnium $M^{(h)}$ maxima sit $M^{(1)}$, et omnium $N^{(h)}$ maxima $N^{(-1)}$, maximi ultimorum duorum huius formulae terminorum propter factores $h-1$ atque $h+1$ evanescunt, duo ultimi igitur huius formulae termini multo minores sunt quam formulae illius termini correspondentes, quare haec formula illi praeferenda videtur. Multiplicationes vero per $(n)\delta z$, quas ultimi termini requirunt, eodem modo perficiendae sunt, quo multiplicationes reliquae.

Quamquam formulis nostris sinum latitudinis dedimus, tamen latitudo ipsa ex tabulis motum Lunae exhibentibus et hac theoria nitentibus facile depromitur. Posito primo et maximo ipsius s termino hoc

$$\sin\Gamma \sin[\bar{f}+(y+\alpha-\eta-\mathrm{v})nt+v+k+\Theta]=\sin B$$

B ipsa loco ipsius $\sin B$ in tabulam redigenda est, quo facto, tabellis, quae perturbationes ipsius s suppeditant, tabella differentiam inter perturbationes ipsius s et ipsius $\operatorname{arc}(\sin=s)$ subministrans facili opera annectitur. Perturbationes vero ipsius s in tabulis conservandae sunt, quia his perturbationes reductionis longitudinis ad planum proiectionis facillima opera computantur.

5.

Transmutata littera x in v, reductio longitudinis ad planum proiectionis secundum aequationem (72) Sect. II. integrali datur formulae huius

$$(8)\ldots\ldots \frac{d(l-V_{\prime})}{dt}=\frac{n(\mathrm{v}-\alpha+\eta)}{\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}-\frac{(q_{\prime\prime}\cos V_{\prime}+p_{\prime\prime}\sin V_{\prime})\left(\frac{dq_{\prime\prime}}{dt}\sin V_{\prime}-\frac{dp_{\prime\prime}}{dt}\cos V_{\prime}\right)}{(1-q_{\prime\prime}\sin V_{\prime}+p_{\prime\prime}\cos V_{\prime})(1+q_{\prime\prime}\sin V_{\prime}-p_{\prime\prime}\cos V_{\prime})\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}$$

ubi

$$(9)\ldots\ldots\left\{\begin{aligned} p_{\prime\prime}&=p_{\prime}\cos[\mathrm{v}nt-\Theta]-q_{\prime}\sin[\mathrm{v}nt-\Theta]\\ q_{\prime\prime}&=q_{\prime}\cos[\mathrm{v}nt-\Theta]+p_{\prime}\sin[\mathrm{v}nt-\Theta]\end{aligned}\right.$$

atque

$$V_{\prime}=\bar{f}+n(y+\alpha-\eta-\mathrm{v})t+v+k+\Theta$$

Iam nunc positis

$$p_{,} = \sin\Gamma\sin[\nu nt - \Theta] + \delta_{,,}p_{,}$$
$$q_{,} = \sin\Gamma\cos[\nu nt - \Theta] + \delta_{,,}q_{,}$$

unde $\delta_{,,}p_{,} = \delta_{,}p_{,}$, $\delta_{,,}q_{,} = \delta_{,}q_{,} + \frac{c}{\nu}\cos\Gamma$ atque

$$s = \sin\Gamma\sin[\bar{f} + (y + \alpha - \eta - \nu)nt + v + k + \Theta] + \delta_{,,}q_{,}\sin V - \delta_{,,}p_{,}\cos V$$

sive

$$s = \sin B + \delta s$$

ubi igitur

$$\delta s = \delta_{,,}q_{,}\sin V - \delta_{,,}p_{,}\cos V$$

habetur adiumento aequationum (9)

$$p_{,,} = \delta_{,,}p_{,}\cos[\nu nt - \Theta] - \delta_{,,}q_{,}\sin[\nu nt - \Theta]$$
$$q_{,,} = \sin\Gamma + \delta_{,,}p_{,}\sin[\nu nt - \Theta] + \delta_{,,}q_{,}\cos[\nu nt - \Theta]$$

Positis porro

$$p_{,,} = (p_{,,}) + \delta p_{,,}\ ,\quad q_{,,} = (q_{,,}) + \delta q_{,,}$$

habetur

$$(p_{,,}) = o\ ;\quad (q_{,,}) = \sin\Gamma$$
$$\delta p_{,,} = \delta_{,,}p_{,}\cos[\nu nt - \Theta] - \delta_{,,}q_{,}\sin[\nu nt - \Theta]$$
$$\delta q_{,,} = \delta_{,,}p_{,}\sin[\nu nt - \Theta] + \delta_{,,}q_{,}\cos[\nu nt - \Theta]$$

et insuper

$$\delta s = \delta q_{,,}\sin V_{,} - \delta p_{,,}\cos V_{,}$$

Si nunc factores, qui in formulae (8) dextra parte continentur, in series infinitas secundum potestates productaque ipsarum $\delta p_{,,}$ atque $\delta q_{,,}$ progredientes evolvimus, habetur

$$\frac{1}{1 - q_{,,}\sin V_{,} + p_{,,}\cos V_{,}} = \frac{1}{1 - \sin\Gamma\sin V_{,}} - \frac{\cos V_{,}}{(1 - \sin\Gamma\sin V_{,})^2}\delta p_{,,} + \frac{\sin V_{,}}{(1 - \sin\Gamma\sin V_{,})^2}\delta q_{,,}$$
$$+ \frac{\cos^2 V_{,}}{(1 - \sin\Gamma\sin V_{,})^3}\delta p_{,,}^2 - 2\frac{\sin V_{,}\cos V_{,}}{(1 - \sin\Gamma\sin V_{,})^3}\delta p_{,,}\delta q_{,,} + \frac{\sin V_{,}}{(1 - \sin\Gamma\sin V_{,})^3}\delta q_{,,}^2$$

$$\frac{1}{1 + q_{,,}\sin V_{,} - p_{,,}\cos V_{,}} = \frac{1}{1 + \sin\Gamma\sin V_{,}} + \frac{\cos V_{,}}{(1 + \sin\Gamma\sin V_{,})^2}\delta p_{,,} - \frac{\sin V_{,}}{(1 + \sin\Gamma\sin V_{,})^2}\delta q_{,,}$$
$$+ \frac{\cos^2 V_{,}}{(1 + \sin\Gamma\sin V_{,})^3}\delta p_{,,}^2 - 2\frac{\sin V_{,}\cos V_{,}}{(1 + \sin\Gamma\sin V_{,})^3}\delta p_{,,}\delta q_{,,} + \frac{\sin V_{,}}{(1 + \sin\Gamma\sin V_{,})^3}\delta q_{,,}^2$$

$$\frac{1}{\sqrt{1 - p_{,,}^2 - q_{,,}^2}} = \frac{1}{\cos\Gamma} + \frac{\sin\Gamma}{\cos^3\Gamma}\delta q_{,,} + \frac{1}{2\cos^3\Gamma}\delta p_{,,}^2 + \frac{1 + 2\sin^2\Gamma}{2\cos^5\Gamma}\delta q_{,,}^2$$

$$q_{,,}\cos V_{,} + p_{,,}\sin V_{,} = \sin\Gamma\cos V_{,} + \sin V_{,}\delta p_{,,} + \cos V_{,}\delta q_{,,}$$

quarum ultima rigorosa, reliquae vero usque ad quantitates tertii ordinis accuratae sunt. Multiplicatis inter se his expressionibus, emergit

$$\frac{q_{,,}\cos V_{,}+p_{,,}\sin V_{,}}{(1-q_{,,}\sin V_{,}+p_{,,}\cos V_{,})(1+q_{,,}\sin V_{,}-p_{,,}\cos V_{,})\sqrt{1-p_{,,}^2-q_{,,}^2}}=\frac{\operatorname{tg}\Gamma\cos V_{,}}{\cos^2 B}$$
$$+C\delta p_{,,}+D\delta q_{,,}+E\delta p_{,,}^2+2F\delta p_{,,}\delta q_{,,}+G\delta q_{,,}^2$$

ubi

$$C=\quad\frac{1-2\sin^2\Gamma+\sin^2\Gamma\sin^2 V_{,}}{\cos\Gamma\cos^4 B}\sin V_{,}$$

$$D=\quad\frac{1+(\sin^2\Gamma-2\sin^4\Gamma)\sin^2 V_{,}}{\cos^3\Gamma\cos^4 B}\cos V_{,}$$

$$E=\quad\sin\Gamma\frac{3-2\sin^2\Gamma-(6-10\sin^2\Gamma+6\sin^4\Gamma)\sin^2 V_{,}-(2\sin^2\Gamma-\sin^4\Gamma)\sin^4 V_{,}}{2\cos^3\Gamma\cos^6 B}\cos V_{,}$$

$$F=-\sin\Gamma\frac{3-2\sin^2\Gamma-(6-10\sin^2\Gamma+6\sin^4\Gamma)\sin^2 V_{,}-(2\sin^2\Gamma-\sin^4\Gamma)\sin^4 V_{,}}{2\cos^3\Gamma\cos^6 B}\sin V_{,}$$

$$G=\quad\sin\Gamma\frac{3+(6-14\sin^2\Gamma+2\sin^4\Gamma)\sin^2 V_{,}+(2\sin^2\Gamma-5\sin^4\Gamma+6\sin^6\Gamma)\sin^4 V_{,}}{2\cos^5\Gamma\cos^6 B}\cos V_{,}$$

atque hinc

$$\frac{(q_{,,}\cos V_{,}+p_{,,}\sin V_{,})(dq_{,,}\sin V_{,}-dp_{,,}\cos V_{,})}{(1-q_{,,}\sin V_{,}+p_{,,}\cos V_{,})(1+q_{,,}\sin V_{,}-p_{,,}\cos V_{,})\sqrt{1-p_{,,}^2-q_{,,}^2}}=\frac{\operatorname{tg}\Gamma\cos V_{,}}{\cos^2 B}(dq_{,,}\sin V_{,}-dp_{,,}\cos V_{,})$$
$$+C\sin V_{,}\delta p_{,,}dq_{,,}+D\sin V_{,}\delta q_{,,}dq_{,,}+E\sin V_{,}\delta p_{,,}^2 dq_{,,}+2F\sin V_{,}\delta p_{,,}\delta q_{,,}dq_{,,}$$
$$+G\sin V_{,}\delta q_{,,}^2 dq_{,,}-C\cos V_{,}\delta p_{,,}dp_{,,}-D\cos V_{,}\delta q_{,,}dp_{,,}-E\cos V_{,}\delta p_{,,}^2 dp_{,,}$$
$$-2F\cos V_{,}\delta p_{,,}\delta q_{,,}dp_{,,}-G\cos V_{,}\delta q_{,,}^2 dp_{,,}$$

Facile vero reperitur identicas esse aequationes has

$$\begin{aligned}
\int\delta p_{,,}dq_{,,} &= \tfrac{1}{2}\delta p_{,,}\delta q_{,,}-\tfrac{1}{2}\int\{\delta q_{,,}dp_{,,}-\delta p_{,,}dq_{,,}\}\\
\int\delta q_{,,}dp_{,,} &= \tfrac{1}{2}\delta p_{,,}\delta q_{,,}+\tfrac{1}{2}\int\{\delta q_{,,}dp_{,,}-\delta p_{,,}dq_{,,}\}\\
\int\delta p_{,,}\delta q_{,,}dq_{,,} &= \tfrac{1}{3}\delta p_{,,}\delta q_{,,}^2-\tfrac{1}{3}\int\delta q_{,,}\{\delta q_{,,}dp_{,,}-\delta p_{,,}dq_{,,}\}\\
\int\delta p_{,,}\delta q_{,,}dp_{,,} &= \tfrac{1}{3}\delta p_{,,}^2\delta q_{,,}+\tfrac{1}{3}\int\delta p_{,,}\{\delta q_{,,}dp_{,,}-\delta p_{,,}dq_{,,}\}\\
\int\delta p_{,,}^2 dq_{,,} &= \tfrac{1}{3}\delta p_{,,}^2\delta q_{,,}-\tfrac{2}{3}\int\delta p_{,,}\{\delta q_{,,}dp_{,,}-\delta p_{,,}dq_{,,}\}\\
\int\delta q_{,,}^2 dp_{,,} &= \tfrac{1}{3}\delta p_{,,}\delta q_{,,}^2+\tfrac{2}{3}\int\delta q_{,,}\{\delta q_{,,}dp_{,,}-\delta p_{,,}dq_{,,}\}
\end{aligned}$$

expressione igitur praecedenti integrata, emergit, facili reductione facta,

$$\int\frac{(q_{\prime\prime}\cos V_{\prime}+p_{\prime\prime}\sin V_{\prime})(dq_{\prime\prime}\sin V_{\prime}-dp_{\prime\prime}\cos V_{\prime})}{(1-q_{\prime\prime}\sin V_{\prime}+p_{\prime\prime}\cos V_{\prime})(1+q_{\prime\prime}\sin V_{\prime}-p_{\prime\prime}\cos V_{\prime})\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}=$$
$$(\delta q_{\prime\prime}\sin V_{\prime}-\delta p_{\prime\prime}\sin V_{\prime})\Big\{\frac{\operatorname{tg}\Gamma\cos V_{\prime}}{\cos^2 B}+\tfrac{1}{2}(C\delta p_{\prime\prime}+D\delta q_{\prime\prime})+\tfrac{1}{3}(E\delta p_{\prime\prime}^2+2F\delta p_{\prime\prime}\delta q_{\prime\prime}+G\delta q_{\prime\prime}^2)\Big\}$$
$$-\tfrac{1}{2}\{C\sin V_{\prime}+D\cos V_{\prime}\}\int\{\delta q_{\prime\prime}dp_{\prime\prime}-\delta p_{\prime\prime}dq_{\prime\prime}\}-\tfrac{2}{3}\{E\sin V_{\prime}+F\cos V_{\prime}\}\int\delta p_{\prime\prime}\{\delta q_{\prime\prime}dp_{\prime\prime}-\delta p_{\prime\prime}dq_{\prime\prime}\}$$
$$-\tfrac{2}{3}\{F\sin V_{\prime}+G\cos V_{\prime}\}\int\delta q_{\prime\prime}\{\delta q_{\prime\prime}dp_{\prime\prime}-\delta p_{\prime\prime}dq_{\prime\prime}\}$$

Porro adiumento evolutionis ipsius $(1-p_{\prime\prime}^2-q_{\prime\prime}^2)^{-\frac{1}{2}}$ modo datae perfacile elicitur

$$\int\frac{n(v-\alpha+\eta)}{\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}\,dt=\frac{v-\alpha+\eta}{\cos\Gamma}\,nt+\sin\Gamma\frac{v-\alpha+\eta}{\cos^3\Gamma}\,n\int\delta q_{\prime\prime}dt$$

Congestis his expressionibus, propter praecedentes ipsarum C, D, E, F atque G expressiones, quae suppeditant

$$E\sin V_{\prime}+F\cos V_{\prime}=o;\ C\sin V_{\prime}+D\cos V_{\prime}=\frac{1}{\cos^3\Gamma};\ F\sin V_{\prime}+G\cos V_{\prime}=\frac{3\sin\Gamma}{2\cos^5\Gamma}$$

expressio (8) integrata evadit

$$\left.\begin{aligned}l=\Pi+R+V_{\prime}-\delta s\Big\{&\frac{\operatorname{tg}\Gamma\cos V_{\prime}}{\cos^2 B}+\tfrac{1}{2}(C\delta p_{\prime\prime}+D\delta q_{\prime\prime})+\tfrac{1}{3}(E\delta p_{\prime\prime}^2+2F\delta p_{\prime\prime}\delta q_{\prime\prime}+G\delta q_{\prime\prime}^2)\Big\}\\&+\frac{v-\alpha+\eta}{\cos\Gamma}\,nt+\sin\Gamma\frac{v-\alpha+\eta}{\cos^3\Gamma}\,n\int\delta q_{\prime\prime}dt+\frac{1}{2\cos^3\Gamma}\int\Big\{\delta q_{\prime\prime}\frac{dp_{\prime\prime}}{dt}-\delta p_{\prime\prime}\frac{dq_{\prime\prime}}{dt}\Big\}dt\\&+\frac{\sin\Gamma}{\cos^5\Gamma}\int\delta q_{\prime\prime}\Big\{\delta q_{\prime\prime}\frac{dp_{\prime\prime}}{dt}-\delta p_{\prime\prime}\frac{dq_{\prime\prime}}{dt}\Big\}dt\end{aligned}\right\}\text{......(10)}$$

ubi

$$\operatorname{tg}R=-\frac{\operatorname{tg}^2\frac{1}{2}\Gamma\sin 2V_{\prime}}{1+\operatorname{tg}^2\frac{1}{2}\Gamma\cos 2V_{\prime}}$$

et Π constans est, quam in art. 34. Sect. II. definivimus.

Quae longitudinis ad planum proiectionis reductae expressio ab expressione eiusdem longitudinis, quam antea dedi, eo discrepat, quod δs factor quoque quantitatum ordinis primi et secundi reddita est, sed in hac forma concinnior et usui accommodatior est quam illa. Praeterea demonstrari potest hanc formam generalem esse, et huic reductioni attribui posse, quantumvis praecisionis gradum adipisci velis. Iam ex calculo modo peracto facile perspicitur, rigorosam esse debere aequationem hanc

$$\int\frac{(q_{\prime\prime}\cos V_{\prime}+p_{\prime\prime}\sin V_{\prime})(dq_{\prime\prime}\sin V_{\prime}-dp_{\prime\prime}\cos V_{\prime})}{(1-q_{\prime\prime}\sin V_{\prime}+p_{\prime\prime}\cos V_{\prime})(1+q_{\prime\prime}\sin V_{\prime}-p_{\prime\prime}\cos V_{\prime})\sqrt{1-p_{\prime\prime}^2-q_{\prime\prime}^2}}=W\delta s+\int X\{\delta q_{\prime\prime}dp_{\prime\prime}-\delta p_{\prime\prime}dq_{\prime\prime}\}$$

designantibus W atque X series infinitas secundum potestates productaque ipsarum $\delta p_{\prime\prime}$ atque $\delta q_{\prime\prime}$ progredientes, quarum W insuper erit functio ipsarum $V_{\prime}$ atque $\Gamma_{\prime}$ et X insuper functio solius Γ. Quae est forma usque ad quantitates quarti ordinis reductioni huic supra attributa.

Maximas reductionis longitudinis ad planum proiectionis perturbationes continet terminus expressionis (10) per δs multiplicatus. Primus quidem factoris ipsius δs terminus in tabulam unius argumenti redigi potest, reliqui vero termini tabulas duplicis argumenti requirunt, sed quum hic factor per δs, cuius valor maximus circiter $\frac{1}{11}$ est, multiplicandus sit, maioribus solummodo ipsarum $\delta p_{\prime\prime}$ atque $\delta q_{\prime\prime}$ terminis in eo opus est. Hinc factum est, ut hic factor ex tabulis ceteroquin apte constructis facili opera desumi possit, et quum ad latitudinem computandam ipsa δs praeterea opus sit, multiplicatio factoris illius per δs ab astronomo loca Lunae ex tabulis computaturo sive logarithmis sive numeris ipsis perfacile absolvetur.

Reliqui expressionis (10) termini minores sunt et in tabulas unius argumenti redigi possunt. Sint

$$\delta p_{\prime\prime} = E \sin[lnt + A] + F \sin[hnt + B]$$
$$\delta q_{\prime\prime} = E' \cos[lnt + A] + F' \cos[hnt + B]$$

duo quivis ipsarum $\delta p_{\prime\prime}$ atque $\delta q_{\prime\prime}$ termini; hinc emergunt

$$\frac{dp_{\prime\prime}}{dt} = nlE \cos[lnt + A] + nhF \cos[hnt + B]$$
$$\frac{dq_{\prime\prime}}{dt} = -nlE' \sin[lnt + A] - nhF' \sin[hnt + B]$$

$$\delta q_{\prime\prime} \frac{dp_{\prime\prime}}{dt} = n\frac{l}{2} EE' \cos[2lnt + 2A] + n\frac{l}{2} EE' + n\frac{hFE' + lF'E}{2} \cos[n(h+l)t + A + B]$$
$$+ n\frac{hFE' + lF'E}{2} \cos[n(h-l)t + B - A] + n\frac{h}{2} FF' \cos[2hnt + 2B] + n\frac{h}{2} FF'$$
$$\delta p_{\prime\prime} \frac{dq_{\prime\prime}}{dt} = n\frac{l}{2} EE' \cos[2lnt + 2A] - n\frac{l}{2} EE' + n\frac{hF'E + lFE'}{2} \cos[n(h+l)t + A + B]$$
$$- n\frac{hF'E + lFE'}{2} \cos[n(h-l)t + B - A] + n\frac{h}{2} FF' \cos[2nht + 2B] - n\frac{h}{2} FF'$$

unde

$$\int\left\{\delta q_{\prime\prime}\frac{dp_{\prime\prime}}{dt}-\delta p_{\prime\prime}\frac{dq_{\prime\prime}}{dt}\right\}dt = nt(lEE'+hFF')+\left.\begin{array}{l}\frac{h-l}{h+l}\cdot\frac{FE'-F'E}{2}\sin[(h+l)nt+A+B]\\ +\frac{h+l}{h-l}\cdot\frac{FE'+F'E}{2}\sin[(h-l)nt+B-A]\end{array}\right\}\ldots\ldots(11)$$

quae huius integralis est generalis forma. Quum generaliter quam proxime habeatur

$$\text{aut } E=E',\ F=F' \text{ aut } E=-E',\ F=-F'$$

terminus aut secundus aut tertius praecedentis expressionis semper fere minutissimum habet coefficientem, quem plerumque negligere licet. Multiplicato praecedentis expressionis differentiali per numericam ipsius $\delta q_{\prime\prime}$ expressionem, post integrationem secundum regulas notas instituendam integrale $\int\delta q_{\prime\prime}\left\{\delta q_{\prime\prime}\frac{dp_{\prime\prime}}{dt}-\delta p_{\prime\prime}\frac{dq_{\prime\prime}}{dt}\right\}dt$ facile invenitur, quod vero integrale nullam fere vim habet. Item terminus expressionis (10) per $\int\delta q_{\prime\prime}dt$ multiplicatus propter factoris $\mathrm{v}-\alpha+\eta$ exiguitatem omnino negligendus est.

Termini expressionis (10) per tempus ipsum multiplicati usque ad terminos minutissimos se mutuo tollunt, nam perfacile demonstratur, maximum huius generis terminum proxime evanescere debere. Habetur enim, si ad maximum ipsarum $\delta p_{\prime\prime}$ et $\delta q_{\prime\prime}$ terminum tantum respicimus,

$$\delta p_{\prime\prime}=-\frac{c}{\mathrm{v}}\cos\Gamma\sin[\mathrm{v}nt-\Theta]\ ;\ \delta q_{\prime\prime}=\frac{c}{\mathrm{v}}\cos\Gamma\cos(\mathrm{v}nt-\Theta)$$

itaque secundum notationes modo introductas

$$E=-\frac{c}{\mathrm{v}}\cos\Gamma,\ E'=\frac{c}{\mathrm{v}}\cos\Gamma$$

Formula igitur (11) in hoc casu, ubi ad secundum ipsarum $\delta p_{\prime\prime}$ et $\delta q_{\prime\prime}$ terminum non respiciendum est, suppeditat

$$\int\left\{\delta q_{\prime\prime}\frac{dp_{\prime\prime}}{dt}-\delta p_{\prime\prime}\frac{dq_{\prime\prime}}{dt}\right\}dt = -nt\frac{c^2}{\mathrm{v}}\cos^2\Gamma$$

Ex art. 31. Sect. II. vero habemus in theoria Lunae, ubi in casu, quem nunc tractamus, $m=o$ statuere licet, quam proxime

$$n\alpha=\frac{m'}{w}\mu\cos^2\tfrac{1}{2}I\ ;\ n\eta=\frac{m'}{w}\mu\sin\tfrac{1}{2}I;\ nc=2\frac{m'}{w}\mu\sin\tfrac{1}{2}I\cos\tfrac{1}{2}I$$

unde sequitur

$$\mathrm{v}=\alpha+\eta;\ \mathrm{v}-\alpha+\eta=2\nu;\ n\frac{c^2}{\mathrm{v}}=4\eta\cos^2\tfrac{1}{2}I$$

Substitutis his valoribus invenitur

$$\frac{\nu - \alpha + \eta}{\cos \Gamma} nt + \frac{1}{2\cos^3 \Gamma} \int \left\{ \delta q_{\prime\prime} \frac{dp_{\prime\prime}}{dt} - \delta p_{\prime\prime} \frac{dq_{\prime\prime}}{dt} \right\} dt = nt \frac{2\eta}{\cos \Gamma} \sin^2 \tfrac{1}{2} I$$

quae est quantitas minutissima. Coefficientes ipsarum $\delta p_{\prime\prime}$ atque $\delta q_{\prime\prime}$, qui post allatos coefficientes $-\frac{e}{\mathrm{v}} \cos I$ atque $\frac{c}{\mathrm{v}} \cos \Gamma$ maximi sunt, valorem circiter $8' \cos \Gamma$ ambo habent; quare, quum horum coefficientium combinationem maximum integralis praecedentis terminum suppeditare debere, manifestum sit, pono

$$E = -\frac{c}{\mathrm{v}} \cos \Gamma, \quad E' = \frac{c}{\mathrm{v}} \cos \Gamma$$
$$F = \quad 8' \cos \Gamma, \quad F' = 8' \cos \Gamma$$

hinc et quum in hoc casu sit

$$l = \alpha + \eta \qquad\qquad = 0{,}0041$$
$$h = 2\frac{\alpha'}{n} + 2\gamma' + \alpha + \eta = 0{,}1549$$
$$\frac{c}{\mathrm{v}} = \sin(5^\circ\ 8')$$

atque habita aequatore plano proiectionis

$$\Gamma = 23^\circ\ 28'$$

expressio (11) suppeditat 17,″5, qui coefficiens est maximus integralis $\frac{1}{2\cos^3 \Gamma} \int \left\{ dq_{\prime\prime} \frac{dp_{\prime\prime}}{dt} - \delta p_{\prime\prime} \frac{dq_{\prime\prime}}{dt} \right\} dt$. Iisdem numeris confirmatur, maximum integralis $\frac{\sin \Gamma}{\cos^5 \Gamma} \int \delta q_{\prime\prime} \left\{ \delta q_{\prime\prime} \frac{dp_{\prime\prime}}{dt} - \delta p_{\prime\prime} \frac{dq_{\prime\prime}}{dt} \right\} dt$ coefficientem vix ad unam minutam secundam ascendere posse.

SECTIO VII.

SOLVTIO PROBLEMATIS QVATVOR CORPORVM BREVITER EXPOSITA.

1.

Theoriam planetarum investigans problema quidem quatuor pluriumve corporum tractavi, sed aliud est problema quatuor corporum, cuius solutionem nunc suscipiam. In illo enim terminos per tempus ipsum multiplicatos admittere licuit, quoniam series infinitas constituunt, quae per longam annorum seriem convergunt, quum hoc, si fieri potest, ita solvere propositum sit, ut termini illi omnino non adsint. Formulae ad huius problematis solutionem spectantes, quas infra evolvemus, ita erunt comparatae, ut formularum ad problema Lunae motus determinandi pertinentium, quas in praecedentibus dedimus, casum generaliorem ponant, et, abscissis terminis ad quartum corpus relatis, has suppeditent. Praeterea, additis terminis analogis ad quintum, sextum, etc. corpus relatis, ex hac problematis quatuor corporum solutione solutionem problematis plurium corporum facile componere poteris; propter singularem vero huius problematis conditionem ex formulis ad problema trium corporum spectantibus formulae problematis quatuor pluriumve corporum componi nequeunt.

Non modo massas quatuor corporum per M, m, m', m'', e quibus M ad corpus primarium, sive ad corpus, circum quod relativi reliquorum corporum motus investigandi sunt, pertinet, designabo, sed etiam quantitates reliquas, quibus utemur, iisdem litteris atque in Sectionibus praecedentibus, aut nullum, aut unum, aut duo commatis signa affigens, quo corpus, ad quod hae quantitates spectent, indicetur, denotabo. In hoc tamen a significatione in praecedenti Lunae theoria adhibita digrediar, quod loco quantitatum ny, $n\alpha$, $n\eta$, etc. et similium ad reliqua corpora spectantium infra simplicius y, α, η, etc., commatibus debitis affixis, ponam.

Praeter quantitates, quae potissimum ad unum corpus spectant, quantitates ad duo corpora relationem habentes aderunt, quae ita designabuntur, ut numerus commatum summae litterae affixorum indicet corpus ad cuius motum pertinent, et numerus commatum imae litterae affixorum corpus alterum relationem habens.

Et massae perturbantes, et excentricitates inclinationesque mutuae in sequentibus quantitates parvulae primi ordinis appellantur, quod ita intelligendum est, ut hae quantitates non maiores esse debeant, quam ut series infinitae, in quas formulae nostrae, quoties applicantur, evolvendae sunt, convergant. Inclinationem vero orbitarum versus planum proiectionis seu fundamentale quamlibet et quidem quantitatem finitam esse, in subsequentibus non minus quam in praecedentibus supponitur. Brevitatis caussa in subsequentibus formulas ad corpora m' atque m'' spectantes saepe suppressurus formulas tantum ad m spectantes apponam, ex quibus, mutatis

quantitatibus omnibus, quae ad m . . . m' . . . m'' spectant,

resp. in quantitates ad m' . . . m'' . . . m pertinentes,

formulae pro m', et postquam in his eaedem mutationes factae erunt, formulae pro m'' semper eliciuntur.

§. I. *Expositio aequationum differentialium finitarum, e quarum integratione problematis solutio pendet.*

2.

Quum secundum praecedentia sit

$$\varrho = \bar{\varrho}\, c^{\beta}\,, \;\; h = (h)\, c^{-(S+\varepsilon)}$$

rigorosa ipsius T expressio in art. 16. Sect. II. data facile transformatur in hanc

$$T = c^{\beta-(S+\varepsilon)}(h)\left\{2\frac{\bar{\varrho}}{r}\cos(v_{,}-\lambda)-1+2\frac{h^{2}\bar{\varrho}}{\varkappa(M+m)}\left[\cos(v_{,}-\lambda)-1\right]\right\}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$+c^{\beta-(S+\varepsilon)}(h)2\frac{\bar{\varrho}}{r}\sin(v_{,}-\lambda)r\left(\frac{d\Omega}{dr}\right)+(c^{\beta}-1)h\left(\frac{d\Omega}{dv_{,}}\right)-\frac{y(h)c^{-(S+\varepsilon)}}{\varkappa(M+m)}\cdot\frac{d.\bar{\varrho}^{2}c^{2\beta}}{d\tau}$$

Si vero ponimus

$$Z = (h)\left\{2\frac{\bar{\varrho}}{r}\cos(v_{,}-\lambda)-1+2\frac{h^{2}\bar{\varrho}}{\varkappa(M+m)}\left[\cos(v_{,}-\lambda)-1\right]\right\}\left(\frac{d\Omega}{dv_{,}}\right)$$
$$+\,2(h)\,\frac{\bar{\varrho}}{r}\sin(v_{,}-\lambda)r\left(\frac{d\Omega}{dr}\right)$$

et perpendimus esse

$$\frac{d\zeta}{d\tau} = c^{S+\varepsilon-2\beta},\;\; h\left(\frac{d\Omega}{dv_{,}}\right) = \frac{dS}{dt} = 2\frac{d\beta}{dt} + \frac{\frac{d^{2}\zeta}{d\tau\, dt}}{\frac{d\zeta}{d\tau}}$$

aequatio praecedens abit in hanc

$$T\frac{d\zeta}{d\tau} = Zc^{-\beta} + 2\frac{d\zeta}{d\tau}\frac{d\beta}{dt}(c^{\beta}-1) + \frac{d^{2}\zeta}{d\tau\, dt}(c^{\beta}-1) - \frac{y}{(n)(a)^{2}\sqrt{1-(e)^{2}}}\left\{\frac{d.\bar{\varrho}^{2}}{d\tau}+2\bar{\varrho}^{2}\frac{d\beta}{d\tau}\right\}$$

Sed $\frac{d^{2}\zeta}{d\tau\, dt}$ et T aequatione iunguntur hac

$$\frac{d^{2}\zeta}{d\tau\, dt} = T\frac{d\zeta}{d\tau} + \frac{\frac{d^{2}\zeta}{d\tau^{2}}\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}$$

itaque

$$\frac{d^{2}\zeta}{d\tau\, dt} = Zc^{-\beta} + 2\frac{d\zeta}{d\tau}\frac{d\beta}{dt}(c^{\beta}-1) + \frac{d^{2}\zeta}{d\tau\, dt}(c^{\beta}-1) + \frac{\frac{d^{2}\zeta}{d\tau^{2}}.\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}} - \frac{y}{(n)(a)^{2}\sqrt{1-(e)^{2}}}\left\{\frac{d.\bar{\varrho}^{2}}{d\tau}+2\bar{\varrho}^{2}\frac{d\beta}{d\tau}\right\}$$

Si haec aequatio per c^{β} multiplicatur, emergit, facili reductione facta, haec

$$\frac{d^2\zeta}{d\tau\, dt} = Z + 2\frac{d\zeta}{d\tau}\frac{d\beta}{dt}(c^\beta-1)c^\beta + \frac{d^2\zeta}{d\tau\, dt}(c^\beta-1)^2 + \frac{\frac{d^2\zeta}{d\tau^2}\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}c^\beta - \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\left\{\frac{d.\bar{\varrho}^2}{d\tau} + 2\bar{\varrho}^2\frac{d\beta}{d\tau}\right\}c^\beta$$

quae, quum secundum et tertium ad dextram positum membrum integrabilia sint, integrata praebet

$$\frac{d\zeta}{d\tau} = \int Z\, dt + \frac{d\zeta}{d\tau}(c^\beta-1)^2 + \int \frac{\frac{d^2\zeta}{d\tau^2}\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}c^\beta dt - \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\int\left\{\frac{d.\bar{\varrho}^2}{d\tau} + 2\bar{\varrho}^2\frac{d\beta}{d\tau}\right\}c^\beta dt$$

cui integrali constantem non addidi, quia in terminis sub integrationis signo existentibus contenta censeri potest. Quum in praecedenti ipsius Z expressione $\bar{\varrho}$ et λ sint functiones solius variabilis ζ, pono secundum theorema Taylorianum

$$Z = (Z) + \frac{d(Z)}{d\tau}\delta\zeta + \tfrac{1}{2}\frac{d^2(Z)}{d\tau^2}\delta\zeta^2 + \frac{1}{2.3}\frac{d^3(Z)}{d\tau^3}\delta\zeta^3 + \text{etc.}$$

et praeterea $(Z) = \frac{dW}{dt} + Y$. Si hi valores in praecedenti pro $\frac{d\zeta}{d\tau}$ inventa aequatione substituuntur, et termini ipsam W et eius quotientes differentiales respectu τ continentes per partes integrantur, elicitur

$$(1)\begin{cases} \frac{d\zeta}{d\tau} = \frac{d\zeta}{d\tau}(c^\beta-1)^2 + W + \frac{dW}{d\tau}\delta\zeta + \tfrac{1}{2}\frac{d^2W}{d\tau^2}\delta\zeta^2 + \frac{1}{2.3}\frac{d^3W}{d\tau^3}\delta\zeta^3 + \text{etc.} \\ + \int \frac{\frac{d^2\zeta}{d\tau^2}\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}c^\beta dt - \int\frac{dW}{d\tau}\frac{d\zeta}{dt}dt - \int\frac{d^2W}{d\tau^2}\delta\zeta\frac{d\zeta}{dt}dt - \tfrac{1}{2}\int\frac{d^3W}{d\tau^3}\delta\zeta^2\frac{d\zeta}{dt}dt - \frac{1}{2.3}\int\frac{d^4W}{d\tau^4}\delta\zeta^3\frac{d\zeta}{dt}dt - \text{etc.} \\ + \int Y dt + \int\frac{dY}{d\tau}\delta\zeta\, dt + \tfrac{1}{2}\int\frac{d^2Y}{d\tau^2}\delta\zeta^2 dt + \frac{1}{2.3}\int\frac{d^3Y}{d\tau^3}\delta\zeta^3 dt + \text{etc.} - \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\int\left\{\frac{d.\bar{\varrho}^2}{d\tau} + 2\bar{\varrho}\frac{d\beta}{d\tau}\right\}c^\beta dt \end{cases}$$

Sit

$$(2)\ldots\ldots\quad Y + \frac{dY}{d\tau}\delta\zeta + \tfrac{1}{2}\frac{d^2Y}{d\tau^2}\delta\zeta^2 + \frac{1}{2.3}\frac{d^3Y}{d\tau^3}\delta\zeta^3 + \text{etc.} = \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\left\{\frac{d.\bar{\varrho}^2}{d\tau} + 2\bar{\varrho}\frac{d\beta}{d\tau}\right\}c^\beta$$

atque

$$(3)\ldots\ldots\quad \frac{d\zeta}{d\tau} = \frac{d\zeta}{d\tau}(c^\beta-1)^2 + W + \frac{dW}{d\tau}\delta\zeta + \tfrac{1}{2}\frac{d^2W}{d\tau^2}\delta\zeta^2 + \frac{1}{2.3}\frac{d^3W}{d\tau^3}\delta\zeta^3 + \text{etc.} + X$$

ubi X est functio adhuc indeterminata. Ad hanc determinandam differentietur aequatio (3) respectu ipsius τ, unde propter aequationem $\left(\frac{d\beta}{d\tau}\right)\left(\frac{d\zeta}{d\tau}\right) = -\tfrac{1}{2}\frac{d^2\zeta}{d\tau^2}$ prodit

$$\left.\begin{aligned}\frac{d^2\zeta}{d\tau^2}c^{\beta} = & \frac{dW}{d\tau}+\frac{d^2W}{d\tau^2}\delta\zeta+\tfrac{1}{2}\frac{d^3W}{d\tau^3}\delta\zeta^2+\frac{1}{2.3}\frac{d^4W}{d\tau^4}\delta\zeta^3+\text{etc.}+\frac{dX}{d\tau}\\ & +\frac{dW}{d\tau}\left(\frac{d\delta\zeta}{d\tau}\right)+\frac{d^2W}{d\tau^2}\delta\zeta\left(\frac{d\delta\zeta}{d\tau}\right)+\tfrac{1}{2}\frac{d^3W}{d\tau^3}\delta\zeta^2\left(\frac{d\delta\zeta}{d\tau}\right)+\frac{1}{2.3}\frac{d^4W}{d\tau^4}\delta\zeta^3\left(\frac{d\delta\zeta}{d\tau}\right)+\text{etc.}\end{aligned}\right\}\quad\ldots\ldots(4)$$

Iam si animadvertimus esse $\frac{d\zeta}{d\tau}=1+\left(\frac{d\delta\zeta}{d\tau}\right)$, atque et hanc et praecedentes aequationes in aequatione (1) substituerimus, nanciscimur ad functionem X determinandam aequationem hanc

$$\left(\frac{dX}{dt}\right)\left(\frac{d\zeta}{d\tau}\right)-\left(\frac{dX}{d\tau}\right)\left(\frac{d\zeta}{dt}\right)=o$$

cuius integrale est $X=f\zeta$, denotante f functionem arbitrariam. In art. 6. Sect. III. vero demonstratum est, aequationis pro $\frac{d^2\zeta}{d\tau\,dt}$ integrali respectu t loco constantis arbitrariae functionem $1-b+A(1-b)\xi+$ etc. addendam esse, quare nanciscimur sive $f\zeta$ sive $X=1-b+A(1-b)\xi+$ etc. Quum vero W in aequatione (1) contenta integratione alia eruenda sit, integer ipsius X modo inventus valor in duas partes, quarum altera ad integrale modo inventum, altera ad integrale, quod ipsam W suppeditaturum sit, pertineat, ad lubitum distribui potest. Sit igitur $X=1$ ea ipsius X pars, quae integrali praecedenti adiicitur: hinc, si aequationem

$$\frac{d\zeta}{d\tau}=c^{S+\varepsilon-2\beta}$$

respexeris, aequatio (3) subministrat hanc

$$\frac{d\zeta}{d\tau}=1+W+\Sigma_1^\infty\frac{1}{1.2\ldots\nu}\frac{d^\nu W}{d\tau^\nu}\delta\zeta^\nu+(1-c^{-\beta})^2c^{S+\varepsilon}\quad\ldots\ldots(5)$$

et aequatio (4) hanc

$$\frac{\frac{d^2\zeta}{d\tau^2}}{\frac{d\zeta}{d\tau}}=c^{-\beta}\left[\frac{dW}{d\tau}+\Sigma_1^\infty\frac{1}{1.2\ldots\nu}\frac{d^{\nu+1}W}{d\tau^{\nu+1}}\delta\zeta^\nu\right]\quad\ldots\ldots(6)$$

quae rigorosae aequationes sunt, quarum termini usque ad quartum ordinem in Sectione tertia evoluti sunt. Iam reliquis in Sectione tertia explicatis computationibus, quibus opus est, peractis, nanciscimur aequationes rigorosas has

$$(7)\ldots\ldots\left\{\begin{aligned}
(n)z &= g+(n)\int\Big\{\overline{W}+\Sigma_{1}^{\infty}\frac{1}{1.2\ldots\nu}\Big(\overline{\frac{d^{\nu}W}{d\gamma^{\nu}}}\Big)(n)\delta z^{\nu}+(1-c^{-w})^{2}c^{S+\varepsilon}\Big\}\,dt\\
&\quad-\frac{y}{\sqrt{1-(e)^{2}}}\int\Big\{\frac{(r)^{2}}{(a)^{2}}+\Sigma_{1}^{\infty}\frac{1}{1.2\ldots\nu}\,\frac{d^{\nu}.(r)^{2}}{(a)^{2}dg^{\nu}}(n)\,\delta z^{\nu}\Big\}\,dt\\
w &= C+\tfrac{1}{2}\varepsilon-\tfrac{1}{2}(n)\int c^{-w}\Big\{\Big(\overline{\frac{dW}{d\gamma}}\Big)+\Sigma_{1}^{\infty}\frac{1}{1.2\ldots\nu}\Big(\overline{\frac{d^{\nu+1}W}{d\gamma^{\nu+1}}}\Big)(n)\,\delta z^{\nu}\Big\}\,dt\\
&\quad+\tfrac{1}{2}\,\frac{y}{\sqrt{1-(e)^{2}}}\int\Big\{\frac{d.(r)^{2}}{(a)^{2}dg}+\Sigma_{1}^{\infty}\frac{1}{1.2\ldots\nu}\,\frac{d^{\nu+1}.(r)^{2}}{(a)^{2}dg^{\nu+1}}(n)\,\delta z^{\nu}\Big\}\,dt
\end{aligned}\right.$$

ubi linea ipsi W et quotientibus eius differentialibus superposita, τ in t sive γ in g mutandam esse, denotat. Praeterea statim ex Sectione tertia habemus

$$(8)\ldots\ldots\quad S+\varepsilon = 2w+l\Big\{\frac{dz}{dt}+\frac{y}{(n)(a)^{2}\sqrt{1-(e)^{2}}}\Big[\frac{(r)^{2}}{(a)^{2}}+\Sigma_{1}^{\infty}\frac{1}{1.2\ldots\nu}\,\frac{d^{\nu}.(r)^{2}}{(a)^{2}dg^{\nu}}(n)\,\delta z^{\nu}\Big]\Big\}$$

ubi l denotat logarithmum hyperbolicum.

Restat ut aequatio differentialis pro W obtineatur. Quum $\bar{\varrho}$ nec non $\bar{\varrho}^{2}$ sit functio solius variabilis ζ, habetur per theorema Taylorianum

$$\bar{\varrho}^{2}=(\varrho)^{2}+\frac{d.(\varrho)^{2}}{d\tau}\delta\zeta+\tfrac{1}{2}\,\frac{d^{2}.(\varrho)^{2}}{d\tau^{2}}\delta\zeta^{2}+\frac{1}{2.3}\,\frac{d^{3}.(\varrho)^{2}}{d\tau^{3}}\delta\zeta^{3}+\text{etc.}$$

hinc emergit

$$\frac{d.\bar{\varrho}^{2}}{d\tau}=\frac{d\zeta}{d\tau}\Big\{\frac{d.(\varrho)^{2}}{d\tau}+\frac{d^{2}.(\varrho)^{2}}{d\tau^{2}}\delta\zeta+\tfrac{1}{2}\,\frac{d^{3}.(\varrho)^{2}}{d\tau^{3}}\delta\zeta^{2}+\frac{1}{2.3}\,\frac{d^{4}.(\varrho)^{2}}{d\tau^{4}}\delta\zeta^{3}+\text{etc.}\Big\}$$

Iam si loco serierum infinitarum et in praecedentibus aequationibus et in aequationibus (2), (5) atque (6) contentarum, quae omnes secundum eandem legem procedunt, ponuntur signa $[Y]$, $[W]$, $\Big[\frac{dW}{d\tau}\Big]$, $[(\varrho)^{2}]$ atque $\Big[\frac{d.(\varrho)^{2}}{d\tau}\Big]$, hae quantitates pro functionibus solius variabilis ζ habendae erunt, quae, posita τ loco ζ, resp. in Y, W, $\frac{dW}{d\tau}$, $(\varrho)^{2}$ atque $\frac{d.(\varrho)^{2}}{d\tau}$ transeunt. Quibus positis, aequatio (2) abit in

$$(9)\ldots\ldots\quad [Y]=\frac{y}{(n)(a)^{2}\sqrt{1-(e)^{2}}}\Big\{\Big[\frac{d.(\varrho)^{2}}{d\tau}\Big]\frac{d\zeta}{d\tau}c^{\beta}-[(\varrho)^{2}]\,\frac{\frac{d^{2}\zeta}{d\tau^{2}}}{\frac{d\zeta}{d\tau}}\,c^{\beta}\Big\}$$

et (5) atque (6) in

$$\frac{d\zeta}{d\tau}c^{\beta}=\frac{1+[W]}{2-c^{\beta}},\quad \frac{\frac{d^{2}\zeta}{d\tau^{2}}}{\frac{d\zeta}{d\tau}}c^{\beta}=\Big[\frac{dW}{d\tau}\Big]$$

quarum prior in aequatione $\frac{d\zeta}{d\tau} = c^{S+\varepsilon-2\beta}$ substituta suppeditat

$$\frac{1}{2-c^{\beta}} = \tfrac{1}{2}\,\frac{1+[W]+c^{S+\varepsilon}}{1+[W]}$$

Substitutis his aequationibus in (9) emergit

$$[Y]_{1} = \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\left\{\tfrac{1}{2}\left[\frac{d.(\varrho)^2}{d\tau}\right]\left\{1+[W]+c^{S+\varepsilon}\right\} - [(\varrho)^2]\left[\frac{dW}{d\tau}\right]\right\}$$

quae aequatio praeter constantes et quantitatem S; quae nec τ nec ζ continet, non nisi functionibus constat, quae functiones solius variabilis ζ censeri possunt, itaque secundum theorema Taylorianum evoluta suppeditare debet

$$Y + \frac{dY}{d\tau}\delta\zeta + \tfrac{1}{2}\,\frac{d^2Y}{d\tau^2}\delta\zeta^2 + \text{etc.} =$$

$$\frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\left\{\tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\left\{1+W+c^{S+\varepsilon}\right\}-(\varrho)^2\frac{dW}{d\tau}\right\} + \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\,\frac{d\left\{\tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\left\{1+W+c^{S+\varepsilon}\right\}-(\varrho)^2\frac{dW}{d\tau}\right\}}{d\tau}\delta\zeta$$

$$+\tfrac{1}{2}\,\frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\,\frac{d^2.\left\{\tfrac{1}{2}\frac{d.(\varrho)^2}{d\tau}\left\{1+W+c^{S+\varepsilon}\right\}-(\varrho)^2\frac{dW}{d\tau}\right\}}{d\tau^2}\delta\zeta^2 + \text{etc.}$$

unde sequitur

$$Y = \frac{y}{(n)(a)^2\sqrt{1-(e)^2}}\left\{\tfrac{1}{2}\,\frac{d.(\varrho)^2}{d\tau}\left\{1+W+c^{S+\varepsilon}\right\} - (\varrho)^2\,\frac{dW}{d\tau}\right\}$$

Substituto hoc ipsius Y valore, nec non valore ipsius (Z) ex praecedentibus desumendo in aequatione $\frac{dW}{dt} = (Z) - Y$, nanciscimur aequationem rigorosam hanc

$$\left.\begin{aligned} \frac{dW}{dt} = {} & \frac{an}{\sqrt{1-e^2}}\left\{2\,\frac{\varrho}{r}\cos(v_{,}-\lambda) - 1 + 2\,\frac{h^2\varrho}{(h)^2 a(1-e^2)}\left[\cos(v_{,}-\lambda)-1\right]\right\}\left(\frac{d\Omega}{dv_{,}}\right) \\ & + 2\,\frac{an}{\sqrt{1-e^2}}\,\frac{\varrho}{r}\sin(v_{,}-\lambda)\,r\left(\frac{d\Omega}{dr}\right) - \frac{y'}{\sqrt{1-e^2}}\,\frac{d.\varrho^2}{a^2\,d\gamma} \\ & + \frac{y}{\sqrt{1-e^2}}\left\{\frac{\varrho^2}{a^2}\,\frac{dW}{d\gamma} - \frac{d.\varrho^2}{2a^2 d\gamma}\left[W + c^{S+\varepsilon} - 1\right]\right\} \end{aligned}\right\} \ldots\ldots(10)$$

ubi brevitatis caussa a, n, e, ϱ, λ loco (a), (n), (e), (ϱ), (λ) scripsi. In hac igitur aequatione quantitates a, n, e, (h) pro constantibus, ϱ atque λ pro functionibus ipsius τ sive γ et illarum constantium, et $v_{,}$, r, h, nec non quantitates reliquae in Ω contentae pro variabilibus et quidem pro

functionibus ipsius t, quae et explicite et implicite in iis continetur, habendae sunt. Perfacile praeterea perspicitur, partem dextram praecedentis aequationis, si terminos in ultima linea positos excipis, esse quantitatem in Sectione tertia $\frac{dX}{dt}$ nominatam.

Iam per se manifestum est, aequationes differentiales Sectionis secundae longitudinem mediam perturbatam et logarithmum radii vectoris suppeditantes ad quatuor corpora extendi, si termini ipsius Ω ad quartum corpus spectantes restituuntur, sive si ponis

(11)...... $$\Omega = \frac{m'}{M+m}\left\{\frac{1}{\Delta_{,}} + \frac{\Delta_{,}^2 - r^2 - r'^2}{2r'^3}\right\} + \frac{m''}{M+m}\left\{\frac{1}{\Delta_{,,}} + \frac{\Delta_{,,}^2 - r^2 - r''^2}{2r''^3}\right\}$$

quamobrem aequationes praecedentes, substituto hoc ipsius Ω valore, ad problema quatuor corporum pertinent, et mutatis mutandis aequationes analogas ad corpora m' atque m'' spectantes subministrant.

3.

Ad integrationem aequationis (10) sublevandam animadverto in Sect. V. art. 6. inductione demonstratum esse, ipsam W in omnibus ad eius verum valorem obtinendum instituendis approximationibus theoremati art. 9. Sect. IV. subiectam esse, unde sequitur W induere debere formam hanc

$$W = \Xi + \Upsilon\left(\frac{\varrho}{a}\cos\varphi + \tfrac{3}{2}e\right) + \Psi\frac{\varrho}{a}\sin\varphi$$

ubi Ξ, Υ et Ψ functiones sunt ab ipsa τ liberae, et φ anomaliam veram pure ellipticam adiumento ipsius τ computandam denotat. Habemus igitur

$$\frac{dW}{dt} = \frac{d\Xi}{dt} + \frac{d\Upsilon}{dt}\left(\frac{\varrho}{a}\cos\varphi + \tfrac{3}{2}e\right) + \frac{d\Psi}{dt}\frac{\varrho}{a}\sin\varphi$$

$$\frac{dW}{d\gamma} = -\Upsilon\frac{\sin\varphi}{\sqrt{1-e^2}} + \Psi\frac{\cos\varphi + e}{\sqrt{1-e^2}}$$

Substitutis his valoribus in aequatione (10), nanciscimur post comparatos terminos eiusdem formae

$$\left.\begin{aligned}
\frac{d\Upsilon}{dt} &= y\Psi+2\frac{an}{\sqrt{1-e^2}}\left\{\left[\left(\frac{a}{r}+\frac{h^2}{(h)^2(1-e^2)}\right)\cos\bar{f}+\frac{h^2 e}{(h)^2(1-e^2)}\right]\left(\frac{d\Omega}{dv_{\prime}}\right)+\frac{a}{r}\sin\bar{f}\,r\left(\frac{d\Omega}{dr}\right)\right\}\\
\frac{d\Psi}{dt} &= -y\Upsilon+2\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{a}{r}+\frac{h^2}{(h)^2(1-e^2)}\right)\sin\bar{f}\left(\frac{d\Omega}{dv_{\prime}}\right)-\frac{a}{r}\cos\bar{f}\,r\left(\frac{d\Omega}{dr}\right)\right\}\\
&\quad -y\frac{e}{1-e^2}\left\{2+\Xi+\tfrac{3}{2}e\Upsilon+c^{S+\varepsilon}-1\right\}\\
\frac{d\Xi}{dt} &= -\frac{an}{\sqrt{1-e^2}}\left\{\left[3\left(\frac{ae}{r}+\frac{h^2 e}{(h)^2(1-e^2)}\right)\cos\bar{f}+1+2\frac{h^2}{(h)^2}+3\frac{h^2e^2}{(h)^2(1-e^2)}\right]\left(\frac{d\Omega}{dv_{\prime}}\right)+3\frac{ae}{r}\sin\bar{f}\,r\left(\frac{d\Omega}{dr}\right)\right\}-\tfrac{3}{2}ey\Psi
\end{aligned}\right\}(12)$$

e quibus, mutatis mutandis, similes aequationes pro $\frac{d\Upsilon'}{dt}$, $\frac{d\Psi'}{dt}$, $\frac{d\Xi'}{dt}$, $\frac{d\Upsilon''}{dt}$, etc: nanciscimur, e quarum integratione perturbationes longitudinis mediae et logarithmi radii vectoris pendent. Itaque quum praecedens ipsius W expressio aequationi (10) revera satisfaciat, demonstratum est, ipsam W theoremati Sect. IV. art. 9. demonstrato rigorose subiectam esse. Istas autem quantitates Υ, Ψ, Ξ, quas, ut iam dixi, ad sublevandam tantum aequationum pro $\frac{dW}{dt}$, $\frac{dW'}{dt}$, $\frac{dW''}{dt}$ integrationem introduximus, integralibus inventis, eliminabuntur, ita ut statim ipsarum W, W', W'' expressiones habeantur. Integrationibus his persequendis inserviunt aequationes (7) et (8) nec non aequationes analogae ad corpora m' et m'' spectantes tanquam aequationes auxiliares.

4.

Ad perturbationes latitudinis versus planum fundamentale, cuius situs in hoc quoque problemate prorsus arbitrarius supponitur, et reductionis longitudinis ad idem planum investigandas easdem aequationes (33) Sect. II. et earum similes pro corporibus m' atque m'' habemus, si in iis expressio (11) ipsius Ω nec non analogae ipsarum Ω' atque Ω'' expressiones substituuntur. Ut in his formulis mutuae orbitarum corporum m, m' atque m'', inclinationes longitudinesque nodorum his respondentes introducantur, quo forma ipsarum Ω, Ω' atque Ω'' computationesque omnes simpliciores reddantur, idem adhibendus est calculus, quem in Sect. II. artt. 22. seqq. respectu generalis trium corporum problematis exposuimus. Quum vero orbitae tres trium corporum m, m', m'' tria forment in spatio triangula sphaerica, necesse est quantitates I, Φ, Ψ, φ, ψ, prout ad hoc vel illud triangulum pertineant, eo modo, quem in art. 1. designavimus, a se

invicem discernantur. Erit igitur nobis $I_{\prime}$ inclinatio mutua orbitarum m atque m', $I_{\prime\prime}$ inclinatio mutua orbitarum m atque m'' et $I'_{\prime\prime}$ inclinatio mutua orbitarum m' atque m'', quae inclinationes etiam resp. per I', I'', $I''_{\prime}$ denotari possunt; erunt porro

$\Phi_{\prime}$ arcus orbitae	m	a nodo ascendenti huius orbitae cum orbita	m'	usque ad nodum ascendentem orbitae	m	cum plano fundamentali rediens,
$\Phi_{\prime\prime}$	m	......	m''	......	m	......
$\Phi'_{\prime\prime}$	m'	......	m''	......	m'	......
Φ'	m'	......	m	......	m'	......
Φ''	m''	......	m	......	m''	......
$\Phi''_{\prime}$	m''	......	m'	......	m''	......
$\Psi_{\prime}$	m'	a nodo descendenti huius orbitae cum orbita	m	......	m'	......
$\Psi_{\prime\prime}$	m''	......	m	......	m''	......
$\Psi'_{\prime\prime}$	m''	......	m'	......	m''	......
Ψ''	m	......	m'	......	m	......
Ψ'	m	......	m''	......	m	......
$\Psi''_{\prime}$	m'	......	m''	......	m'	......

denique

$$(13)\ldots\ldots\begin{cases}\varphi_{\prime}=\Phi_{\prime}+\chi-\omega\,, & \varphi'_{\prime\prime}=\Phi'_{\prime\prime}+\chi'-\omega'\,, \quad \varphi''=\Phi''+\chi''-\omega''\\ \psi_{\prime}=\Psi_{\prime}+\chi'-\omega'\,, & \psi'_{\prime\prime}=\Psi'_{\prime\prime}+\chi''-\omega''\,, \quad \psi''=\Psi''+\chi-\omega\\ \varphi_{\prime\prime}=\Phi_{\prime\prime}+\chi-\omega\,, & \varphi'=\Phi'+\chi'-\omega'\,, \quad \varphi''_{\prime}=\Phi''_{\prime}+\chi''-\omega''\\ \psi_{\prime\prime}=\Psi_{\prime\prime}+\chi''-\omega''\,, & \psi'=\Psi'+\chi-\omega\,, \quad \psi''_{\prime}=\Psi''_{\prime}+\chi'-\omega'\end{cases}$$

Iam quum nodus ascendens orbitae m in orbita m' ab eiusdem orbitae nodo descendente in orbita m' arcu 180 graduum semper distet, et eadem relatio inter reliquos nodos locum habeat, habemus aequationes conditionales has

$$(14)\ldots\ldots\begin{cases}\varphi_{\prime}-\psi''=180^\circ\,, & \varphi'_{\prime\prime}-\psi''_{\prime}=180^\circ\,, \quad \varphi''-\psi_{\prime\prime}=180^\circ\\ \varphi_{\prime\prime}-\psi''=180^\circ\,, & \varphi'-\psi_{\prime}=180^\circ\,, \quad \varphi''_{\prime}-\psi'_{\prime\prime}=180^\circ\end{cases}$$

Quibus statutis, aequationes (41) Sect. II. abeunt in

$$(15)\ldots\ldots\begin{cases}dI_{\prime}=\sin\varphi_{\prime}\dfrac{dp}{\cos i}+\cos\varphi_{\prime}\dfrac{dq}{\cos i}-\sin\psi_{\prime}\dfrac{dp'}{\cos i'}-\cos\psi_{\prime}\dfrac{dq'}{\cos i'}\\ d\varphi_{\prime}=\operatorname{cotg}I_{\prime}\left\{\cos\varphi_{\prime}\dfrac{dp}{\cos i}-\sin\varphi_{\prime}\dfrac{dq}{\cos i}\right\}-\operatorname{cosec}I_{\prime}\left\{\cos\psi_{\prime}\dfrac{dp'}{\cos i'}-\sin\psi_{\prime}\dfrac{dq'}{\cos i'}\right\}\\ d\psi_{\prime}=\operatorname{cosec}I_{\prime}\left\{\cos\varphi_{\prime}\dfrac{dp}{\cos i}-\sin\varphi_{\prime}\dfrac{dq}{\cos i}\right\}-\operatorname{cotg}I_{\prime}\left\{\cos\psi_{\prime}\dfrac{dp'}{\cos i'}-\sin\psi_{\prime}\dfrac{dq'}{\cos i'}\right\}\end{cases}$$

e quibus mutatis mutandis similes aequationes pro $dI_{\prime\prime}$, $d\varphi_{\prime\prime}$, $d\psi_{\prime\prime}$, $d\Gamma_{\prime\prime}$, etc. obtinemus.

5.

Sint porro sicut in art. 25. Sect. II.

$$\left.\begin{array}{ll}
\pi+\pi'-(\varphi_{\prime}+\psi_{\prime})-2\alpha_{\prime}t=2N_{\prime}\,, & \pi-\pi'-(\varphi_{\prime}-\psi_{\prime})+2\eta_{\prime}t=2K_{\prime} \\
\pi+\pi''-(\varphi_{\prime\prime}+\psi_{\prime\prime})-2\alpha_{\prime\prime}t=2N_{\prime\prime}\,, & \pi-\pi''-(\varphi_{\prime\prime}-\psi_{\prime\prime})+2\eta_{\prime\prime}t=2K_{\prime\prime} \\
\pi'+\pi''-(\varphi'_{\prime\prime}+\psi'_{\prime\prime})-2\alpha'_{\prime\prime}t=2N'_{\prime\prime}\,, & \pi'-\pi''-(\varphi'_{\prime\prime}-\psi'_{\prime\prime})+2\eta'_{\prime\prime}t=2K'_{\prime\prime} \\
\pi'+\pi-(\varphi'+\psi')-2\alpha't=2N'\,, & \pi'-\pi-(\varphi'-\psi')+2\eta't=2K' \\
\pi''+\pi-(\varphi''+\psi'')-2\alpha''t=2N''\,, & \pi''-\pi-(\varphi''-\psi'')+2\eta''t=2K'' \\
\pi''+\pi'-(\varphi''_{\prime}+\psi''_{\prime})-2\alpha''_{\prime}t=2N''_{\prime}\,, & \pi''-\pi'-(\varphi''_{\prime}-\psi''_{\prime})+2\eta''_{\prime}t=2K''_{\prime}
\end{array}\right\}\text{......(16)}$$

quae aequationes suppeditant

$$\pi-\varphi_{\prime}-(\alpha_{\prime}-\eta_{\prime})t=N_{\prime}+K_{\prime}\,,\quad \pi'-\varphi'-(\alpha'-\eta')t=N'+K'$$
$$\pi'-\psi_{\prime}-(\alpha_{\prime}+\eta_{\prime})t=N_{\prime}-K_{\prime}\,,\quad \pi-\psi'-(\alpha'+\eta')t=N'-K'$$

unde

$$\psi'-\varphi_{\prime}=N_{\prime}-N'+K_{\prime}+K'+(\alpha_{\prime}-\alpha'-\eta_{\prime}-\eta')t$$
$$\varphi'-\psi_{\prime}=N_{\prime}-N'-K_{\prime}-K'+(\alpha_{\prime}-\alpha'+\eta_{\prime}+\eta')t$$

Hinc nanciscimur propter aequationes conditionales (14)

$$N_{\prime}-N'=180^\circ\,,\quad K_{\prime}=-K'\,,\quad \alpha_{\prime}=\alpha'\,,\quad \eta_{\prime}=-\eta'$$

et eodem modo

$$N_{\prime\prime}-N''=180^\circ\,,\quad K_{\prime\prime}=-K''\,,\quad \alpha_{\prime\prime}=\alpha''\,,\quad \eta_{\prime\prime}=-\eta''$$
$$N'_{\prime\prime}-N''_{\prime}=180^\circ\,,\quad K'_{\prime\prime}=-K''_{\prime}\,,\quad \alpha'_{\prime\prime}=\alpha''_{\prime}\,,\quad \eta'_{\prime\prime}=-\eta''_{\prime}$$

Sint porro

$$\left.\begin{array}{lll}
P_{\prime}=2\sin\tfrac{1}{2}I_{\prime}\sin(N_{\prime}-\nu_{\prime})\,, & P'_{\prime\prime}=2\sin\tfrac{1}{2}I'_{\prime\prime}\sin(N'_{\prime\prime}-\nu'_{\prime\prime})\,, & P_{\prime\prime}=2\sin\tfrac{1}{2}I_{\prime\prime}\sin(N_{\prime\prime}-\nu_{\prime\prime}) \\
Q_{\prime}=2\sin\tfrac{1}{2}I_{\prime}\cos(N_{\prime}-\nu_{\prime})\,, & Q'_{\prime\prime}=2\sin\tfrac{1}{2}I'_{\prime\prime}\cos(N'_{\prime\prime}-\nu'_{\prime\prime})\,, & Q_{\prime\prime}=2\sin\tfrac{1}{2}I_{\prime\prime}\cos(N_{\prime\prime}-\nu_{\prime\prime}) \\
P'=2\sin\tfrac{1}{2}I'\sin(N'-\nu')\,, & P''_{\prime}=2\sin\tfrac{1}{2}I''_{\prime}\sin(N''_{\prime}-\nu''_{\prime})\,, & P''=2\sin\tfrac{1}{2}I''\sin(N''-\nu'') \\
Q'=2\sin\tfrac{1}{2}I'\cos(N'-\nu')\,, & Q''_{\prime}=2\sin\tfrac{1}{2}I''_{\prime}\cos(N''_{\prime}-\nu''_{\prime})\,, & Q''=2\sin\tfrac{1}{2}I''\cos(N''-\nu'')
\end{array}\right\}\text{......(17)}$$

ubi $\nu_{\prime}$, ν', $\nu'_{\prime\prime}$, etc. constantes sunt in valoribus ipsarum $N_{\prime}$, N', $N'_{\prime\prime}$, etc. contentae. Hinc sequitur, propter aequationes conditionales praecedentes esse debere

$$\nu_{\prime}-\nu'=180^\circ\,,\quad \nu'_{\prime\prime}-\nu''_{\prime}=180^\circ\,,\quad \nu''-\nu_{\prime\prime}=180^\circ.$$

unde emergit

$$P_{,} = P', \; Q_{,} = Q', \; P'_{,,} = P''_{,}, \; Q'_{,,} = Q''_{,}, \; P_{,,} = P'', \; Q_{,,} = Q''$$

Itaque quum inter duodeviginti quantitates P, Q et K hae novem aequationes conditionales adsint, in computationibus subsequentibus non nisi novem harum quantitatum considerentur oportet, et quidem has novem

$$P_{,}, \; Q_{,}, \; K_{,}, \; P'_{,,}, \; Q'_{,,}, \; K'_{,,}, \; P'', \; Q'', \; K''$$

eligemus.

6.

Computatio in art. 25. Sect. II. peracta suppeditat, posita $\alpha_{,}$ loco $(n)\alpha$ et $\eta_{,}$ loco $(n)\eta$,

$$\begin{aligned}
\frac{dP_{,}}{dt} &= -\alpha_{,} Q_{,} + \frac{dI_{,}}{dt}\cos\tfrac{1}{2}I_{,}\sin(N_{,}-v_{,}) - \left(\frac{d\varphi_{,}}{dt}+\frac{d\psi_{,}}{dt}\right)\sin\tfrac{1}{2}I_{,}\cos(N_{,}-v_{,})\\
\frac{dQ_{,}}{dt} &= \alpha_{,} P_{,} + \frac{dI_{,}}{dt}\cos\tfrac{1}{2}I_{,}\cos(N_{,}-v_{,}) + \left(\frac{d\varphi_{,}}{dt}+\frac{d\psi_{,}}{dt}\right)\sin\tfrac{1}{2}I_{,}\sin(N_{,}-v_{,})\\
\frac{dK_{,}}{dt} &= \eta_{,} - \tfrac{1}{2}\left(\frac{d\varphi_{,}}{dt}-\frac{d\psi_{,}}{dt}\right)
\end{aligned}$$

unde, substitutis aequationibus (15), elicitur

$$\begin{aligned}
\frac{dP_{,}}{dt} &= -\alpha_{,} Q_{,} - \cos\tfrac{1}{2}I_{,}\left\{\cos H_{,}\frac{dp}{\cos i\,dt} - \sin H_{,}\frac{dq}{\cos i\,dt} - \cos G_{,}\frac{dp'}{\cos i'\,dt} + \sin G_{,}\frac{dq'}{\cos i'\,dt}\right\}\\
\frac{dQ_{,}}{dt} &= \alpha_{,} P_{,} + \cos\tfrac{1}{2}I_{,}\left\{\sin H_{,}\frac{dp}{\cos i\,dt} + \cos H_{,}\frac{dq}{\cos i\,dt} - \sin G_{,}\frac{dp_{,}}{\cos i'\,dt} - \cos G_{,}\frac{dq'}{\cos i'\,dt}\right\}\\
\frac{dK_{,}}{dt} &= \eta_{,} + \tfrac{1}{4}\sec\tfrac{1}{2}I_{,}\left\{\begin{aligned} &[Q_{,}\cos H_{,} + P_{,}\sin H_{,}]\frac{dp}{\cos i\,dt} - [Q_{,}\sin H_{,} - P_{,}\cos H_{,}]\frac{dq}{\cos i\,dt}\\ &+[Q_{,}\cos G_{,} + P_{,}\sin G_{,}]\frac{dp'}{\cos i'\,dt} - [Q_{,}\sin G_{,} - P_{,}\cos G_{,}]\frac{dq'}{\cos i'\,dt}\end{aligned}\right\}
\end{aligned}$$

ubi brevitatis caussa

$$(18)\ldots\ldots \quad H_{,} = \varphi_{,} + N_{,} - v_{,}, \quad G_{,} = \psi_{,} + N_{,} - v_{,}$$

scripsi, et e quibus mutatis mutandis nanciscimur has

$$\frac{dP'_{\prime\prime}}{dt} = -\alpha'_{\prime\prime}Q'_{\prime\prime} - \cos\tfrac{1}{2}I'_{\prime\prime}\left\{\cos H'_{\prime\prime}\frac{dp'}{\cos i'\,dt} - \sin H'_{\prime\prime}\frac{dq'}{\cos i'\,dt} - \cos G'_{\prime\prime}\frac{dp''}{\cos i''\,dt} + \sin G'_{\prime\prime}\frac{dq''}{\cos i''\,dt}\right\}$$

$$\frac{dQ'_{\prime\prime}}{dt} = \alpha'_{\prime\prime}P'_{\prime\prime} + \cos\tfrac{1}{2}I'_{\prime\prime}\left\{\sin H'_{\prime\prime}\frac{dp'}{\cos i'\,dt} + \cos H'_{\prime\prime}\frac{dq'}{\cos i'\,dt} - \sin G'_{\prime\prime}\frac{dp''}{\cos i''\,dt} - \cos G'_{\prime\prime}\frac{dq''}{\cos i''\,dt}\right\}$$

$$\frac{dK'_{\prime\prime}}{dt} = \eta'_{\prime\prime} + \tfrac{1}{4}\sec\tfrac{1}{2}I'_{\prime\prime}\left\{\begin{array}{l}[Q'_{\prime\prime}\cos H'_{\prime\prime} + P'_{\prime\prime}\sin H'_{\prime\prime}]\dfrac{dp'}{\cos i'\,dt} - [Q'_{\prime\prime}\sin H'_{\prime\prime} - P'_{\prime\prime}\cos H'_{\prime\prime}]\dfrac{dq'}{\cos i'\,dt}\\ +[Q'_{\prime\prime}\cos G'_{\prime\prime} + P'_{\prime\prime}\sin G'_{\prime\prime}]\dfrac{dp''}{\cos i''\,dt} - [Q'_{\prime\prime}\sin G'_{\prime\prime} - P'_{\prime\prime}\cos G'_{\prime\prime}]\dfrac{dq''}{\cos i''\,dt}\end{array}\right\}$$

$$\frac{dP''}{dt} = -\alpha''Q'' - \cos\tfrac{1}{2}I''\left\{\cos H''\frac{dp''}{\cos i''\,dt} - \sin H''\frac{dq''}{\cos i''\,dt} - \cos G''\frac{dp}{\cos i\,dt} + \sin G''\frac{dq}{\cos i\,dt}\right\}$$

$$\frac{dQ''}{dt} = \alpha''P'' + \cos\tfrac{1}{2}I''\left\{\sin H''\frac{dp''}{\cos i''\,dt} + \cos H''\frac{dq''}{\cos i''\,dt} - \sin G''\frac{dp}{\cos i\,dt} - \cos G''\frac{dq}{\cos i\,dt}\right\}$$

$$\frac{dK''}{dt} = \eta'' + \tfrac{1}{4}\sec\tfrac{1}{2}I''\left\{\begin{array}{l}[Q''\cos H'' + P''\sin H'']\dfrac{dp''}{\cos i''\,dt} - [Q''\sin H'' - P''\cos H'']\dfrac{dq''}{\cos i''\,dt}\\ +[Q''\cos G'' + P''\sin G'']\dfrac{dp}{\cos i\,dt} - [Q''\sin G'' - P''\cos G'']\dfrac{dq}{\cos i\,dt}\end{array}\right\}$$

Quae aequationes monstrant, novem quantitates $P_{\prime}$, $Q_{\prime}$, $K_{\prime}$, $P'_{\prime\prime}$, etc. e sex quantitatibus p, q, p', etc. pendere, quamobrem inter illas quantitates tres aequationes conditionales existere debent. Quas eruamus ante omnia oportet. Multiplicata prima aequatione praecedente per factorem indeterminatum $a_{\prime}\sec\tfrac{1}{2}I_{\prime}$, secunda per $b_{\prime}\sec\tfrac{1}{2}I_{\prime}$, tertia per $4c_{\prime}\cos\tfrac{1}{2}I_{\prime\prime}$, quarta per $a'_{\prime\prime}\sec\tfrac{1}{2}I'_{\prime\prime}$, quinta per $b'_{\prime\prime}\sec\tfrac{1}{2}I'_{\prime\prime}$, sexta per $4c'_{\prime\prime}\cos\tfrac{1}{2}I'_{\prime\prime}$, septima per $a''\sec\tfrac{1}{2}I''$, octava per $b''\sec\tfrac{1}{2}I''$ et nona per $4c''\cos\tfrac{1}{2}I''$, omnia producta addantur et coefficientes ipsarum $\frac{dp}{\cos i\,dt}$, $\frac{dq}{\cos i\,dt}$, $\frac{dp'}{\cos i'\,dt}$, etc. singuli cifrae aequales statuantur. Quibus factis, nanciscimur septem aequationes has

$$\left.\begin{array}{l} o = a_{\prime}\left(\dfrac{dP_{\prime}}{dt} + \alpha_{\prime}Q_{\prime}\right)\sec\tfrac{1}{2}I_{\prime} + b_{\prime}\left(\dfrac{dQ_{\prime}}{dt} - \alpha_{\prime}P_{\prime}\right)\sec\tfrac{1}{2}I_{\prime} + 4c_{\prime}\left(\dfrac{dK_{\prime}}{dt} - \eta_{\prime}\right)\cos\tfrac{1}{2}I_{\prime}\\ \quad + a'_{\prime\prime}\left(\dfrac{dP'_{\prime\prime}}{dt} + \alpha'_{\prime\prime}Q'_{\prime\prime}\right)\sec\tfrac{1}{2}I'_{\prime\prime} + b'_{\prime\prime}\left(\dfrac{dQ'_{\prime\prime}}{dt} - \alpha'_{\prime\prime}P'_{\prime\prime}\right)\sec\tfrac{1}{2}I'_{\prime\prime} + 4c'_{\prime\prime}\left(\dfrac{dK'_{\prime\prime}}{dt} - \eta'_{\prime\prime}\right)\cos\tfrac{1}{2}I'_{\prime\prime}\\ \quad + a''\left(\dfrac{dP''}{dt} + \alpha''Q''\right)\sec\tfrac{1}{2}I'' + b''\left(\dfrac{dQ''}{dt} - \alpha''P''\right)\sec\tfrac{1}{2}I'' + 4c''\left(\dfrac{dK''}{dt} - \eta''\right)\cos\tfrac{1}{2}I'' \end{array}\right\}\ldots\ldots(19)$$

$$o = -a_{\prime}\cos H_{\prime} + b_{\prime}\sin H_{\prime} + c_{\prime}[Q_{\prime}\cos H_{\prime} + P_{\prime}\sin H_{\prime}] + a''\cos G'' - b''\sin G'' + c''[Q''\cos G'' + P''\sin G'']$$
$$o = a_{\prime}\sin H_{\prime} + b_{\prime}\cos H_{\prime} - c_{\prime}[Q_{\prime}\sin H_{\prime} - P_{\prime}\cos H_{\prime}] - a''\sin G'' - b''\cos G'' - c''[Q''\sin G'' - P''\cos G'']$$
$$o = a_{\prime}\cos G_{\prime} - b_{\prime}\sin G_{\prime} + c_{\prime}[Q_{\prime}\cos G_{\prime} + P_{\prime}\sin G_{\prime}] - a'_{\prime\prime}\cos H'_{\prime\prime} + b'_{\prime\prime}\sin H'_{\prime\prime} + c'_{\prime\prime}[Q'_{\prime\prime}\cos H'_{\prime\prime} + P'_{\prime\prime}\sin H'_{\prime\prime}]$$
$$o = -a_{\prime}\sin G_{\prime} - b_{\prime}\cos G_{\prime} - c_{\prime}[Q_{\prime}\sin G_{\prime} - P_{\prime}\cos G_{\prime}] + a'_{\prime\prime}\sin H'_{\prime\prime} + b'_{\prime\prime}\cos H'_{\prime\prime} - c'_{\prime\prime}[Q'_{\prime\prime}\sin H'_{\prime\prime} - P'_{\prime\prime}\cos H'_{\prime\prime}]$$
$$o = a'_{\prime\prime}\cos G'_{\prime\prime} - b'_{\prime\prime}\sin G'_{\prime\prime} + c'_{\prime\prime}[Q'_{\prime\prime}\cos G'_{\prime\prime} + P'_{\prime\prime}\sin G'_{\prime\prime}] - a''\cos H'' + b''\sin H'' + c''[Q''\cos H'' + P''\sin H'']$$
$$o = -a'_{\prime\prime}\sin G'_{\prime\prime} - b'_{\prime\prime}\cos G'_{\prime\prime} - c'_{\prime\prime}[Q'_{\prime\prime}\sin G'_{\prime\prime} - P'_{\prime\prime}\cos G'_{\prime\prime}] + a''\sin H'' + b''\cos H'' - c''[Q''\sin H'' - P''\cos H'']$$

Quodsi sex quantitates indeterminatae ex. gr. $a_{,}$, $b_{,}$, $a'_{,,}$, $b'_{,,}$, a'', b'', eliminatione functiones reliquarum trium $c_{,}$, $c'_{,,}$, c'' ex ultimis sex harum aequationum factae in prima aequatione substituuntur, haec formam induit hanc

$$o = Ac_{,} + Bc'_{,,} + Cc''$$

et propter indeterminatas $c_{,}$, $c'_{,,}$, c'' statim suppeditat

$$A = o\,,\quad B = o\,,\quad C = o$$

quae requisitae sunt aequationes conditionales. Eliminatione instituta, facili opera inveni

$$\begin{aligned}
a_{,}\sin\tfrac{1}{2}(G_{,}+G'_{,,}+G''-H_{,}-H'_{,,}-H'') &= c_{,}P_{,}\cos\tfrac{1}{2}(G_{,}+G'_{,,}+G''-H_{,}-H'_{,,}-H'')\\
&-c'_{,,}[Q'_{,,}\sin\tfrac{1}{2}(G'_{,,}-G_{,}+G''+H'_{,,}-H_{,}-H'')-P'_{,,}\cos\tfrac{1}{2}(G'_{,,}-G_{,}+G''+H'_{,,}-H_{,}-H'')]\\
&+c''[Q''\sin\tfrac{1}{2}(G_{,}-G''+G'_{,,}+H_{,}-H''-H'_{,,})+P''\cos\tfrac{1}{2}(G_{,}-G''+G'_{,,}+H_{,}-H''-H'_{,,})]\\
b_{,}\sin\tfrac{1}{2}(G_{,}+G'_{,,}+G''-H_{,}-H'_{,,}-H'') &= c_{,}Q_{,}\cos\tfrac{1}{2}(G_{,}+G'_{,,}+G''-H_{,}-H'_{,,}-H'')\\
&+c'_{,,}[Q'_{,,}\cos\tfrac{1}{2}(G'_{,,}-G_{,}+G''+H'_{,,}-H_{,}-H'')+P'_{,,}\sin\tfrac{1}{2}(G'_{,,}-G_{,}+G''+H'_{,,}-H_{,}-H'')]\\
&+c''[Q''\cos\tfrac{1}{2}(G_{,}-G''+G'_{,,}+H_{,}-H''-H'_{,,})-P''\sin\tfrac{1}{2}(G_{,}-G''+G'_{,,}+H_{,}-H''-H'_{,,})]
\end{aligned}$$

e quibus mutatis mutandis analogi ipsarum $a'_{,,}$, $b'_{,,}$, a'', b'' valores emergunt.

Si habita ratione aequationum (16) et (18), hi ipsarum $a_{,}$, $b_{,}$, $a'_{,,}$ etc. valores in aequatione (19) substituuntur, et brevitatis caussa statuerimus

$$\begin{aligned}
\tfrac{1}{2}(G''-G'_{,,}+G_{,}+H''-H'_{,,}-H_{,}) &= (\alpha'_{,,}-\alpha''-\eta_{,})t+\nu'_{,,}-\nu''+K_{,} = L_{,}\\
\tfrac{1}{2}(G_{,}-G''+G'_{,,}+H_{,}-H''-H'_{,,}) &= (\alpha''-\alpha_{,}-\eta'_{,,})t+\nu''-\nu_{,}+K'_{,,} = L'_{,,}\\
\tfrac{1}{2}(G'_{,,}-G_{,}+G''+H'_{,,}-H_{,}-H'') &= (\alpha_{,}-\alpha'_{,,}-\eta'')t+\nu_{,}-\nu'_{,,}+K'' = L''
\end{aligned}$$

unde

$$\tfrac{1}{2}(G_{,}+G'_{,,}+G''-H_{,}-H'_{,,}-H'') = K_{,}+K'_{,,}+K''-(\eta_{,}+\eta'_{,,}+\eta'')t = L_{,}+L'_{,,}+L''$$

aequationem nanciscimur hanc

$$\begin{aligned}
o = 4\left\{\frac{dK_{,}}{dt}-\eta_{,}\right\}&\cos\tfrac{1}{2}I_{,}\sin(L_{,}+L'_{,,}+L'')+\left\{P_{,}\frac{dP_{,}}{dt}+Q_{,}\frac{dQ_{,}}{dt}\right\}\sec\tfrac{1}{2}I_{,}\cos(L_{,}+L'_{,,}+L'')\\
&+\left\{P_{,}\frac{dP'_{,,}}{dt}+Q_{,}\frac{dQ'_{,,}}{dt}+\alpha'_{,,}[P_{,}Q'_{,,}-Q_{,}P'_{,,}]\right\}\sec\tfrac{1}{2}I'_{,,}\cos L''\\
&-\left\{P_{,}\frac{dQ'_{,,}}{dt}-Q_{,}\frac{dP'_{,,}}{dt}-\alpha'_{,,}[P_{,}P'_{,,}+Q_{,}Q'_{,,}]\right\}\sec\tfrac{1}{2}I'_{,,}\sin L''\\
&+\left\{P_{,}\frac{dP''}{dt}+Q_{,}\frac{dQ''}{dt}+\alpha''[P_{,}Q''-Q_{,}P'']\right\}\sec\tfrac{1}{2}I''\cos L'_{,,}\\
&+\left\{P_{,}\frac{dQ''}{dt}-Q_{,}\frac{dP''}{dt}-\alpha''[P_{,}P''+Q_{,}Q'']\right\}\sec\tfrac{1}{2}I''\sin L'_{,,}
\end{aligned}$$

et praeterea duas alias aequationes, quas, quum mutatis mutandis ex hac inveniri possint, adscribere non est opus.

Hae tres aequationes a se invicem independentes requisitae sunt. Annotandum est eas relationes pure geometricas inter quantitates $P_{,}$, $Q_{,}$, $K_{,}$, $P_{,,}$, etc. constituere et locum habere, quibuscunque relationibus mechanicis hae quantitates subiiciantur.

7.

Computatio, qua in Sect. II. artt. 23. et 29. p atque q primum in functiones ipsarum I, φ atque ψ, et deinde in functiones ipsarum P, Q atque K convertimus, in quatuor quoque corporum problemate adhiberi potest, unde statim habemus

$$\begin{aligned}
dp = {} & \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dQ_{,}}\Big)\cos\tfrac{1}{2}I_{,} + \Big(\frac{d\Omega}{dK_{,}}\Big)\frac{P_{,}}{4\cos\frac{1}{2}I_{,}}\Big\}\cos[\pi - v_{,} - K_{,} - (\alpha_{,} - \eta_{,})t]\,dt \\
& + \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dP_{,}}\Big)\cos\tfrac{1}{2}I_{,} - \Big(\frac{d\Omega}{dK_{,}}\Big)\frac{Q_{,}}{4\cos\frac{1}{2}I_{,}}\Big\}\sin[\pi - v_{,} - K_{,} - (\alpha_{,} - \eta_{,})t]\,dt \\
& + \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dQ_{,,}}\Big)\cos\tfrac{1}{2}I_{,,} + \Big(\frac{d\Omega}{dK_{,,}}\Big)\frac{P_{,,}}{4\cos\frac{1}{2}I_{,,}}\Big\}\cos[\pi - v_{,,} - K_{,,} - (\alpha_{,,} - \eta_{,,})t]\,dt \\
& + \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dP_{,,}}\Big)\cos\tfrac{1}{2}I_{,,} - \Big(\frac{d\Omega}{dK_{,,}}\Big)\frac{Q_{,,}}{4\cos\frac{1}{2}I_{,,}}\Big\}\sin[\pi - v_{,,} - K_{,,} - (\alpha_{,,} - \eta_{,,})t]\,dt
\end{aligned}$$

$$\begin{aligned}
dq = {} & -\cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dP_{,}}\Big)\cos\tfrac{1}{2}I_{,} - \Big(\frac{d\Omega}{dK_{,}}\Big)\frac{Q_{,}}{4\cos\frac{1}{2}I_{,}}\Big\}\cos[\pi - v_{,} - K_{,} - (\alpha_{,} - \eta_{,})t]\,dt \\
& + \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dQ_{,}}\Big)\cos\tfrac{1}{2}I_{,} + \Big(\frac{d\Omega}{dK_{,}}\Big)\frac{P_{,}}{4\cos\frac{1}{2}I_{,}}\Big\}\sin[\pi - v_{,} - K_{,} - (\alpha_{,} - \eta_{,})t]\,dt \\
& - \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dP_{,,}}\Big)\cos\tfrac{1}{2}I_{,,} - \Big(\frac{d\Omega}{dK_{,,}}\Big)\frac{Q_{,,}}{4\cos\frac{1}{2}I_{,,}}\Big\}\cos[\pi - v_{,,} - K_{,,} - (\alpha_{,,} - \eta_{,,})t]\,dt \\
& + \cos i \frac{an}{\sqrt{1-e^2}}\Big\{\Big(\frac{d\Omega}{dQ_{,,}}\Big)\cos\tfrac{1}{2}I_{,,} + \Big(\frac{d\Omega}{dK_{,,}}\Big)\frac{P_{,,}}{4\cos\frac{1}{2}I_{,,}}\Big\}\sin[\pi - v_{,,} - K_{,,} - (\alpha_{,,} - \eta_{,,})t]\,dt
\end{aligned}$$

e quibus mutatis mutandis emergunt

$$\begin{aligned}
dp' = {} & \cos i' \frac{a'n'}{\sqrt{1-e'^2}}\Big\{\Big(\frac{d\Omega'}{dQ'}\Big)\cos\tfrac{1}{2}I' + \Big(\frac{d\Omega'}{dK'}\Big)\frac{P'}{4\cos\frac{1}{2}I'}\Big\}\cos[\pi' - v' - K' - (\alpha' - \eta')t]\,dt \\
& + \cos i' \frac{a'n'}{\sqrt{1-e'^2}}\Big\{\Big(\frac{d\Omega'}{dP'}\Big)\cos\tfrac{1}{2}I' - \Big(\frac{d\Omega'}{dK'}\Big)\frac{Q'}{4\cos\frac{1}{2}I'}\Big\}\sin[\pi' - v' - K' - (\alpha' - \eta')t]\,dt \\
& + \cos i' \frac{a'n'}{\sqrt{1-e'^2}}\Big\{\Big(\frac{d\Omega'}{dQ_{,,}'}\Big)\cos\tfrac{1}{2}I_{,,}' + \Big(\frac{d\Omega'}{dK_{,,}'}\Big)\frac{P_{,,}'}{4\cos\frac{1}{2}I_{,,}'}\Big\}\cos[\pi' - v_{,,}' - K_{,,}' - (\alpha_{,,}' - \eta_{,,}')t]\,dt \\
& + \cos i' \frac{a'n'}{\sqrt{1-e'^2}}\Big\{\Big(\frac{d\Omega'}{dP_{,,}'}\Big)\cos\tfrac{1}{2}I_{,,}' - \Big(\frac{d\Omega'}{dK_{,,}'}\Big)\frac{Q_{,,}'}{4\cos\frac{1}{2}I_{,,}'}\Big\}\sin[\pi' - v_{,,}' - K_{,,}' - (\alpha_{,,}' - \eta_{,,}')t]\,dt
\end{aligned}$$

$$dq' = -\cos i' \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dP'}\right) \cos \tfrac{1}{2} I' - \left(\frac{d\Omega'}{dK'}\right) \frac{Q'}{4 \cos \frac{1}{2} I'} \right\} \cos[\pi' - \nu' - K' - (\alpha' - \eta')t]\, dt$$
$$+ \cos i' \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dQ'}\right) \cos \tfrac{1}{2} I' + \left(\frac{d\Omega'}{dK'}\right) \frac{P'}{4 \cos \frac{1}{2} I'} \right\} \sin[\pi' - \nu' - K' - (\alpha' - \eta')t]\, dt$$
$$- \cos i' \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dP_{,,}'}\right) \cos \tfrac{1}{2} I_{,,}' - \left(\frac{d\Omega'}{dK_{,,}'}\right) \frac{Q_{,,}'}{4 \cos \frac{1}{2} I_{,,}'} \right\} \cos[\pi' - \nu_{,,}' - K_{,,}' - (\alpha_{,,}' - \eta_{,,}')t]\, dt$$
$$+ \cos i' \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dQ_{,,}'}\right) \cos \tfrac{1}{2} I_{,,}' + \left(\frac{d\Omega'}{dK_{,,}'}\right) \frac{P_{,,}'}{4 \cos \frac{1}{2} I_{,,}'} \right\} \sin[\pi' - \nu_{,,}' - K_{,,}' - (\alpha_{,,}' - \eta_{,,}')t]\, dt$$

e quibus etiam mutatis mutandis aequationes pro dp'' atque dq'' derivari possunt. Substitutis his ipsarum dp, dq, dp' atque dq' valoribus in aequationibus art. praec. valores ipsarum $dP_{,}$, $dQ_{,}$ atque $dK_{,}$ per illas quantitates exprimentibus, elicitur

$$\frac{dP_{,}}{dt} = -\alpha_{,} Q_{,} - \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dQ_{,}}\right) \cos \tfrac{1}{2} I_{,} + \left(\frac{d\Omega}{dK_{,}}\right) \frac{P_{,}}{4 \cos \frac{1}{2} I_{,}} \right\} \cos \tfrac{1}{2} I_{,}$$
$$- \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dQ'}\right) \cos \tfrac{1}{2} I' + \left(\frac{d\Omega'}{dK'}\right) \frac{P'}{4 \cos \frac{1}{2} I'} \right\} \cos \tfrac{1}{2} I'$$
$$- \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dQ_{,,}}\right) \cos \tfrac{1}{2} I_{,,} + \left(\frac{d\Omega}{dK_{,,}}\right) \frac{P_{,,}}{4 \cos \frac{1}{2} I_{,,}} \right\} \cos \tfrac{1}{2} I_{,} \cos[(\alpha_{,} - \alpha_{,,} - \eta_{,} + \eta_{,,})t + \nu_{,} - \nu_{,,} + K_{,} - K_{,,}]$$
$$- \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dP_{,,}}\right) \cos \tfrac{1}{2} I_{,,} - \left(\frac{d\Omega}{dK_{,,}}\right) \frac{Q_{,,}}{4 \cos \frac{1}{2} I_{,,}} \right\} \cos \tfrac{1}{2} I_{,} \sin[(\alpha_{,} - \alpha_{,,} - \eta_{,} + \eta_{,,})t + \nu_{,} - \nu_{,,} + K_{,} - K_{,,}]$$
$$- \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dQ_{,,}'}\right) \cos \tfrac{1}{2} I_{,,}' + \left(\frac{d\Omega'}{dK_{,,}'}\right) \frac{P_{,,}'}{4 \cos \frac{1}{2} I_{,,}'} \right\} \cos \tfrac{1}{2} I_{,} \cos[(\alpha' - \alpha_{,,}' - \eta' + \eta_{,,}')t + \nu' - \nu_{,,}' + K' - K_{,,}']$$
$$- \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dP_{,,}'}\right) \cos \tfrac{1}{2} I_{,,}' - \left(\frac{d\Omega'}{dK_{,,}'}\right) \frac{Q_{,,}'}{4 \cos \frac{1}{2} I_{,,}'} \right\} \cos \tfrac{1}{2} I_{,} \sin[(\alpha' - \alpha_{,,}' - \eta' + \eta_{,,}')t + \nu' - \nu_{,,}' + K' - K_{,,}']$$

$$\frac{dQ_{,}}{dt} = \alpha_{,} P_{,} + \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dP_{,}}\right) \cos \tfrac{1}{2} I_{,} - \left(\frac{d\Omega}{dK_{,}}\right) \frac{Q_{,}}{4 \cos \frac{1}{2} I_{,}} \right\} \cos \tfrac{1}{2} I_{,}$$
$$+ \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dP'}\right) \cos \tfrac{1}{2} I' - \left(\frac{d\Omega'}{dK'}\right) \frac{Q'}{4 \cos \frac{1}{2} I'} \right\} \cos \tfrac{1}{2} I'$$
$$+ \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dP_{,,}}\right) \cos \tfrac{1}{2} I_{,,} - \left(\frac{d\Omega}{dK_{,,}}\right) \frac{Q_{,,}}{4 \cos \frac{1}{2} I_{,,}} \right\} \cos \tfrac{1}{2} I_{,} \cos[(\alpha_{,} - \alpha_{,,} - \eta_{,} + \eta_{,,})t + \nu_{,} - \nu_{,,} + K_{,} - K_{,,}]$$
$$- \frac{an}{\sqrt{1-e^2}} \left\{ \left(\frac{d\Omega}{dQ_{,,}}\right) \cos \tfrac{1}{2} I_{,,} + \left(\frac{d\Omega}{dK_{,,}}\right) \frac{P_{,,}}{4 \cos \frac{1}{2} I_{,,}} \right\} \cos \tfrac{1}{2} I_{,} \sin[(\alpha_{,} - \alpha_{,,} - \eta_{,} + \eta_{,,})t + \nu_{,} - \nu_{,,} + K_{,} - K_{,,}]$$
$$+ \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dP_{,,}'}\right) \cos \tfrac{1}{2} I_{,,}' - \left(\frac{d\Omega'}{dK_{,,}'}\right) \frac{Q_{,,}'}{4 \cos \frac{1}{2} I_{,,}'} \right\} \cos \tfrac{1}{2} I_{,} \cos[(\alpha' - \alpha_{,,}' - \eta' + \eta_{,,}')t + \nu' - \nu_{,,}' + K' - K_{,,}']$$
$$- \frac{a'n'}{\sqrt{1-e'^2}} \left\{ \left(\frac{d\Omega'}{dQ_{,,}'}\right) \cos \tfrac{1}{2} I_{,,}' + \left(\frac{d\Omega'}{dK_{,,}'}\right) \frac{P_{,,}'}{4 \cos \frac{1}{2} I_{,,}'} \right\} \cos \tfrac{1}{2} I_{,} \sin[(\alpha' - \alpha_{,,}' - \eta' + \eta_{,,}')t + \nu' - \nu_{,,}' + K' - K_{,,}']$$

$$= \frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP_{,}}\right)\frac{P_{,}}{4}+\left(\frac{d\Omega}{dQ_{,}}\right)\frac{Q_{,}}{4}\right\}-\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left(\frac{d\Omega'}{dP'}\right)\frac{P'}{4}+\left(\frac{d\Omega'}{dQ'}\right)\frac{Q'}{4}\right\}$$

$$= \frac{an}{\sqrt{1-e^2}}\left\{\left[\left(\frac{d\Omega}{dP_{,,}}\right)\cos\tfrac{1}{2}I_{,,}-\left(\frac{d\Omega}{dK_{,,}}\right)\frac{Q_{,,}}{4\cos\frac{1}{2}I_{,,}}\right]\frac{P_{,}}{4}+\left[\left(\frac{d\Omega}{dQ_{,,}}\right)\cos\tfrac{1}{2}I_{,,}+\left(\frac{d\Omega}{dK_{,,}}\right)\frac{P_{,,}}{4\cos\frac{1}{2}I_{,,}}\right]\frac{Q_{,}}{4}\right\}\sec\tfrac{1}{2}I_{,}\cos[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k$$

$$+\frac{an}{\sqrt{1-e^2}}\left\{\left[\left(\frac{d\Omega}{dP_{,,}}\right)\cos\tfrac{1}{2}I_{,,}-\left(\frac{d\Omega}{dK_{,,}}\right)\frac{Q_{,,}}{4\cos\frac{1}{2}I_{,,}}\right]\frac{Q_{,}}{4}-\left[\left(\frac{d\Omega}{dQ_{,,}}\right)\cos\tfrac{1}{2}I_{,,}+\left(\frac{d\Omega}{dK_{,,}}\right)\frac{P_{,,}}{4\cos\frac{1}{2}I_{,,}}\right]\frac{P_{,}}{4}\right\}\sec\tfrac{1}{2}I_{,}\sin[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k$$

$$\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left[\left(\frac{d\Omega'}{dP_{,,}'}\right)\cos\tfrac{1}{2}I_{,,}'-\left(\frac{d\Omega'}{dK_{,,}'}\right)\frac{Q_{,,}'}{4\cos\frac{1}{2}I_{,,}'}\right]\frac{P'}{4}+\left[\left(\frac{d\Omega'}{dQ_{,,}'}\right)\cos\tfrac{1}{2}I_{,,}'+\left(\frac{d\Omega'}{dK_{,,}'}\right)\frac{P_{,,}'}{4\cos\frac{1}{2}I_{,,}'}\right]\frac{Q'}{4}\right\}\sec\tfrac{1}{2}I_{,}\cos[(\alpha'-\alpha_{,,}'-\eta'+\eta_{,,}')t+\nu'-\nu_{,,}'+K$$

$$\frac{a'n'}{\sqrt{1-e'^2}}\left\{\left[\left(\frac{d\Omega'}{dQ_{,,}'}\right)\cos\tfrac{1}{2}I_{,,}'-\left(\frac{d\Omega'}{dK_{,,}'}\right)\frac{Q_{,,}'}{4\cos\frac{1}{2}I_{,,}'}\right]\frac{Q'}{4}-\left[\left(\frac{d\Omega'}{dQ_{,,}'}\right)\cos\tfrac{1}{2}I_{,,}'+\left(\frac{d\Omega'}{dK_{,,}'}\right)\frac{P_{,,}'}{4\cos\frac{1}{2}I_{,,}'}\right]\frac{P'}{4}\right\}\sec\tfrac{1}{2}I_{,}\sin[(\alpha'-\alpha_{,,}'-\eta'+\eta_{,,}')t+\nu'-\nu_{,,}'+K$$

e quibus mutatis mutandis aequationes pro $\frac{dP_{,,}'}{dt}$ etc. eliciuntur; quae integratae valores perturbatos ipsarum $P_{,}$, $Q_{,}$, $K_{,}$, $P_{,,}$, etc. suppeditant. Iam vide discrimen essentiale inter problema trium, et problema plurium corporum. Aequationes quidem (12), si pro tribus tantum corporibus evolutae essent, statim ad plura corpora extendi possent, quia discrimen in eo tantum consistit, quod in functione perturbatrice Ω termini analogi addendi sunt, sed in aequationibus praecedentibus res secus se habet. Hae enim, additis terminis analogis, ad problema quotlibet corporum extendi possunt et, abscissis terminis ad quartum corpus spectantibus, problematis trium corporum aequationes subministrant, sed aequationes pro $\frac{dP}{dt}$, $\frac{dQ}{dt}$ atque $\frac{dK}{dt}$ ad problema trium corporum pertinentes additione terminorum analogorum ad problema plurium corporum extendi nequeunt.

8.

Ad aequationes art. praec. pro dp, dq, etc. integrandas eandem, quam in problemate trium corporum in usum vocavimus, transformationem adhibebo. Positis igitur

$$f_{,}t = -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ_{,}}\right)\cos\tfrac{1}{2}I_{,}+\left(\frac{d\Omega}{dK_{,}}\right)\frac{P_{,}}{4\cos\frac{1}{2}I_{,}}\right\}\cos(K_{,}-k_{,})+\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP_{,}}\right)\cos\tfrac{1}{2}I_{,}-\left(\frac{d\Omega}{dK_{,}}\right)\frac{Q_{,}}{4\cos\frac{1}{2}I_{,}}\right\}\sin(K_{,}-k_{,})$$

$$\varphi_{,}t = -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP_{,}}\right)\cos\tfrac{1}{2}I_{,}-\left(\frac{d\Omega}{dK_{,}}\right)\frac{Q_{,}}{4\cos\frac{1}{2}I_{,}}\right\}\cos(K_{,}-k_{,})-\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ_{,}}\right)\cos\tfrac{1}{2}I_{,}+\left(\frac{d\Omega}{dK_{,}}\right)\frac{P_{,}}{4\cos\frac{1}{2}I_{,}}\right\}\sin(K_{,}-k_{,})$$

$$f_{,,}t = -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ_{,,}}\right)\cos\tfrac{1}{2}I_{,,}+\left(\frac{d\Omega}{dK_{,,}}\right)\frac{P_{,,}}{4\cos\frac{1}{2}I_{,,}}\right\}\cos(K_{,,}-k_{,,})+\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP_{,,}}\right)\cos\tfrac{1}{2}I_{,,}-\left(\frac{d\Omega}{dK_{,,}}\right)\frac{Q_{,,}}{4\cos\frac{1}{2}I_{,,}}\right\}\sin(K_{,,}-k_{,,})$$

$$\varphi_{,,}t = -\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dP_{,,}}\right)\cos\tfrac{1}{2}I_{,,}-\left(\frac{d\Omega}{dK_{,,}}\right)\frac{Q_{,,}}{4\cos\frac{1}{2}I_{,,}}\right\}\cos(K_{,,}-k_{,,})-\frac{an}{\sqrt{1-e^2}}\left\{\left(\frac{d\Omega}{dQ_{,,}}\right)\cos\tfrac{1}{2}I_{,,}+\left(\frac{d\Omega}{dK_{,,}}\right)\frac{P_{,,}}{4\cos\frac{1}{2}I_{,,}}\right\}\sin(K_{,,}-k_{,,})$$

computatio eadem, quae in art. 29. et 30. Sect. II. est peracta, scriptis $\alpha_{,}-\eta_{,}$ loco $(n)\,\varepsilon$, et $-\pi+\nu_{,}+k_{,}$ loco D, suppeditat

$$\frac{dp_{\circ}}{dt}=(\alpha_{,}-\eta_{,})\,q_{\circ}-\Big\{f_{,}t+f_{,,}t\cos\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]+\varphi_{,,}t\sin\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]\Big\}u$$

$$\frac{dq_{\circ}}{dt}=-(\alpha_{,}-\eta_{,})\,p_{\circ}-\Big\{\varphi_{,}t+\varphi_{,,}t\cos\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]-f_{,,}t\sin\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]\Big\}u$$

$$\frac{du}{dt}=\Big\{f_{,}t+f_{,,}t\cos\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]+\varphi_{,,}t\sin\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]\Big\}p_{\circ}$$
$$+\Big\{\varphi_{,}t+\varphi_{,,}t\cos\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]-f_{,,}t\sin\left[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t+\nu_{,}-\nu_{,,}+k_{,}-k_{,,}\right]\Big\}q_{\circ}$$

(ubi u denotat $\cos i$) quibus aequationibus respondent valores finiti ipsarum $p_{\circ}$ atque $q_{\circ}$ hi

$$p_{\circ}=\sin i\sin[\chi-\omega-\pi+\nu_{,}+k_{,}+(\alpha_{,}-\eta_{,})t]$$
$$q_{\circ}=\sin i\cos[\chi-\omega-\pi+\nu_{,}+k_{,}+(\alpha_{,}-\eta_{,})t]$$

Mutatis mutandis, ex his aequationibus analogas pro ipsis $p'_{\circ}$, $q'_{\circ}$, u', $p''_{\circ}$, $q''_{\circ}$, u'' atque $\frac{dp'_{\circ}}{dt}$, etc. nanciscimur. Eadem porro computatio, positis $\alpha_{,,}-\eta_{,,}$ loco $(n)\,\varepsilon$ et $-\pi+\nu_{,,}+k_{,,}+180°$ loco D, suppeditat

$$\frac{dp_{\circ\circ}}{dt}=(\alpha_{,,}-\eta_{,,})q_{\circ\circ}+\Big\{f_{,,}t+f_{,}t\cos\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]+\varphi_{,}t\sin\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]\Big\}u$$

$$\frac{dq_{\circ\circ}}{dt}=-(\alpha_{,,}-\eta_{,,})p_{\circ\circ}+\Big\{\varphi_{,,}t+\varphi_{,}t\cos\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]-f_{,}t\sin\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]\Big\}u$$

$$\frac{du}{dt}=-\Big\{f_{,,}t+f_{,}t\cos\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]+\varphi_{,}t\sin\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]\Big\}p_{\circ\circ}$$
$$-\Big\{\varphi_{,,}t+\varphi_{,}t\cos\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]-f_{,}t\sin\left[(\alpha_{,,}-\alpha_{,}-\eta_{,,}+\eta_{,})t+\nu_{,,}-\nu_{,}+k_{,,}-k_{,}\right]\Big\}q_{\circ\circ}$$

quibus correspondent valores finiti ipsarum $p_{\circ\circ}$ atque $q_{\circ\circ}$ hi

$$p_{\circ\circ}=-\sin i\sin[\chi-\omega-\pi+\nu_{,,}+k_{,,}+(\alpha_{,,}-\eta_{,,})t]$$
$$q_{\circ\circ}=-\sin i\cos[\chi-\omega-\pi+\nu_{,,}+k_{,,}+(\alpha_{,,}-\eta_{,,})t]$$

quae aequationes mutatis mutandis valores analogos ipsarum $p'_{\circ\circ}$, $q'_{\circ\circ}$, u', $p''_{\circ\circ}$, $q''_{\circ\circ}$, u'' atque $\frac{dq'_{\circ\circ}}{dt}$, etc. subministrant. Eadem denique computatio, positis $\alpha_{,,}-\eta_{,,}-\eta_{,}$ loco $(n)\,\varepsilon$ et $-\pi+\nu_{,,}+k_{,,}+k_{,}$ loco D, suppeditat

cc

$$(\alpha'_{,,}-\alpha_{,,}-\eta_{,})q_{000} = \left\{\begin{array}{l} f_{,}t\cos[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}]-\varphi_{,}t\sin[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}] \\ +f_{,,}t\cos[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}]+\varphi_{,,}t\sin[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}] \end{array}\right\}u$$

$$-(\alpha'_{,,}-\alpha_{,,}-\eta_{,})p_{000} = \left\{\begin{array}{l} \varphi_{,}t\cos[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}]+f_{,}t\sin[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}] \\ +\varphi_{,,}t\cos[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}]-f_{,,}t\sin[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}] \end{array}\right\}u$$

$$\left\{\begin{array}{l} f_{,}t\cos[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}]-\varphi_{,}t\sin[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}] \\ +f_{,,}t\cos[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}]+\varphi_{,,}t\sin[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}] \end{array}\right\}p_{000}$$

$$+\left\{\begin{array}{l} \varphi_{,}t\cos[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}]+f_{,}t\sin[(\alpha_{,}-\alpha'_{,,}+\eta_{,,})t+\nu_{,}-\nu'_{,,}-k_{,,}] \\ +\varphi_{,,}t\cos[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}]-f_{,,}t\sin[(\alpha'_{,,}-\alpha_{,,}-\eta_{,})t+\nu'_{,,}-\nu_{,,}+k_{,}] \end{array}\right\}q_{000}$$

quibus valores correspondent finiti ipsarum p_{000} atque q_{000} hi

$$p_{000} = \sin i \sin[\chi-\omega-\pi+\nu'_{,,}+k_{,,}+k_{,}+(\alpha'_{,,}-\eta_{,,}-\eta_{,})t]$$
$$q_{000} = \sin i \cos[\chi-\omega-\pi+\nu'_{,,}+k_{,,}+k_{,}+(\alpha'_{,,}-\eta_{,,}-\eta_{,})t]$$

e quibus aequationibus, mutatis mutandis, valores analogi ipsarum p'_{000}, q'_{000}, u', p''_{000}, q''_{000}, u'' atque $\frac{dp'_{000}}{dt}$, etc. emergunt.

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§. II. *Praemissa aequationum prioris paragraphi, qua termini primi ordinis eliciuntur, integratio.*

9.

Ante omnia termini variabilium prioris paragraphi quantitatum maximi, qui sunt primi ordinis respectu elementorum et o^{ti} ordinis respectu massarum, integratione sunt indagandi. Quem in finem in aequationibus differentialibus ii secernendi sunt termini, qui respectu et massarum et elementorum primi ordinis et simul aut constantes aut cum sinu aut cosinu arcus $at+b$ multiplicati sunt, ubi a est ordinis massarum. Quum W, $n\delta z$, $n'\delta z'$, $n''\delta z''$, w, w', w'' non minus quam Υ et Ψ sint quantitates ordinis primi, potestates et producta harum quantitatum necessario sunt aut secundi aut etiam altioris ordinis, unde sequitur in evolvendis aequationibus (12) terminos per potestates productave illarum quantitatum multiplicatos in hoc calculo omittendos esse. Quum in his aequationibus n et h sint variabiles, habemus

$$\frac{1}{r} = \frac{1}{\bar{r}}(1-w), \quad \frac{h^2}{(h)^2} = 1-2(\delta+\epsilon)$$

et, substitutis his valoribus, coefficientes ipsarum $n\delta z$, $n'\delta z'$, $n''\delta z''$, $\delta P_{,}$ etc. sunt resp. quotientes differentiales expressionum (12) respectu g, g', g'', $P_{,}$, etc., et coefficientes ipsarum w', w''; nec non altera coefficientis ipsius w pars resp. per r', r'' et r multiplicati quotientes differentiales earundem expressionum respectu r', r'' et r. Quibus praemissis, adhibitisque aequationibus his

$$\frac{d\Omega}{dg}=\frac{df}{dg}\left(\frac{d\Omega}{dv_{,}}\right)+\frac{dlr}{dg}r\left(\frac{d\Omega}{dr}\right),\quad \frac{d\Omega}{de}=\frac{df}{de}\left(\frac{d\Omega}{dv_{,}}\right)+\frac{dlr}{de}r\left(\frac{d\Omega}{dr}\right)$$

$$\frac{df}{dg}=\frac{a^2}{r^2}\sqrt{1-e^2}=\frac{1}{\sqrt{1-e^2}}\left\{\frac{a}{r}+\frac{ae}{r}\cos f\right\}=\frac{e}{\sqrt{1-e^2}}\left\{\left(\frac{a}{r}+\frac{1}{1-e^2}\right)\cos f+\frac{1}{e(1-e)^2}\right\}$$

$$\frac{dlr}{dg}=\frac{ae\sin f}{r\sqrt{1-e^2}},\quad \frac{df}{de}=\left\{\frac{a}{r}+\frac{1}{1-e^2}\right\}\sin f,\quad \frac{dlr}{de}=-\frac{a}{r}\cos f,\quad r\left(\frac{d\Omega}{dr}\right)=a\left(\frac{d\Omega}{da}\right)$$

aequationes (12) post evolutionem peractam facile ita exhibentur

$$(20)\left\{\begin{aligned}
\frac{d\Upsilon}{dt}= {} & y\Psi+2\frac{an}{e}\left\{\left(\frac{d\Omega}{dg}\right)-\frac{1}{\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)\right\}+2\frac{an}{e}\left\{\left(\frac{d^2\Omega}{dg^2}\right)-\frac{1}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{dv_{,}dg}\right)\right\}n\delta z\\
&+2\frac{an}{e}\left\{\left(\frac{d^2\Omega}{dg\,dg'}\right)-\frac{1}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{dv_{,}.dg'}\right)\right\}n'\delta z'+2\frac{an}{e}\left\{\left(\frac{d^2\Omega}{dg\,dg''}\right)-\frac{1}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{dv_{,}.dg''}\right)\right\}n''\delta z''\\
&+2\frac{an}{e}\left\{a\left(\frac{d^2\Omega}{dg.da}\right)-\frac{1}{\sqrt{1-e^2}}a\left(\frac{d^2\Omega}{dv_{,}.da}\right)-\left(\frac{d\Omega}{dg}\right)+\frac{a}{r\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)\right\}w\\
&+2\frac{an}{e}\left\{a'\left(\frac{d^2\Omega}{dg.da'}\right)-\frac{1}{\sqrt{1-e^2}}a'\left(\frac{d^2\Omega}{dv_{,}.da'}\right)\right\}w'+2\frac{an}{e}\left\{a''\left(\frac{d^2\Omega}{dg.da''}\right)-\frac{1}{\sqrt{1-e^2}}a''\left(\frac{d^2\Omega}{dv_{,}.da''}\right)\right\}w''\\
&-4\frac{an}{(1-e^2)^{\frac{3}{2}}}(\cos f+e)\left(\frac{d\Omega}{dv_{,}}\right)(S+\varepsilon)+2\frac{an}{e}\left\{\left(\frac{d^2\Omega}{dg.dP_{,}}\right)-\frac{1}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{dv_{,}.dP_{,}}\right)\right\}\delta P_{,}+\text{etc.}\\
\frac{d\Psi}{dt}= {} & -y\Upsilon-y\frac{e}{1-e^2}\left\{2+\Xi+\tfrac{3}{2}e\Upsilon+c^{S+\varepsilon}-1\right\}\\
&+2\frac{an}{\sqrt{1-e^2}}\left(\frac{d\Omega}{de}\right)+2\frac{an}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{de\,dg}\right)n\delta z+2\frac{an}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{de.dg'}\right)n'\delta z'+2\frac{an}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{de.dg''}\right)n''\delta z''\\
&+2\frac{an}{\sqrt{1-e^2}}\left\{a\left(\frac{d^2\Omega}{de\,da}\right)-\left(\frac{d\Omega}{de}\right)+\frac{\sin f}{1-e^2}\left(\frac{d\Omega}{dv_{,}}\right)\right\}w+2\frac{an}{\sqrt{1-e^2}}a'\left(\frac{d\Omega}{de\,da'}\right)w'+2\frac{an}{\sqrt{1-e^2}}a''\left(\frac{d^2\Omega}{de\,da''}\right)w''\\
&-4\frac{an}{(1-e^2)^{\frac{3}{2}}}\sin f\left(\frac{d\Omega}{dv_{,}}\right)(S+\varepsilon)+2\frac{an}{\sqrt{1-e^2}}\left(\frac{d^2\Omega}{de.dP_{,}}\right)\delta P_{,}+\text{etc.}\\
\frac{d\Xi}{dt}= {} & -\tfrac{3}{2}ey\Psi-3an\left(\frac{d\Omega}{dg}\right)-3an\left(\frac{d^2\Omega}{dg^2}\right)n\delta z-3an\left(\frac{d^2\Omega}{dg.dg'}\right)n'\delta z'-3an\left(\frac{d^2\Omega}{dg.dg''}\right)n''\delta z''\\
&-3an\left\{a\left(\frac{d^2\Omega}{dg.da}\right)-\left(\frac{d\Omega}{dg}\right)+\frac{a}{r\sqrt{1-e^2}}\left(\frac{d\Omega}{dv_{,}}\right)\right\}w-3ana'\left(\frac{d^2\Omega}{dg.da'}\right)w'-3ana''\left(\frac{d^2\Omega}{dg.da''}\right)w''\\
&+2\frac{an}{\sqrt{1-e^2}}\left\{3\frac{e\cos f+e^2}{1-e^2}+2\right\}\left(\frac{d\Omega}{dv_{,}}\right)(S+\varepsilon)-3an\left(\frac{d^2\Omega}{dg.dP_{,}}\right)\delta P_{,}+\text{etc.}
\end{aligned}\right.$$

ubi ubique valores pure elliptici substituendi sunt.

Quae aequationes statim monstrant Ξ esse quantitatem secundi ordinis, iidem enim termini ipsius $\frac{d\Xi}{dt}$ in $\frac{dT}{dt}$ per e divisi continentur. Quum igitur evolutae quantitatis $\frac{r}{a}\cos\varphi + \frac{3}{2}e$ maximus terminus sit $\cos\gamma$ et evolutae quantitatis $\frac{r}{a}\sin\varphi$ maximus terminus $\sin\gamma$, habemus, si non nisi terminos primi ordinis respicimus,

$$W = \Upsilon\cos\gamma + \Psi\sin\gamma$$

porro

$$n\delta z = n\int \overline{W}\,dt,\ w = -\tfrac{1}{2}n\int\left(\frac{\overline{dW}}{d\gamma}\right)dt,\ S + s = 2w + \frac{d\delta z}{dt}$$

nam omnes reliquos aequationum (7) et (8) terminos ad minimum secundi ordinis necessario esse debere, ex praecedentibus facile reperitur. Expressio praecedens ipsius $\overline{W}$ suppeditat

$$\overline{W} = \Upsilon\cos g + \Psi\sin g$$

$$\left(\frac{\overline{dW}}{d\gamma}\right) = -\Upsilon\sin g + \Psi\cos g$$

hinc

$$n\delta z = \Upsilon\sin g - \Psi\cos g - \int\frac{d\Upsilon}{dt}\sin g\,dt + \int\frac{d\Psi}{dt}\cos g\,dt$$

$$w = -\tfrac{1}{2}\Upsilon\cos g - \tfrac{1}{2}\Psi\sin g + \tfrac{1}{2}\int\frac{d\Upsilon}{dt}\cos g\,dt + \tfrac{1}{2}\int\frac{d\Psi}{dt}\sin g\,dt$$

Sed facile perspicitur, terminos harum expressionum sub signo integrationis contentos ordinis massarum esse, itaque omitti debere. Habemus igitur in hoc calculo

$$n\delta z = \Upsilon\sin g - \Psi\cos g;\ w = -\tfrac{1}{2}\Upsilon\cos g - \tfrac{1}{2}\Psi\sin g$$

quarum ope aequatio $S + s = 2w + \frac{d\delta z}{dt}$ suppeditat $S + s = o$, et eodem modo invenitur

$$n\delta z' = \Upsilon'\sin g' - \Psi'\cos g',\ w' = -\tfrac{1}{2}\Upsilon'\cos g' - \tfrac{1}{2}\Psi'\sin g',$$
$$n\delta z'' = \Upsilon''\sin g'' - \Psi''\cos g'',\ w'' = -\tfrac{1}{2}\Upsilon''\cos g'' - \tfrac{1}{2}\Psi''\sin g''.$$

qui valores in aequationibus praecedentibus pro $\frac{d\Upsilon}{dt}$ atque $\frac{d\Psi}{dt}$ substituendi sunt.

10.

Ut termini designati in ipsarum $n\delta z$, $n'\delta z'$, etc. aequationum (20) coefficientibus investigari possint, necesse est indoles functionis Ω consideretur. Quem in finem primum appono expressiones finitas distantiarum mutuarum $\Delta_{,}$, $\Delta_{,,}$ et Δ' secundum art. 25. Sect. II. has

$$\begin{aligned}
\Delta_{,}^2 = r^2 + r'^2 &- 2rr' \cos^2 \tfrac{1}{2} I_{,} \cos[\bar f - \bar f' + (y - y' - 2\eta_{,})t + 2K_{,}] \\
&- 2rr' \sin^2 \tfrac{1}{2} I_{,} \cos[\bar f + \bar f' + (y + y' + 2\alpha_{,})t + 2N_{,}] \\
\Delta_{,,}'^2 = r'^2 + r''^2 &- 2r'r'' \cos^2 \tfrac{1}{2} I_{,,}' \cos[\bar f' - \bar f'' + (y' - y'' - 2\eta_{,,}')t + 2K_{,,}'] \\
&- 2r'r'' \sin^2 \tfrac{1}{2} I_{,,}' \cos[\bar f' + \bar f'' + (y' + y'' + 2\alpha_{,,}')t + 2N_{,,}'] \\
\Delta''^2 = r''^2 + r^2 &- 2r''r \cos^2 \tfrac{1}{2} I'' \cos[\bar f'' - \bar f + (y'' - y - 2\eta'')t + 2K''] \\
&- 2r''r \sin^2 \tfrac{1}{2} I'' \cos[\bar f'' + \bar f + (y'' + y + 2\alpha'')t + 2N'']
\end{aligned}$$

quas ita quoque exprimere nobis licet

$$\begin{aligned}
\Delta'^2 = r'^2 + r^2 &- 2r'r \cos^2 \tfrac{1}{2} I' \cos[\bar f' - \bar f + (y' - y - 2\eta')t + 2K'] \\
&- 2r'r \sin^2 \tfrac{1}{2} I' \cos[\bar f' + \bar f + (y' + y + 2\alpha')t + 2N'] \\
\Delta_{,}''^2 = r''^2 + r'^2 &- 2r''r' \cos^2 \tfrac{1}{2} I_{,}'' \cos[\bar f'' - \bar f' + (y'' - y' - 2\eta_{,}'')t + 2K_{,}''] \\
&- 2r''r' \sin^2 \tfrac{1}{2} I_{,}'' \cos[\bar f'' + \bar f' + (y'' + y' + 2\alpha_{,}'')t + 2N_{,}''] \\
\Delta_{,,}^2 = r^2 + r''^2 &- 2rr'' \cos^2 \tfrac{1}{2} I_{,,} \cos[\bar f - \bar f'' + (y - y'' - 2\eta_{,,})t + 2K_{,,}] \\
&- 2rr'' \sin^2 \tfrac{1}{2} I_{,,} \cos[\bar f + \bar f'' + (y + y'' + 2\alpha_{,,})t + 2N_{,,}]
\end{aligned}$$

Hinc sequitur, Ω in seriem infinitam evolvi posse, cuius terminus generalis formam induat hanc

$$\begin{aligned}
&X_{,} \cos\{ig + i'g' + i''[(y - y' - 2\eta_{,})t + 2k_{,}] + i'''[(y + y' + 2\alpha_{,})t + 2\nu_{,}]\} \\
+ &X_{,,} \cos\{ig + i''g'' + i''[(y - y'' - 2\eta_{,,})t + 2k_{,,}] + i'''[(y + y'' + 2\alpha_{,,})t + 2\nu_{,,}]\}
\end{aligned}$$

ubi indicibus i, i', i'', i''' omnes valores integri a $-\infty$ usque ad $+\infty$ singulatim attribuendi sunt, et mutatis mutandis hanc expressionem evolutarum Ω' et Ω'' formam suppeditare. Quibus positis, notam esse suppono formam analyticam evolutae quantitatis Ω, quae typis iam excusa in libris compluribus reperitur; tum facili comparatione instituta invenitur

$$\begin{aligned}
X_{,} &= e^{\pm(i - i'' - i''')} \,.\, e'^{\pm(i' + i'' - i''')} \,.\, \sin^{\pm 2i'''} \tfrac{1}{2} I_{,} \{i, i', i'', i'''\}_{,} \\
X_{,,} &= e^{\pm(i - i'' - i''')} \,.\, e''^{\pm(i' + i'' - i''')} \,.\, \sin^{\pm 2i'''} \tfrac{1}{2} I_{,,} \{i, i', i'', i'''\}_{,,}
\end{aligned}$$

$$\{i, i', i'', i'''\}_, = l_1 + l_2 e^2 + l_3 e'^2 + l_4 \sin^2 \tfrac{1}{2} I_, + l_5 e^4 + l_6 e^2 e'^2 + \text{etc.}$$
$$\{i, i', i'', i'''\}_{,,} = h_1 + h_2 e^2 + h_3 e''^2 + h_4 \sin^2 \tfrac{1}{2} I_{,,} + h_5 e^4 + h_6 e^2 e''^2 + \text{etc.}$$

ubi l_1, l_2, etc. functiones sunt semiaxium maiorum a atque a' et indicum i, i' etc., h_1, h_2, etc. vero functiones semiaxium maiorum a atque a'' et indicum i, i'', etc. et ambigua exponentium signa semper ita sumenda sunt, ut exponens evadat positivus. Quae expressiones ante omnia monstrant, quotientes differentiales ipsius Ω respectu $P_,$, $Q_,$ etc. ad minimum per $\sin \frac{1}{2} I$ multiplicatos, itaque primi ordinis esse, quare et quum $\delta P_,$, $\delta Q_,$ etc. ipsae sint primi ordinis, sequitur terminos aequationum (20) per $\delta P_,$ etc. multiplicatos terminos primi ordinis proferre non posse, itaque reiiciendos esse.

Iam advertamus ad primos aequationum (20) terminos hos

$$2\frac{an}{e}\left\{\left(\frac{d\Omega}{dg}\right) - \left(\frac{d\Omega}{dv_,}\right)\right\} \text{ atque } 2an\left(\frac{d\Omega}{de}\right)$$

quorum prior propter $\frac{dv_,}{d.nt} = 1$, quae praebet

$$\left(\frac{d\Omega}{dv_,}\right) = \left(\frac{d\Omega}{d.nt}\right)$$

abit in

$$2\frac{an}{e}\left\{\left(\frac{d\Omega}{dg}\right) - \left(\frac{d\Omega}{d.nt}\right)\right\}.$$

Quum haec expressio per e divisa, et illa respectu e differentiata sit, termini secundi ordinis in ipsa Ω terminos primi ordinis in his expressionibus producent, unde ponere possumus

$$i - i' = o, \quad i + i' - i'' = o, \quad i''' = \pm 1$$

Sed quum termini indagandi ab ipsis g, g' et g'' liberi esse debeant, necessario esse debet $i = o$ atque $i' = o$, unde sequitur aequationes praecedentes una existere non posse, itaque talem terminum in Ω locum non habere. Hinc concluditur in terminis omnibus indagandis esse debere $i''' = o$, unde sequitur aut

$$i' - i'' = o, \quad i' + i'' = o$$

aut

$$\pm(i' - i'') = 1, \quad \pm(i' + i'') = 1$$

aut

$$i - i' = o, \quad \pm(i' + i') = 2$$

aut

$$\pm(i - i') = 2, \quad i' + i' = o,$$

Iam aequationes $i = o$ atque $i' = o$ in tertiis et quartis coniunctis aequationibus substitutae monstrant, has aequationes una existere non posse, unde tales terminos in ipsa Ω omnino non adesse concluditur. Aequationibus vero $i = o$ atque $i' = o$ in primis coniunctis aequationibus substitutis, nanciscimur $i' = o$, et iisdem in secundis substitutis, obtinemus $i' = \mp 1$, unde habemus in ipsa Ω terminos hos

$$\Omega = \{o,o,o,o\}_, + \{o,o,o,o\}_{,,} + 2ee'\{o,o,1,o\}_, \cos[(y - y' - 2\eta_,)t + 2k_,]$$
$$+ 2ee''\{o,o,1,o\}_{,,} \cos[(y - y'' - 2\eta_{,,})t + 2k_{,,}]$$

qui soli sunt termini eius generis et ordinis, qui requiruntur. Qui termini suppeditant

$$2\frac{an}{e}\left\{\left(\frac{d\Omega}{dg}\right) - \left(\frac{d\Omega}{d.yt}\right)\right\} = 4ane'\{o,o,1,o\}_, \sin[(y - y' - 2\eta_,)t + 2k_,]$$
$$+ 4ane''\{o,o,1,o\}_{,,} \sin[(y - y'' - 2\eta_{,,})t + 2k_{,,}]$$

$$2an\left(\frac{d\Omega}{de}\right) = 2an\frac{d.\{o,o,o,o\}_,}{de} + 2an\frac{d.\{o,o,o,o\}_{,,}}{de} + 4ane'\{o,o,1,o\}_, \cos[(y - y' - 2\eta_,)t + 2k_,]$$
$$+ 4ane''\{o,o,1,o\}_{,,} \cos[(y - y'' - 2\eta_{,,})t + 2k_{,,}]$$

Revertamur ad terminos aequationum (20) per ndz et w multiplicatos. Quum secundum praecedentia termini maximi ipsarum ndz atque w per $\cos g$ atque $\sin g$ multiplicati sint, in terminis ipsius Ω nunc inveniendis esse debet $i = \pm 1$, atque $i' = o$; quum porro termini maximi ipsarum ndz atque w ipsas Υ et Ψ contineant, quae primi ordinis sunt, et termini respectivi aequationum (20) aut per e dividantur aut differentiationi respectu e subiiciantur: termini ipsius Ω nunc inveniendi primi ordinis respectu ipsius e esse debent. Habemus igitur

$$i' = \pm o, \quad i' + i' = o, \quad \pm(i - i') = 1$$

unde $i' = o$ sequitur. Termini igitur quaesiti sunt

$$\Omega = 2e\{1,o,o,o\}_, \cos g + 2e\{1,o,o,o\}_{,,} \cos g$$

hinc elicitur

$$2\frac{an}{e}\left\{\left(\frac{d^2\Omega}{dg^2}\right)-\left(\frac{d^2\Omega}{dv_{,}\,dg}\right)\right\}=-4an\{1,0,0,0\}_{,}\cos g-4an\{1,0,0,0\}_{,,}\cos g$$

$$2\frac{an}{e}\left\{a\left(\frac{d^2\Omega}{dg\,.\,da}\right)-\left(\frac{d^2\Omega}{dv_{,}\,dg}\right)-\left(\frac{d\Omega}{dg}\right)\right\}=-4an\,.\,a\frac{d\{1,0,0,0\}_{,}}{da}\sin g+4an\{1,0,0,0\}_{,}\sin g$$
$$-4an\,.\,a\frac{d\{1,0,0,0\}_{,,}}{da}\sin g+4an\{1,0,0,0\}_{,,}\sin g$$

$$2an\left(\frac{d^2\Omega}{de\,dg}\right)=-4an\{1,0,0,0\}_{,}\sin g-4an\{1,0,0,0\}_{,,}\sin g$$

$$2an\left\{a\left(\frac{d^2\Omega}{de\,da}\right)-\left(\frac{d\Omega}{de}\right)\right\}=4an\,.\,a\frac{d\{1,0,0,0\}_{,}}{da}\cos g-4an\{1,0,0,0\}_{,}\cos g$$
$$+4an\,.\,a\frac{d\{1,0,0,0\}_{,,}}{da}\cos g-4an\{1,0,0,0\}_{,,}\cos g$$

nec terminos $\frac{a}{r}\left(\frac{d\Omega}{dv_{,}}\right)$ atque $\frac{\sin f}{1-e^2}\left(\frac{d\Omega}{dv_{,}}\right)$ in coefficientibus ipsius w contentos ad quaesitos terminos quidquam addere posse, facile perspicitur. Eodem quo supra modo concluditur in aequationum (20) coefficientibus ipsarum $n'\delta z'$ atque w' ante omnia esse debere $i=o$ atque $i'=\pm1$, quatenus ad priorem ipsius Ω partem pertineant, porro

$$i'''=o\,,\quad i'+i''=o\,,\quad \pm(i-i')=1$$

unde $i''=\mp1$ sequitur, ita ut $i''=-1$ correspondeat ipsi $i'=+1$, et $i''=+1$ ipsi $i'=-1$. Hinc elicitur

$$\Omega=2e\{0,1,-1,0\}_{,}\cos[g'-(y-y'-2\eta_{,})t+2k_{,}]$$

et

$$2\frac{an}{e}\left\{\left(\frac{d^2\Omega}{dg\,dg'}\right)-\left(\frac{d^2\Omega}{dv_{,}\,dg'}\right)\right\}=-4an\{0,1,-1,0\}_{,}\cos[g'-(y-y'-2\eta_{,})t+2k_{,}]$$

$$2\frac{an}{e}\left\{a'\left(\frac{d^2\Omega}{dg\,da'}\right)-a'\left(\frac{d^2\Omega}{dv_{,}\,da'}\right)\right\}=-4an\,.\,a'\frac{d\{0,1,-1,0\}_{,}}{da}\sin[g'-(y-y'-2\eta_{,})t+2k_{,}]$$

$$2an\left(\frac{d^2\Omega}{de\,dg'}\right)=-4an\{0,1,-1,0\}_{,}\sin[g'-(y-y'-2\eta_{,})t+2k_{,}]$$

$$2an\,a'\left(\frac{d^2\Omega}{de\,da'}\right)=4an\,.\,a'\frac{d\{0,1,-1,0\}_{,}}{da'}\cos[g'-(y-y'-2\eta_{,})t+2k_{,}]$$

quibus coefficientes ipsarum $n''\delta z''$ atque w'' plane similes sunt. Hi sunt termini quaesiti, reliqui autem aequationum (20) termini altioris ordinis generisve alius sunt.

11.

Antequam coefficientium in art. praec. inventorum valores in aequationibus (20) substituimus, ex re est relationes, quae inter hos coefficientes existere debeant, investigare. Habemus

$$(21)\ldots\ldots\quad \left(\frac{d\Omega}{de}\right) = \frac{df}{de}\left(\frac{d\Omega}{d.\,yt}\right) + \frac{dlr}{de}\,a\left(\frac{d\Omega}{da}\right)$$

et maximi atque ad propositum nostrum spectantes termini ipsarum $\frac{df}{de}$ et $\frac{dlr}{de}$ sunt hi

$$\frac{df}{de} = 2\sin g\,,\quad \frac{dlr}{de} = \tfrac{1}{2}e - \cos g$$

Quodsi ponimus

$$\left(\frac{d\Omega}{de}\right) = \frac{d\{0,0,0,0\}_{,}}{de}$$

et eundem terminum in dextro aequationis praecedentis membro reproducere volumus, ponenda est

$$\Omega = \{0,0,0,0\}_{,} + 2e\{1,0,0,0\}_{,}\cos g$$

alio enim modo terminus ille reproduci nequit. Substitutis vero his valoribus in aequatione (21), nanciscimur

$$\frac{d\{0,0,0,0\}_{,}}{de} = \tfrac{1}{2}e\,a\,\frac{d\{0,0,0,0\}_{,}}{da} - e\,a\,\frac{d\{1,0,0,0\}_{,}}{da}.$$

Posita porro

$$\left(\frac{d\Omega}{de}\right) = 2\{1,0,0,0\}_{,}\cos g$$

quae non minus quam illae convenit cum generali ipsius Ω forma supra allata, eaedem aequationes praecedentes subministrant

$$2\{1,0,0,0\}_{,} = -a\,\frac{d\{0,0,0,0\}_{,}}{da}$$

unde

$$(22)\ldots\ldots\left\{\begin{array}{l} \dfrac{d\{0,0,0,0\}_{,}}{de} = -e\{1,0,0,0\}_{,} - e\,a\,\dfrac{d\{1,0,0,0\}_{,}}{da} \\ \text{et eodem modo invenitur} \\ \dfrac{d\{0,0,0,0\}_{,,}}{de} = -e\{1,0,0,0\}_{,,} - e\,a\,\dfrac{d\{1,0,0,0\}_{,,}}{da} \end{array}\right.$$

Aequationi (21) similis est haec

$$\left(\frac{d\Omega}{de'}\right)=\frac{df'}{de'}\left(\frac{d\Omega}{d\,.\,y't}\right)+\frac{dlr'}{de'}a'\left(\frac{d\Omega}{da'}\right)$$

quare, si ponimus

$$\left(\frac{d\Omega}{de'}\right)=2e\,\{0,0,1,0\}_{,}\cos\left[(y-y'-2\eta_{,})t+2k_{,}\right]$$

quae cum generali ipsius Ω forma convenit, in dextro aequationis praecedentis membro ponendae sunt hae

$$\Omega=2e\,\{0,1,-1,0\}_{,}\cos\left[g'-(y-y'-2\eta_{,})t+2k_{,}\right]$$

$$\frac{df'}{de'}=2\sin g'\;,\quad\frac{dlr'}{de'}=-\cos g'$$

ut ille terminus reproduci possit, qui alio modo omnino reproduci nequit. Substitutis vero his aequationibus invenitur

$$\left.\begin{array}{l}\{0,0,1,0\}_{,}=-\{0,1,-1,0\}_{,}-\tfrac{1}{2}a'\dfrac{d\{0,1,-1,0\}_{,}}{da'}\\[2ex]\text{eodemque modo}\\[2ex]\{0,0,1,0\}_{,,}=-\{0,1,-1,0\}_{,,}-\tfrac{1}{2}a''\dfrac{d\{0,1,-1,0\}_{,,}}{da''}\end{array}\right\}\ldots\ldots(23)$$

Iam substitutis coefficientium in art. praec. evolutorum nec non ipsarum $n\delta z$, w, etc. valoribus in aequationibus (20), omissisque post substitutionem terminis per $\sin 2g$, $\cos 2g$, $\sin 2g'$, etc. multiplicatis, quia alius generis sunt quam quaesiti, nanciscimur propter aequationes conditionales (22) et (23)

$$\left.\begin{array}{l}\dfrac{d\Upsilon}{dt}=(y-\mu_{,}-\mu_{,,})\Psi+\sigma_{,}(\Upsilon'+2e')\sin[(y-y'-2\eta_{,})t+2k_{,}]-\sigma_{,}\Psi'\cos[(y-y'-2\eta_{,})t+2k_{,}]\\ \qquad\qquad+\sigma_{,,}(\Upsilon''+2e'')\sin[(y-y''-2\eta_{,,})t+2k_{,,}]-\sigma_{,,}\Psi''\cos[(y-y''-2\eta_{,,})t+2k_{,,}]\\ \dfrac{d\Psi}{dt}=-(y-\mu_{,}-\mu_{,,})(\Upsilon+2e)+\sigma_{,}(\Upsilon'+2e')\cos[(y-y'-2\eta_{,})t+2k_{,}]+\sigma_{,}\Psi'\sin[(y-y'-2\eta_{,})t+2k_{,}]\\ \qquad\qquad+\sigma_{,,}(\Upsilon''+2e'')\cos[(y-y''-2\eta_{,,})t+2k_{,,}]+\sigma_{,,}\Psi''\sin[(y-y''-2\eta_{,,})t+2k_{,,}]\end{array}\right\}(24)$$

ubi brevitatis caussa feci

$$an\frac{d\{0,0,0,0\}_{,}}{de}=e\mu_{,}\;,\qquad an\frac{d\{0,0,0,0\}_{,,}}{de}=e\mu_{,,}$$

$$2an\,\{0,0,1,0\}_{,}=\sigma_{,}\;,\quad 2an\,\{0,0,0,0\}_{,,}=\sigma_{,,}$$

Ex his aequationibus nanciscimur, mutatis mutandis, aequationes pro $\frac{d\Upsilon'}{dt}$, etc.

12.

Revertamur ad aequationes art. 7. pro $\frac{dP_{,}}{dt}$, etc. Primo oculorum obtutu perspicitur $\frac{dK_{,}}{dt}$ esse ordinis secundi, itaque in hoc calculo omittendam. Quum ipsius Ω quotientes differentiales respectu $P_{,}$, $Q_{,}$, etc. e quotientibus differentialibus respectu ipsarum $I_{,}$, $N_{,}$ etc. pendeant, in generali ipsius Ω expressione primum ponimus

$$i''' = \pm 1, \text{ atque } i - i'' - i''' = o, \ i' + i'' - i''' = o$$

quum vero in terminis quaesitis etiam esse debeat $i = o$ atque $i' = o$, aequationibus praecedentibus satisfieri non potest, unde concluditur terminum huius formae in Ω non adesse. Restat igitur solummodo, ut ponamus

$$i = i' = i'' = i''' = o$$

unde

$$\Omega = \{o,o,o,o\}_{,} + \{o,o,o,o\}_{,,}$$

sive

$$\Omega = l_4 \sin^2 \tfrac{1}{2} I_{,} + \text{etc.} + h_4 \sin^2 \tfrac{1}{2} I_{,,} + \text{etc.}$$

quae, quum sint $P_{,}^2 + Q_{,}^2 = 4 \sin^2 \frac{1}{2} I_{,}$ et $P_{,,}^{\,2} + Q_{,,}^{\,2} = 4 \sin^2 \frac{1}{2} I_{,,}$, abit in

$$\Omega = \tfrac{1}{4} l_4 (P_{,}^2 + Q_{,}^2) + \tfrac{1}{4} h_4 (P_{,,}^{\,2} + Q_{,,}^{\,2})$$

quare

$$\left(\frac{d\Omega}{dP_{,}}\right) = \tfrac{1}{2} l_4 P_{,}, \quad \left(\frac{d\Omega}{dQ_{,}}\right) = \tfrac{1}{2} l_4 Q_{,}, \quad \left(\frac{d\Omega}{dP_{,,}}\right) = \tfrac{1}{2} h_4 P_{,,}$$

$$\left(\frac{d\Omega}{dQ_{,,}}\right) = \tfrac{1}{2} h_4 Q_{,,}, \quad \left(\frac{d\Omega}{dK_{,}}\right) = \left(\frac{d\Omega}{dK_{,,}}\right) = o$$

Omissis igitur terminis secundi et altioris ordinis, unde constantes $k_{,}$, $k_{,,}$, etc. loco $K_{,}$, $K_{,,}$, etc. ponere et $\cos \frac{1}{2} I_{,}$, etc. omittere nobis licet, aequationes art. 7. suppeditant

$$(25)\left\{\begin{aligned}
\frac{dP_{,}}{dt} = & -(\alpha_{,}-\pi_{,}-\pi')Q_{,} + \pi_{,,} Q_{,,} \cos[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t + \nu_{,}-\nu_{,,}+k_{,}-k_{,,}] + \pi_{,,} P_{,,} \sin[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t + \nu_{,}-\nu_{,,}+k_{,}-k_{,,}] \\
& + \pi'_{,,} Q'_{,,} \cos[(\alpha'-\alpha'_{,,}-\eta'+\eta'_{,,})t + \nu'-\nu'_{,,}+k'-k'_{,,}] + \pi'_{,,} P'_{,,} \sin[(\alpha'-\alpha'_{,,}-\eta'+\eta'_{,,})t + \nu'-\nu'_{,,}+k'-k'_{,,}] \\
\frac{dQ_{,}}{dt} = & (\alpha_{,}-\pi_{,}-\pi')P_{,} - \pi_{,,} P_{,,} \cos[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t + \nu_{,}-\nu_{,,}+k_{,}-k_{,,}] + \pi_{,,} Q_{,,} \sin[(\alpha_{,}-\alpha_{,,}-\eta_{,}+\eta_{,,})t + \nu_{,}-\nu_{,,}+k_{,}-k_{,,}] \\
& - \pi'_{,,} P'_{,,} \cos[(\alpha'-\alpha'_{,,}-\eta'+\eta'_{,,})t + \nu'-\nu'_{,,}+k'-k'_{,,}] + \pi'_{,,} Q'_{,,} \sin[(\alpha'-\alpha'_{,,}-\eta'+\eta'_{,,})t + \nu'-\nu'_{,,}+k'-k'_{,,}]
\end{aligned}\right.$$

ubi brevitatis caussa posui

$$\frac{anl_4}{2\sqrt{1-e^2}} = -\pi_{,},\quad \frac{a'n'k'_4}{2\sqrt{1-e'^2}} = -\pi',\quad \frac{anh_4}{2\sqrt{1-e^2}} = -\pi_{,,},\quad \frac{a'n'l'_4}{2\sqrt{1-e'^2}} = -\pi'_{,,}$$

e quibus aequationibus mutatis mutandis analogas pro $\frac{dP'_{,,}}{dt}$, etc. nanciscimur.

Determinationem igitur terminorum primi ordinis, quos incognitae nostrae continent, ad integrationem duorum systematum aequationum differentialium linearium cum coefficientibus variabilibus reduximus, aequationum (24) et similium et aequationum (25) et similium dico integrationem. Quae quidem aequationes propter quantitates $\eta_{,}$, $k_{,}$, etc., quas continent, in aequationes lineares cum coefficientibus constantibus transferri nequeunt, factis vero $\eta_{,}$, $k_{,}$, etc. cifrae aequalibus, positisque

$Y = \Upsilon \cos yt - \Psi \sin yt$, $Z = \Upsilon \sin yt + \Psi \cos yt$, etc.

$H_{,} = P_{,} \cos(\alpha_{,}t + v_{,}) + Q_{,} \sin(\alpha_{,}t + v_{,})$, $L_{,} = -P_{,} \sin(\alpha_{,}t + v_{,}) + Q_{,} \cos(\alpha_{,}t + v_{,})$, etc.

abeunt in

$$\frac{dY}{dt} = -(\mu_{,} + \mu_{,,}) Z - \sigma_{,} Z' - \sigma_{,,} Z''$$

$$\frac{dZ}{dt} = (\mu_{,} + \mu_{,,}) Y + \sigma_{,} Y' + \sigma_{,,} Y''$$

$$\frac{dH_{,}}{dt} = (\pi_{,} + \pi') L_{,} + \pi_{,,} L_{,,} - \pi'_{,,} L'_{,,}$$

$$\frac{dL_{,}}{dt} = -(\pi_{,} + \pi') H_{,} - \pi_{,,} H_{,,} + \pi'_{,,} H'_{,,}$$

quae aequationibus notis ab ill. Lagrange primo datis similes sunt.

Attamen licitum non est in aequationibus (24) et (25) quantitates $\eta_{,}$, $k_{,}$, etc. cifrae aequales ponere, quod si fecissemus, in secunda et subsequentibus approximationibus aut terminos per tempus ipsum multiplicatos inveniremus, aut terminos in approximationes crescentes, qui series divergentes constituerent.

13.

Iam ante omnia necesse est, relationes inter $P_{,}$, $Q_{,}$, $K_{,}$, $P'_{,,}$, etc. indagentur, quem in finem aequationes conditionales in art. 6. evolutae integrandae sunt. Quum vero earum per $P_{,}$, $Q_{,}$, etc. expressa forma admodum implicita videatur, e re est eas transformare et ope aequationum (17)

ipsas $I_{,}$, $N_{,}$, etc. loco $P_{,}$, $Q_{,}$, etc. introducere. Quibus factis, aequationes nostrae conditionales, postquam prima per $\frac{1}{2}\cos\frac{1}{2}I'_{,,}\cos\frac{1}{2}I''\,dt$, secunda per $\frac{1}{2}\cos\frac{1}{2}I_{,}\cos\frac{1}{2}I''\,dt$ et tertia per $\frac{1}{2}\cos\frac{1}{2}I_{,}\cos\frac{1}{2}I'_{,,}\,dt$ multiplicata est, ita se habent

$$\begin{aligned}
o = {} & 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(L_{,}+L'_{,,}+L'')(dK_{,}-\eta_{,}dt)+\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(L_{,}+L'_{,,}+L'')\,dI_{,}\\
& +\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(M_{,}-M'_{,,}+L'')dI'_{,,}+2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(M_{,}-M'_{,,}+L'')(dM'_{,,}+\alpha'_{,,}dt)\\
& +\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(M''-M_{,}+L'_{,,})dI''-2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M''-M_{,}+L'_{,,})(dM''+\alpha''dt)
\end{aligned}$$

$$\begin{aligned}
o = {} & 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(L_{,}+L'_{,,}+L'')(dK'_{,,}-\eta'_{,,}dt)+\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(L_{,}+L'_{,,}+L'')\,dI'_{,,}\\
& +\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(M'_{,,}-M''+L_{,})dI''+2\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M'_{,,}-M''+L_{,})(dM''+\alpha''dt)\\
& +\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(M_{,}-M'_{,,}+L'')dI_{,}-2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(M_{,}-M'_{,,}+L'')(dM_{,}+\alpha_{,}dt)
\end{aligned}$$

$$\begin{aligned}
o = {} & 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(L_{,}+L'_{,,}+L'')(dK''-\eta''dt)+\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(L_{,}+L'_{,,}+L'')\,dI''\\
& +\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(M''-M_{,}+L'_{,,})dI_{,}+2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M''-M_{,}+L'_{,,})(dM_{,}+\alpha_{,}dt)\\
& +\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(M'_{,,}-M''+L_{,})dI'_{,,}-2\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M'_{,,}-M''+L_{,})(dM'_{,,}+\alpha'_{,,}dt)
\end{aligned}$$

ubi brevitatis caussa posui $M_{,}=N_{,}-\nu_{,}$, etc. Multiplicata harum aequationum prima per $\operatorname{tg}\frac{1}{2}I'_{,,}\operatorname{tg}\frac{1}{2}I''\sin(M'_{,,}-M''+L_{,})$, secunda per $\operatorname{tg}\frac{1}{2}I_{,}\operatorname{tg}\frac{1}{2}I''\sin(M''-M_{,}+L'_{,,})$ et tertia per $\operatorname{tg}\frac{1}{2}I_{,}\operatorname{tg}\frac{1}{2}I'_{,,}\sin(M_{,}-M'_{,,}+L'')$, evadit, additis productis, aequatio haec

$$\begin{aligned}
o = {} & 2\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M'_{,,}-M''+L_{,})(dK_{,}-\eta_{,}dt)+\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(M'_{,,}-M''+L_{,})\,dI_{,}\\
& +2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M''-M_{,}+L'_{,,})(dK'_{,,}-\eta'_{,,}dt)+\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(M''-M_{,}+L'_{,,})\,dI'_{,,}\\
& +2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(M_{,}-M'_{,,}+L'')(dK''-\eta''dt)+\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(M_{,}-M'_{,,}+L'')\,dI''
\end{aligned}$$

Quodsi nunc haecce aequatio a praecedentium aequationum conditionalium aggregato subtracta erit, nanciscimur propter aequationes has

$$\begin{aligned}
dL_{,} &= (\alpha'_{,,}-\alpha''-\eta_{,})\,dt+dK_{,}\\
dL'_{,,} &= (\alpha''-\alpha_{,}-\eta'_{,,})\,dt+dK'_{,,}\\
dL'' &= (\alpha_{,}-\alpha'_{,,}-\eta'')\,dt+dK''
\end{aligned}$$

aequationem hanc

$$
\begin{aligned}
o = {} & 2\cos\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\sin(L_{\prime}+L'_{\prime\prime}+L'')\,d\,.\,(L_{\prime}+L'_{\prime\prime}+L'') \\
& +[\sin\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI_{\prime}+\cos\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI'_{\prime\prime}+\cos\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI'']\cos(L_{\prime}+L'_{\prime\prime}+L'') \\
& -[\sin\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI_{\prime}-\cos\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI'_{\prime\prime}-\cos\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI'']\cos(M'_{\prime\prime}-M''+L_{\prime}) \\
& \qquad -2\cos\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\sin(M'_{\prime\prime}-M''+L_{\prime})\,d\,.\,(M'_{\prime\prime}-M''+L_{\prime}) \\
& +[\cos\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI_{\prime}-\sin\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI'_{\prime\prime}+\sin\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI'']\cos(M''-M_{\prime}+L'_{\prime\prime}) \\
& \qquad -2\sin\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\sin(M''-M_{\prime}+L'_{\prime\prime})\,d\,.\,(M''-M_{\prime}+L'_{\prime\prime}) \\
& +[\cos\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI_{\prime}+\sin\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI'_{\prime\prime}-\sin\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI'']\cos(M_{\prime}-M'_{\prime\prime}+L'') \\
& \qquad -2\sin\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\sin(M_{\prime}-M'_{\prime\prime}+L'')\,d\,.\,(M_{\prime}-M'_{\prime\prime}+L'')
\end{aligned}
$$

quae per se integrabilis est integrataque suppeditat

$$
\begin{aligned}
o = {} & \text{const.}-2\cos\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(L_{\prime}+L'_{\prime\prime}+L'')+2\cos\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'_{\prime\prime}-M''+L_{\prime}) \\
& +2\sin\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M''-M_{\prime}+L'_{\prime\prime})+2\sin\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(M_{\prime}-M'_{\prime\prime}+L'')
\end{aligned}
$$

Aequatio pro $\frac{dK_{\prime}}{dt}$ in art. 7. evoluta eiusque similes pro $\frac{dK'_{\prime\prime}}{dt}$ atque $\frac{dK''}{dt}$ monstrant, factis inclinationibus mutuis cifrae aequalibus, quantitates quoque $\eta_{\prime}$, $\eta'_{\prime\prime}$ atque η'' cifrae aequales fieri, quare aequationes (16) in hoc casu praebent

$$L_{\prime}+L'_{\prime\prime}+L''=K_{\prime}+K'_{\prime\prime}+K''=-\tfrac{1}{2}(\varphi_{\prime}-\psi_{\prime})-\tfrac{1}{2}(\varphi'_{\prime\prime}-\psi'_{\prime\prime})-\tfrac{1}{2}(\varphi''-\psi'')$$

si vero inclinationes mutuae evanescunt, differentia quoque inter $\varphi_{\prime}$ atque $\psi_{\prime}$, $\varphi'_{\prime\prime}$ atque $\psi'_{\prime\prime}$ et φ'' atque ψ'' in nihilum recedit, unde $L_{\prime}+L'_{\prime\prime}+L''=o$. Factis igitur in integrali praecedente inclinationibus mutuis cifrae aequalibus, emergit const. $=2$, et integrale nostrum abit in

$$
\begin{aligned}
o = {} & 1-\cos\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(L_{\prime}+L'_{\prime\prime}+L'')+\cos\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'_{\prime\prime}-M''+L_{\prime}) \\
& +\sin\tfrac{1}{2}I_{\prime}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M''-M_{\prime}+L'_{\prime\prime})+\sin\tfrac{1}{2}I_{\prime}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(M_{\prime}-M'_{\prime\prime}+L'')\ldots(26)
\end{aligned}
$$

Aequationes nostrae conditionales per se quidem integrabiles non sunt, sed facile invenitur functio, quae, postquam ad unam earum e. g. ad primam aequationem addita erit, efficiet, ut haec aequatio integrabilis evadat. Posita

$$o = dX_{,} + \left[\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI_{,} - \sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI''\right]\cos(M_{,} - M'_{\prime\prime} + L'')$$
$$-2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\sin(M_{,} - M'_{\prime\prime} + L'')(dM_{,} + \alpha_{,}dt + dK'' - \eta''dt)$$
$$+\left[\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI_{,} - \sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI'_{\prime\prime}\right]\cos(M'' - M_{,} + L'_{\prime\prime})$$
$$+2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\sin(M'' - M_{,} + L'_{\prime\prime})(dM_{,} + \alpha_{,}dt - dK'_{\prime\prime} + \eta'_{\prime\prime}dt)$$
$$+\left[\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\,dI'_{\prime\prime} + \cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\,dI''\right]\cos(L_{,} + L'_{\prime\prime} + L'')$$
$$+2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I_{\prime\prime}\cos\tfrac{1}{2}I''\sin(L_{,} + L'_{\prime\prime} + L'')(dK'_{\prime\prime} - \eta'_{\prime\prime}dt + dK'' - \eta''dt)$$

ubi $dX_{,}$ est functio ignota, quae ita determinanda est, ut haec aequatio revera locum habeat, aggregatum ex hac et prima aequatione conditionali formatum per se integrabile est, et integratum integrale suppeditat hoc

$$o = X_{,} - 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(L_{,} + L'_{\prime\prime} + L'') + 2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(M_{,} - M'_{\prime\prime} + L'')$$
$$+2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'' - M_{,} + L'_{\prime\prime})$$

ubi constantem non adieci, quia in functione $X_{,}$ contenta censeri potest. Eodem modo sive mutatis mutandis duae reliquae aequationes conditionales subministrant

$$o = X'_{\prime\prime} - 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(L_{,} + L'_{\prime\prime} + L'') + 2\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'_{\prime\prime} - M'' + L_{,})$$
$$+2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(M_{,} - M'_{\prime\prime} + L'')$$
$$o = X'' - 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(L_{,} + L'_{\prime\prime} + L'') + 2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'' - M_{,} + L'_{\prime\prime})$$
$$+2\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'_{\prime\prime} - M'' + L_{,})$$

et si aequationes conditionales praeter integrale (26) adhuc duo integralia revera habent, functiones $dX_{,}$, $dX_{\prime\prime}'$ atque dX'' necessario differentialia perfecta esse debent. Combinatis praecedentibus integralibus cum integrali (26), aequationes facile derivantur hae

$$(27)\ldots\ldots\left\{\begin{aligned} o &= X_{,} - 2 - 2\cos\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'_{\prime\prime} - M'' + L_{,}) \\ o &= X'_{\prime\prime} - 2 - 2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\sin\tfrac{1}{2}I''\cos(M'' - M_{,} + L'_{\prime\prime}) \\ o &= X'' - 2 - 2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(M_{,} - M'_{\prime\prime} + L'') \\ o &= X_{,} + X'_{\prime\prime} + X'' - 4 - 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{\prime\prime}\cos\tfrac{1}{2}I''\cos(L_{,} + L'_{\prime\prime} + L'') \end{aligned}\right.$$

e quibus emergunt

$$\left.\begin{aligned}
2\sin\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''(M''-M_{,}+L'_{,,}) &= \sqrt{4\sin^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I'_{,,}\sin^2\tfrac{1}{2}I''-(X'_{,,}-2)^2}\\
2\sin\tfrac{1}{2}I_{,}\sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''(M_{,}-M'_{,,}+L'') &= \sqrt{4\sin^2\tfrac{1}{2}I_{,}\sin^2\tfrac{1}{2}I'_{,,}\cos^2\tfrac{1}{2}I''-(X''-2)^2}\\
2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(L_{,}-L'_{,,}+L'') &= \sqrt{4\cos^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I'_{,,}\cos^2\tfrac{1}{2}I''-(X_{,}+X'_{,,}+X''-4)^2}
\end{aligned}\right\}\ldots\ldots(28)$$

Substitutis his valoribus in aequatione praecedente pro $dX_{,}$, emergit

$$\begin{aligned}
o=2\,dX_{,}&+\operatorname{cotg}\tfrac{1}{2}I_{,}(X'_{,,}+X''-4)\,dI_{,}+\operatorname{tg}\tfrac{1}{2}I'_{,,}(X_{,}+X''-2)\,dI'_{,,}+\operatorname{tg}\tfrac{1}{2}I''(X_{,}+X'_{,,}-2)\,dI''\\
&+2\left\{\sqrt{4\sin^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\sin^2\tfrac{1}{2}I''-(X_{,,}'-2)^2}-\sqrt{4\sin^2\tfrac{1}{2}I_{,}\sin^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X''-2)^2}\right\}(dM_{,}+\alpha_{,}dt)\\
&+2\left\{\sqrt{4\cos^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X_{,}+X_{,,}'+X''-4)^2}-\sqrt{4\sin^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\sin^2\tfrac{1}{2}I''-(X_{,,}'-2)^2}\right\}(dL'_{,,}-\eta'_{,,}dt)\\
&+2\left\{\sqrt{4\cos^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X_{,}+X_{,,}'+X''-4)^2}-\sqrt{4\sin^2\tfrac{1}{2}I_{,}\sin^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X''-2)^2}\right\}(dL''-\eta''dt)
\end{aligned}$$

quum vero sit $X_{,}$ functio ipsarum $I_{,}$, $I_{,,}'$, I'', $M_{,}+\alpha_{,}t$, etc., habemus

$$dX_{,}=\frac{dX_{,}}{dI_{,}}dI_{,}+\frac{dX_{,}}{dI_{,,}'}dI'_{,,}+\frac{dX_{,}}{dI''}dI''+\frac{dX_{,}}{dM_{,}}(dM_{,}+\alpha_{,}dt)+\frac{dX_{,}}{dL_{,,}'}(dL'_{,,}-\eta'_{,,}dt)+\frac{dX_{,}}{dL''}(dL''-\eta''dt)$$

quae in praecedente substituta hanc identicam reddere debet. Quo facto nanciscimur aequationes has

$$\left.\begin{aligned}
&o=2\frac{dX_{,}}{dI_{,}}\operatorname{tg}\tfrac{1}{2}I_{,}+X'_{,,}+X''-4,\quad o=2\frac{dX_{,}}{dI_{,,}'}\operatorname{cotg}\tfrac{1}{2}I'_{,,}+X_{,}+X''-2,\quad o=2\frac{dX_{,}}{dI''}\operatorname{cotg}\tfrac{1}{2}I''+X_{,}+X'_{,,}-2\\
&o=\frac{dX_{,}}{dM_{,}}+\sqrt{4\sin^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\sin^2\tfrac{1}{2}I''-(X_{,,}'-2)^2}-\sqrt{4\sin^2\tfrac{1}{2}I_{,}\sin^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X''-2)^2}\\
&o=\frac{dX_{,}}{dL_{,,}'}+\sqrt{4\cos^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X_{,}+X_{,,}'+X''-4)^2}-\sqrt{4\sin^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\sin^2\tfrac{1}{2}I''-(X_{,,}'-2)^2}\\
&o=\frac{dX_{,}}{dL''}+\sqrt{4\cos^2\tfrac{1}{2}I_{,}\cos^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X_{,}+X_{,,}'+X''-4)^2}-\sqrt{4\sin^2\tfrac{1}{2}I_{,}\sin^2\tfrac{1}{2}I_{,,}'\cos^2\tfrac{1}{2}I''-(X''-2)^2}
\end{aligned}\right\}(29)$$

quarum tres priores subministrant

$$o=X_{,}-\operatorname{tg}\tfrac{1}{2}I_{,}\frac{dX_{,}}{dI_{,}}+\operatorname{cotg}\tfrac{1}{2}I'_{,,}\frac{dX_{,}}{dI_{,,}'}+\operatorname{cotg}\tfrac{1}{2}I''\frac{dX_{,}}{dI''}$$

quae aequatio differentialis inter differentialia partialia ipsius $X_{,}$ integrata prodit

$$X_{,}=\sin^2\tfrac{1}{2}I_{,}\,\varphi\left(\frac{\cos^2\frac{1}{2}I_{,,}'}{\sin^2\frac{1}{2}I_{,}},\ \frac{\cos^2\frac{1}{2}I''}{\sin^2\frac{1}{2}I_{,}}\right)$$

denotante φ arbitrariam quantitatum quibus praefixa est functionem, quae praeterea quantitates $M_{,}+\alpha_{,}t$, $L_{,,}'-\eta_{,,}'t$ atque $L''-\eta''t$ continere potest. Aequationum vero (29) tres posteriores integrabiles non sunt, unde concluditur $X_{,}$ inclinationes mutuas, et φ praeter quantitates indicatas constantes solummodo continere posse. Forma quidem simplicissima, quae ex praecedente ipsius $X_{,}$ valore emergit, est haec

$$X_{,} = A_{,} \sin^2 \tfrac{1}{2} I_{,} + A_{,,}' \cos^2 \tfrac{1}{2} I_{,,}' + A'' \cos^2 \tfrac{1}{2} I''$$

denotantibus $A_{,}$, $A_{,,}'$, A'' constantes. Substituta hac expressione nec non analogis expressionibus pro $X_{,,}'$ atque X'' in tres posteriores aequationes (29), postquam quotientes differentiales $\frac{dX_{,}}{dM_{,}}$, $\frac{dX_{,}}{dL'_{,,}}$, $\frac{dX_{,}}{dL''}$ cifrae aequati sunt: elicitur $A_{,} = A_{,,}' = A'' = 1$, unde denique

$$X_{,} = \sin^2 \tfrac{1}{2} I_{,} + \cos^2 \tfrac{1}{2} I_{,,}' + \cos^2 \tfrac{1}{2} I''$$

Quo ipsius $X_{,}$ valore nec non valoribus analogis ipsarum $X_{,,}'$ atque X'' in aequationibus praecedentibus substitutis, magnum nanciscimur aequationum numerum, quae tamen non nisi tribus a se invicem independentibus aequivalent. Electis quarta aequatione (27), differentia duarum primarum aequationum (28), et secundae tertiaeque aequationum (27) aggregato, habemus

$$(30)\ldots\ldots\begin{cases} o = 2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(L_{,}+L'_{,,}+L'') - 2 + \sin^2\tfrac{1}{2}I_{,} + \sin^2\tfrac{1}{2}I'_{,,} + \sin^2\tfrac{1}{2}I'' \\ o = \cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\sin(M''-M_{,}+L'_{,,}) - \sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\sin(M_{,}-M'_{,,}+L'') \\ o = \sin\tfrac{1}{2}I_{,} + \cos\tfrac{1}{2}I'_{,,}\sin\tfrac{1}{2}I''\cos(M''-M_{,}+L'_{,,}) + \sin\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(M_{,}-M'_{,,}+L'') \end{cases}$$

quae sunt integralia simplicissima aequationum conditionalium in huius articuli initio allatarum. Restitutis adiumento aequationum (17) $P_{,}$, $Q_{,}$, $P_{,,}'$, etc. in praecedentibus aequationibus, emergunt

$$(31)\ldots\ldots\; o = 8\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(L_{,}+L'_{,,}+L'') - 8 + P_{,}^2 + Q_{,}^2 + P_{,,}'^2 + Q_{,,}'^2 + P''^2 + Q''^2$$

$$\begin{aligned} o = {} & \{(P_{,}P'' + Q_{,}Q'')\sin L'_{,,} - (P_{,}Q'' - Q_{,}P'')\cos L'_{,,}\}\cos\tfrac{1}{2}I'_{,,} \\ & - \{(P_{,}P'_{,,} + Q_{,}Q'_{,,})\sin L'' + (P_{,}Q'_{,,} - Q_{,}P'_{,,})\cos L''\}\cos\tfrac{1}{2}I'' \\ o = {} & P_{,}^2 + Q_{,}^2 + \{(P_{,}P'' + Q_{,}Q'')\cos L'_{,,} + (P_{,}Q'' - Q_{,}P'')\sin L'_{,,}\}\cos\tfrac{1}{2}I'_{,,} \\ & + \{(P_{,}P'_{,,} + Q_{,}Q'_{,,})\cos L'' - (P_{,}Q'_{,,} - Q_{,}P'_{,,})\sin L''\}\cos\tfrac{1}{2}I'' \end{aligned}$$

Multiplicatis et harum aequationum secunda per $Q_{,}$ atque tertia per $P_{,}$, et secunda per $-P_{,}$ atque tertia per $Q_{,}$, emergunt, additis resp. productis, hae

$$(32)\ldots\ldots\begin{cases} o = P_{,} + \cos\tfrac{1}{2}I'_{,,}P''\cos L'_{,,} + \cos\tfrac{1}{2}I'_{,,}Q''\sin L'_{,,} + \cos\tfrac{1}{2}I''P'_{,,}\cos L'' - \cos\tfrac{1}{2}I''Q'_{,,}\sin L'' \\ o = Q_{,} + \cos\tfrac{1}{2}I'_{,,}Q''\cos L'_{,,} - \cos\tfrac{1}{2}I'_{,,}P''\sin L'_{,,} + \cos\tfrac{1}{2}I''Q'_{,,}\cos L'' + \cos\tfrac{1}{2}I''P'_{,,}\sin L'' \end{cases}$$

quae una cum aequatione (31) rigorosa et simplicissima aequationum conditionalium art. 6. integralia sunt.

Hae aequationes conditionales ad problema plurium corporum perfacile extenduntur. Ex forma enim earum art. 6. aequationum, e quibus per eliminationem ipsarum dp, dq, etc. aequationes conditionales inter differentialia ipsarum $P_{,}$, $Q_{,}$, $K_{,}$, etc. emergebant, perspicuum est, in problemate plurium corporum easdem aequationes conditionales locum habere et praeterea omnes, quae fieri possint, reliquas corporum m, m', etc. ternas combinationes similes aequationes conditionales suppeditaturas esse, quae tamen non omnes a se invicem independentes sint.

Quibus aequationibus integratis, eadem integralia (31) et (32), nec non similia ad reliquas corporum m, m', etc. ternas combinationes spectantia integralia habemus, quae etiam non omnia a se invicem independentia sunt, applicationem vero facillimam admittunt.

14.

Difficilis fortasse esset aequationum (24) et (25) integratio, si $\eta_{,}$, $\eta'_{,,}$ et η'' a se invicem independentes essent, sed perfacile perficitur, si hoc in usum vocatur

Theorema:

Quantaecunque sunt inclinationes mutuae, modo valores earum perturbati periodicae temporis functiones, limitibus quibusdam circumclusae, sive functiones temporis sint, quae terminis per tempus ipsum multiplicatis careant, rigorosae aequationes hae

$$\eta_{,}+\eta_{,,}'+\eta'' = o \text{ atque } k_{,}+k_{,,}'+k'' = o$$

semper locum habent.

Demonstratio

huius theorematis ut obtineatur, resumo aequationum (30) primam, quam ita exhibeo

$$2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I''\cos(L_{,}+L'_{,,}+L'') = -1+\cos^2\tfrac{1}{2}I_{,}+\cos^2\tfrac{1}{2}I'_{,,}+\cos^2\tfrac{1}{2}I'' \quad \text{......(33)}$$

Facto $\cos(L_{,}+L'_{,,}+L'') = 1$, haec aequatio abit in

$$2\cos\tfrac{1}{2}I_{,}\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I'' = -1+\cos^2\tfrac{1}{2}I_{,}+\cos^2\tfrac{1}{2}I'_{,,}+\cos^2\tfrac{1}{2}I''$$

quae facile reperitur esse perfectum quadratum, cuius radix est

aut $\cos\frac{1}{2}I_, = \cos\frac{1}{2}(I'_{,,} \pm I'')$, aut $\cos\frac{1}{2}I'_{,,} = \cos\frac{1}{2}(I'' \pm I_,)$, aut $\cos\frac{1}{2}I'' = \cos\frac{1}{2}(I_, \pm I'_{,,})$ quae aequationes in unica continentur aequatione hac

$$o = I_, \pm I'_{,,} \pm I''$$

ubi signa superiora et cum superioribus et cum inferioribus reliquis signis coniungere licet. Facto $\cos(L_, + L'_{,,} + L'') = -1$, emergit

$$-2\cos\tfrac{1}{2}I_,\cos\tfrac{1}{2}I'_{,,}\cos\tfrac{1}{2}I'' = -1 + \cos^2\tfrac{1}{2}I_, + \cos^2\tfrac{1}{2}I'_{,,} + \cos^2\tfrac{1}{2}I''$$

quae

aut $\cos\frac{1}{2}I_, = -\cos\frac{1}{2}(I'_{,,} \pm I'')$, aut $\cos\frac{1}{2}I'_{,,} = -\cos\frac{1}{2}(I'' \pm I_,)$, aut $\cos\frac{1}{2}I'' = -\cos\frac{1}{2}(I_, \pm I'_{,,})$ suppeditat, quae in unica continentur aequatione hac

$$180° = \tfrac{1}{2}I_, + \tfrac{1}{2}I'_{,,} + \tfrac{1}{2}I''$$

Hinc sequitur, $\cos(L_, + L'_{,,} + L'')$ inter valores $+1$ atque -1 oscillare, si per virium perturbantium effectus inclinationes mutuae omnes, qui fieri possint, valores deinceps accipiant, itaque arcum $L_, + L'_{,,} + L''$ totam peripheriam perlustrare. Sin autem inclinationes mutuae functiones temporis sunt, quae, limitibus quibusdam circumclusae, omnes, qui fieri possunt, valores accipere nequeunt, eaedem aequationes monstrant, cosinum arcus $L_, + L'_{,,} + L''$ inter valores maximos et minimos a $+1$ atque -1 diversos oscillare, unde sequitur arcum $L_, + L'_{,,} + L''$ totam peripheriam percurrere non posse, sed inter limites quosdam etiam oscillare debere. Ergo quum

$$L_, + L_{,,}' + L'' = K_, + K'_{,,} + K'' - (\eta_, + \eta_{,,}' + \eta'')t$$

et $\eta_,$, $\eta_{,,}'$ atque η'' ita determinentur, ut in ipsis $K_,$, $K_{,,}'$ atque K'' termini per tempus ipsum multiplicati desint, necessario esse debet

$$\eta_, + \eta_{,,}' + \eta'' = o$$

Q. E. D.

Praeterea quum indoles quantitatum $K_,$, $K_{,,}'$ atque K'' per aequationes (16) explicata ex inclinationibus mutuis non pendeat, constantes arbitrariae $k_,$, $k_{,,}'$ atque k'' integralibus $\int dK_,$, $\int dK'_{,,}$ atque $\int dK''$ addendae etiam ab inclinationibus mutuis independentes esse debent; positis vero $I_, = I_{,,}' = I'' = o$ in aequatione (33) sequitur $L_, + L_{,,}' + L'' = o$, unde necessario esse debet

$$k_, + k_{,,}' + k'' = o$$

Q. E. D.

Quum quatuor res quatuor admittant ternas combinationes, secundum ea, quae in fine articuli praecedentis protulimus, in problemate quinque corporum habemus aequationes has

$$\eta_{,}+\eta'_{,,}+\eta''=o\ ;\quad \eta_{,,}+\eta''_{,,,}+\eta'''=o$$
$$\eta_{,}+\eta'_{,,,}+\eta'''=o\ ;\quad \eta'_{,,}+\eta''_{,,,}+\eta'''_{,}=o$$

et quatuor similes inter quantitates k appellatas, utrumque vero harum aequationum systema non nisi tribus a se invicem independentibus aequationibus aequivalet. Prima enim aequatione ad tertiam addita, subductaque secunda et quarta, aequationem identicam nancisceris.

15.

Iam ad aequationes (24) et (25) integrandas suppono inclinationes terminis per tempus ipsum multiplicatis carere, unde aequationes locum habent hae

$$\eta_{,}+\eta'_{,,}+\eta''=o\ ,\quad k_{,}+k'_{,,}+k''=o$$

Si, integrationibus peractis, tales inclinationum mutuarum valores evadunt, haec suppositio legitima est. Harum nec non aequationum $\eta_{,}=-\eta'$, etc., $k_{,}=-k'$, etc. adiumento aequationes (24) earumque similes abeunt in has

$$\left.\begin{aligned}
\frac{d\Upsilon}{dt} &= (y-\mu_{,}-\mu_{,,})\,\Psi + \sigma_{,}(\Upsilon'+2e')\sin(h_{,}t+2k_{,}) - \sigma_{,}\Psi'\cos(h_{,}t+2k_{,})\\
&\quad -\sigma_{,,}(\Upsilon''+2e'')\sin(h''t+2k'') - \sigma_{,,}\Psi''\cos(h''t+2k'')\\
\frac{d\Psi}{dt} &= -(y-\mu_{,}-\mu_{,,})(\Upsilon+2e) + \sigma_{,}(\Upsilon'+2e')\cos(h_{,}t+2k_{,}) + \sigma_{,}\Psi'\sin(h_{,}t+2k_{,})\\
&\quad +\sigma_{,,}(\Upsilon''+2e'')\cos(h''t+2k'') - \sigma_{,,}\Psi''\sin(h''t+2k'')\\
\frac{d\Upsilon'}{dt} &= (y'-\mu'_{,,}-\mu')\,\Psi' + \sigma'_{,,}(\Upsilon''+2e'')\sin(h'_{,,}t+2k'_{,,}) - \sigma'_{,,}\Psi''\cos(h'_{,,}t+2k'_{,,})\\
&\quad -\sigma'(\Upsilon+2e)\sin(h_{,}t+2k_{,}) - \sigma'\Psi\cos(h_{,}t+2k_{,})\\
\frac{d\Psi'}{dt} &= -(y'-\mu'_{,,}-\mu')(\Upsilon'+2e') + \sigma'_{,,}(\Upsilon''+2e'')\cos(h'_{,,}t+2k'_{,,}) + \sigma'_{,,}\Psi''\sin(h'_{,,}t+2k'_{,,})\\
&\quad +\sigma'(\Upsilon+2e)\cos(h_{,}t+2k_{,}) - \sigma'\Psi\sin(h_{,}t+2k_{,})\\
\frac{d\Upsilon''}{dt} &= (y''-\mu''-\mu''_{,})\,\Psi'' + \sigma''(\Upsilon+2e)\sin(h''t+2k'') - \sigma''\Psi\cos(h''t+2k'')\\
&\quad -\sigma''_{,}(\Upsilon'+2e')\sin(h'_{,,}t+2k'_{,,}) - \sigma''_{,}\Psi'\cos(h'_{,,}t+2k'_{,,})\\
\frac{d\Psi''}{dt} &= -(y''-\mu''-\mu''_{,})(\Upsilon''+2e'') + \sigma''(\Upsilon+2e)\cos(h''t+2k'') + \sigma''\Psi\sin(h''t+2k'')\\
&\quad +\sigma''_{,}(\Upsilon'+2e')\cos(h'_{,,}t+2k'_{,,}) - \sigma''_{,}\Psi'\sin(h'_{,,}t+2k'_{,,})
\end{aligned}\right\}\ldots\ldots(34)$$

ubi brevitatis caussa scripsi

$$
(35)\ldots\ldots\left\{\begin{aligned}
y-y'-2\eta_{,} &= h_{,}\\
y'-y''-2\eta'_{,,} &= h'_{,,}\\
y''-y-2\eta'' &= h''
\end{aligned}\right.
$$

e quibus prodit $h_{,}+h'_{,,}+h''=o$.

Quibus positis, facili substitutione manifestatur, his aequationibus differentialibus valores variabilium satisfacere hos

$$
(36)\ldots\ldots\left\{\begin{aligned}
\Upsilon+2e &= 2e\xi\cos D+2e's\cos(h_{,}t+2k_{,}-D')+2e''f\cos(h''t+2k''+D'')\\
\Psi &= 2e\xi\sin D-2e's\sin(h_{,}t+2k_{,}-D')+2e''f\sin(h''t+2k''+D'')\\
\Upsilon'+2e' &= 2e'\xi'\cos D'+2e''s'\cos(h'_{,,}t+2k'_{,,}-D'')+2ef'\cos(h_{,}t+2k_{,}+D)\\
\Psi' &= 2e'\xi'\sin D'-2e''s'\sin(h'_{,,}t+2k'_{,,}-D'')+2ef'\sin(h_{,}t+2k_{,}+D)\\
\Upsilon''+2e'' &= 2e''\xi''\cos D''+2es''\cos(h''t+2k''-D)+2e'f''\cos(h'_{,,}t+2k'_{,,}+D')\\
\Psi'' &= 2e''\xi''\sin D''-2es''\sin(h''t+2k''-D)+2e'f''\sin(h'_{,,}t+2k'_{,,}+D')
\end{aligned}\right.
$$

si coefficientes ξ, s, f, etc. et y, y' atque y'' ex aequationibus determinantur his

$$
(37)\ldots\ldots\left\{\begin{aligned}
o &= (y-\mu_{,}-\mu_{,,})\xi-\sigma_{,,}s''-\sigma_{,}f'\\
o &= (y-2\eta_{,}-\mu'_{,,}-\mu')f'-\sigma_{,}\xi-\sigma'_{,,}s''\\
o &= (y+2\eta''-\mu''-\mu''_{,})s''-\sigma''_{,}f'-\sigma''\xi\\
o &= (y'-\mu'_{,,}-\mu')\xi'-\sigma's-\sigma'_{,,}f''\\
o &= (y'-2\eta'_{,,}-\mu''-\mu''_{,})f''-\sigma''_{,}\xi'-\sigma''s\\
o &= (y'+2\eta_{,}-\mu_{,}-\mu_{,,})s-\sigma_{,,}f''-\sigma_{,}\xi'\\
o &= (y''-\mu''-\mu''_{,})\xi''-\sigma''_{,}s'-\sigma''f\\
o &= (y''-2\eta''-\mu_{,}-\mu_{,,})f-\sigma_{,,}\xi''-\sigma_{,}s'\\
o &= (y''+2\eta'_{,,}-\mu'_{,,}-\mu')s'-\sigma'f-\sigma'_{,,}\xi''
\end{aligned}\right.
$$

Constantes sex e, e', e'', D, D' atque D'' arbitrariae remanent, unde aequationes praecedentes integra aequationum nostrarum differentialium integralia sunt. Constantes vero D, D' atque D'' cifrae aequales ponere nobis licet, quia constantibus, quae periheliorum longitudines denotant, se aggregant. Habemus igitur

$$\left.\begin{array}{l}
\Upsilon = 2e(\xi-1)+2e's\cos(h_{,}t+2k_{,})+2e''f\cos(h''t+2k'') \\
\Psi = \qquad\qquad -2e's\sin(h_{,}t+2k_{,})+2e''f\sin(h''t+2k'') \\
\Upsilon' = 2e'(\xi'-1)+2e''s'\cos(h'_{,,}t+2k'_{,,})+2ef'\cos(h_{,}t+2k_{,}) \\
\Psi' = \qquad\qquad -2e''s'\sin(h'_{,,}t+2k'_{,,})+2ef'\sin(h_{,}t+2k_{,}) \\
\Upsilon'' = 2e''(\xi''-1)+2es''\cos(h''t+2k'')+2e'f''\cos(h'_{,,}t+2k'_{,,}) \\
\Psi'' = \qquad\qquad -2es''\sin(h''t+2k'')+2e'f''\sin(h'_{,,}t+2k'_{,,})
\end{array}\right\}\ldots\ldots(38)$$

aequationibus tamen (36) infra utemur.

Eliminatis s'' atque f' inter tres priores aequationes (37), ξ sua sponte evanescit et aequatio tertii gradus pro y emergit; eliminatis porro s atque f'' inter tres medias aequationes (37), ξ' sua sponte evanescit, et aequatio tertii gradus pro y', alio atque illa modo composita, elicitur; eliminatis denique f atque s' inter tres posteriores aequationes (37), ξ'' sua sponte evanescit, et aequationem tertii gradus pro y'' aliter quam illas compositam nanciscimur, unde quantitates cunctae y, y' atque y'' novem valores habere videntur, qui quidem loco trium in expressionibus ipsarum Υ, Ψ, etc. introductorum angulorum novem angulos efficerent. Hanc vero conditionem secus se habere ita demonstratur. Sint x, x' atque x'' quantitates novae aequationibus determinandae his

$$\left.\begin{array}{l}
y = x - 2\eta'' \\
y' = x' + 2\eta'_{,,} \\
y'' = x''
\end{array}\right\}\ldots\ldots(39)$$

quibus in aequationibus (37) substitutis, emergunt adiumento theorematis art. 14. hae

$$\left.\begin{array}{l}
o = (x-2\eta''-\mu_{,}-\mu_{,,})\xi-\sigma_{,,}s''-\sigma_{,}f' \\
o = (x+2\eta'_{,,}-\mu'_{,,}-\mu')f'-\sigma'\xi-\sigma'_{,,}s'' \\
o = (x \qquad\quad -\mu''-\mu''_{,})s''-\sigma''_{,}f'-\sigma''\xi \\
o = (x'+2\eta'_{,,}-\mu'_{,,}-\mu')\xi'-\sigma's-\sigma'_{,,}f'' \\
o = (x' \qquad\quad -\mu''-\mu''_{,})f''-\sigma''_{,}\xi'-\sigma''s \\
o = (x'-2\eta''-\mu_{,}-\mu_{,,})s-\sigma_{,,}f''-\sigma_{,}\xi' \\
o = (x'' \qquad\quad -\mu''-\mu''_{,})\xi''-\sigma''_{,}s'-\sigma''f \\
o = (x''-2\eta''-\mu_{,}-\mu_{,,})f-\sigma_{,,}\xi''-\sigma_{,}s' \\
o = (x''+2\eta'_{,,}-\mu'_{,,}-\mu')s'-\sigma'f-\sigma'_{,,}\xi''
\end{array}\right\}\ldots\ldots(40)$$

Ex his aequationibus primo oculorum obtutu patet, tres priores post s'' atque f'' eliminatas aequationem tertii gradus pro x, tres medias eandem pro x' et denique tres posteriores eandem pro x'' suppedituras esse, unde concluditur x, x' atque x'' esse radices tres unius aequationis tertii gradus, itaque ipsas Υ, Ψ, Υ', etc. tres tantum angulos diversos continere. Substitutis enim aequationibus (39) in (35), nanciscimur

$$h_{,} = x - x', \quad h'_{,,} = x' - x'', \quad h'' = x'' - x$$

Quantitates $\mu_{,}$, $\mu_{,,}$, $\mu'_{,,}$, etc. atque $\sigma_{,}$, $\sigma_{,,}$, $\sigma'_{,,}$, etc. adiumento valorum massarum m, m', m'' et semiaxium maiorum a, a', a'', qui secundum tertiam legem Kepplerianam ex motibus mediis computari possunt, computantur; quomodo autem $\eta_{,}$, $\eta'_{,,}$, η'' computandae sint, infra monstrabitur.

Excentricitates e, e', e'' observationibus determinandae sunt, quantitates vero ξ, ξ', ξ'', quae prorsus indeterminatae remanent, ad lubitum determinari possunt. Facile tamen perspicitur, sive hunc sive illum valorem ipsis ξ, ξ', ξ'' attribueris, effici, ut observationes alios excentricitatum valores prodituræ sint. Quum ξ, ξ', ξ'' arbitrariae sint, $\xi = \xi' = \xi'' = 1$ statuere optimo iure liceret, unde termini $2e(\xi - 1)$, $2e'(\xi' - 1)$, $2e''(\xi'' - 1)$ ex ipsarum Υ, Υ', Υ'' expressionibus evanescerent, sed rei accommodatius est, quantitates ξ, ξ', ξ'' ita determinare, ut longitudines mediae perturbatae nz, $n'z'$, $n''z''$ terminos formae $A \sin g$, $A' \sin g'$, $A'' \sin g''$ non contineant, sicut iam in praecedentibus in Lunae theoria fecimus. Hinc factum est, ut ξ, ξ', ξ'' sint quantitates, quae ab unitate quantitatibus ordinis massarum perturbantium tantum diversae sunt, et in prima approximatione unitati aequales poni possunt.

Facile denique perspicitur $e(\xi - 1)$ identicam esse cum quantitate ξ in art. 13. Sect. II. in Lunae motu introducta.

16.

Adiumento aequationum per theorema art. 14. suppeditatarum et aequationum conditionalium harum

$$\alpha_{,} = \alpha', \text{ etc.} \quad \eta_{,} = -\eta', \text{ etc.} \quad v_{,} - v' = 180°, \text{ etc.} \quad k_{,} = -k', \text{ etc.}$$

aequationes (25) et earum similes transeunt in has

$$\left.\begin{aligned}
\frac{dP_{\prime}}{dt} &= -(\alpha_{\prime}-\pi_{\prime}-\pi')Q_{\prime}-\pi'_{\prime\prime}Q'_{\prime\prime}\cos(l''t+\lambda'')-\pi'_{\prime\prime}P'_{\prime\prime}\sin(l''t+\lambda'')-\pi_{\prime\prime}Q''\cos(l'_{\prime\prime}t+\lambda'_{\prime\prime})+\pi_{\prime\prime}P''\sin(l'_{\prime\prime}t+\lambda'_{\prime\prime})\\
\frac{dQ_{\prime}}{dt} &= (\alpha_{\prime}-\pi_{\prime}-\pi')P_{\prime}-\pi'_{\prime\prime}Q'_{\prime\prime}\sin(l''t+\lambda'')+\pi'_{\prime\prime}P'_{\prime\prime}\cos(l''t+\lambda'')+\pi_{\prime\prime}Q''\sin(l'_{\prime\prime}t+\lambda'_{\prime\prime})+\pi_{\prime\prime}P''\cos(l'_{\prime\prime}t+\lambda'_{\prime\prime})\\
\frac{dP'_{\prime\prime}}{dt} &= -(\alpha'_{\prime\prime}-\pi'_{\prime\prime}-\pi''_{\prime})Q'_{\prime\prime}-\pi''Q''\cos(l_{\prime}t+\lambda_{\prime})-\pi''P''\sin(l_{\prime}t+\lambda_{\prime})-\pi'Q_{\prime}\cos(l''t+\lambda'')+\pi'P_{\prime}\sin(l''t+\lambda'')\\
\frac{dQ'_{\prime\prime}}{dt} &= (\alpha'_{\prime\prime}-\pi'_{\prime\prime}-\pi''_{\prime})P'_{\prime\prime}-\pi''Q''\sin(l_{\prime}t+\lambda_{\prime})+\pi''P''\cos(l_{\prime}t+\lambda_{\prime})+\pi'Q_{\prime}\sin(l''t+\lambda'')+\pi'P_{\prime}\cos(l''t+\lambda'')\\
\frac{dP''}{dt} &= -(\alpha''-\pi''-\pi_{\prime\prime})Q''-\pi_{\prime}Q_{\prime}\cos(l'_{\prime\prime}t+\lambda'_{\prime\prime})-\pi_{\prime}P_{\prime}\sin(l'_{\prime\prime}t+\lambda'_{\prime\prime})-\pi''_{\prime}Q'_{\prime\prime}\cos(l_{\prime}t+\lambda_{\prime})+\pi''_{\prime}P'_{\prime\prime}\sin(l_{\prime}t+\lambda_{\prime})\\
\frac{dQ''}{dt} &= (\alpha''-\pi''-\pi_{\prime\prime})P''-\pi_{\prime}Q_{\prime}\sin(l'_{\prime\prime}t+\lambda'_{\prime\prime})+\pi_{\prime}P_{\prime}\cos(l'_{\prime\prime}t+\lambda'_{\prime\prime})+\pi''_{\prime}Q'_{\prime\prime}\sin(l_{\prime}t+\lambda_{\prime})+\pi''_{\prime}P'_{\prime\prime}\cos(l_{\prime}t+\lambda_{\prime})
\end{aligned}\right\}\text{(41)}$$

ubi brevitatis caussa feci

$$\left.\begin{aligned}
\alpha'_{\prime\prime}-\alpha''-\eta_{\prime} &= l_{\prime}\,, & \nu'_{\prime\prime}-\nu''+k_{\prime} &= \lambda_{\prime}\\
\alpha''-\alpha_{\prime}-\eta'_{\prime\prime} &= l'_{\prime\prime}\,, & \nu''-\nu_{\prime}+k'_{\prime\prime} &= \lambda'_{\prime\prime}\\
\alpha_{\prime}-\alpha'_{\prime\prime}-\eta'' &= l''\,, & \nu_{\prime}-\nu'_{\prime\prime}+k'' &= \lambda''
\end{aligned}\right\}\text{......(42)}$$

e quibus prodeunt $l_{\prime}+l'_{\prime\prime}+l''=o\,,\quad \lambda_{\prime}+\lambda'_{\prime\prime}+\lambda''=o$

Iam facili substitutione facta, reperitur his aequationibus differentialibus satisfacere valores variabilium hos

$$\left.\begin{aligned}
P_{\prime} &= C_{\prime}\sin D_{\prime}-b_{\prime}C'_{\prime\prime}\sin(l''t+\lambda''-D'_{\prime\prime})+f_{\prime}C''\sin(l'_{\prime\prime}t+\lambda'_{\prime\prime}+D'')\\
Q_{\prime} &= C_{\prime}\cos D_{\prime}+b_{\prime}C'_{\prime\prime}\cos(l''t+\lambda''-D'_{\prime\prime})+f_{\prime}C''\cos(l'_{\prime\prime}t+\lambda'_{\prime\prime}+D'')\\
P'_{\prime\prime} &= C'_{\prime\prime}\sin D'_{\prime\prime}-b'_{\prime\prime}C''\sin(l_{\prime}t+\lambda_{\prime}-D'')+f'_{\prime\prime}C_{\prime}\sin(l''t+\lambda''+D_{\prime})\\
Q'_{\prime\prime} &= C'_{\prime\prime}\cos D'_{\prime\prime}+b'_{\prime\prime}C''\cos(l_{\prime}t+\lambda_{\prime}-D'')+f'_{\prime\prime}C_{\prime}\cos(l''t+\lambda''+D_{\prime})\\
P'' &= C''\sin D''-b''C_{\prime}\sin(l'_{\prime\prime}t+\lambda'_{\prime\prime}-D_{\prime})+f''C'_{\prime\prime}\sin(l_{\prime}t+\lambda_{\prime}+D'_{\prime\prime})\\
Q'' &= C''\cos D''+b''C_{\prime}\cos(l'_{\prime\prime}t+\lambda'_{\prime\prime}-D_{\prime})+f''C'_{\prime\prime}\cos(l_{\prime}t+\lambda_{\prime}+D'_{\prime\prime})
\end{aligned}\right\}\text{......(43)}$$

si coefficientes $b_{\prime}$, $f_{\prime}$, $b'_{\prime\prime}$, etc. atque $\alpha_{\prime}$, $\alpha'_{\prime\prime}$, α'' ex aequationibus computantur his

$$\left.\begin{aligned}
o &= \alpha_{\prime}-\pi_{\prime}-\pi'+\pi'_{\prime\prime}f'_{\prime\prime}+\pi_{\prime\prime}b''\\
o &= (\alpha_{\prime}-\eta''-\pi'_{\prime\prime}-\pi''_{\prime})f'_{\prime\prime}+\pi''b''+\pi'\\
o &= (\alpha_{\prime}+\eta'_{\prime\prime}-\pi''-\pi_{\prime\prime})b''+\pi_{\prime}+\pi''_{\prime}f'_{\prime\prime}\\
o &= \alpha'_{\prime\prime}-\pi'_{\prime\prime}-\pi''_{\prime}+\pi''f''+\pi'b_{\prime}\\
o &= (\alpha'_{\prime\prime}-\eta_{\prime}-\pi''-\pi_{\prime\prime})f''+\pi_{\prime}b_{\prime}+\pi''_{\prime}\\
o &= (\alpha'_{\prime\prime}+\eta''-\pi_{\prime}-\pi')b_{\prime}+\pi'_{\prime\prime}+\pi_{\prime\prime}f''\\
o &= \alpha''-\pi''-\pi_{\prime\prime}+\pi_{\prime}f_{\prime}+\pi''_{\prime}b'_{\prime\prime}\\
o &= (\alpha''-\eta'_{\prime\prime}-\pi_{\prime}-\pi')f_{\prime}+\pi'_{\prime\prime}b'_{\prime\prime}+\pi_{\prime\prime}\\
o &= (\alpha''+\eta_{\prime}-\pi'_{\prime\prime}-\pi''_{\prime})b'_{\prime\prime}+\pi''+\pi'f_{\prime}
\end{aligned}\right\}\text{......(44)}$$

Constantes sex $C_{,}$, $C'_{,,}$, C'', $D_{,}$, $D'_{,,}$, D'' arbitrariae remanent, unde aequationes praecedentes integra aequationum nostrarum differentialium integralia sunt. Constantes vero $D_{,}$, $D'_{,,}$, D'', quae constantibus $\nu_{,}$, $\nu'_{,,}$, ν'' se aggregant, cifrae aequales ponere nobis licet, unde

$$(45)\ldots\left\{\begin{aligned}
P_{,} &= \phantom{C_{,}} -b_{,}C'_{,,}\sin(l''t+\lambda'')+f_{,}C''\sin(l'_{,,}t+\lambda'_{,,})\\
Q_{,} &= C_{,}+b_{,}C'_{,,}\cos(l''t+\lambda'')+f_{,}C''\cos(l'_{,,}t+\lambda'_{,,})\\
P'_{,,} &= \phantom{C_{,}} -b'_{,,}C''\sin(l_{,}t+\lambda_{,})+f'_{,,}C_{,}\sin(l''t+\lambda'')\\
Q'_{,,} &= C'_{,,}+b'_{,,}C''\cos(l_{,}t+\lambda_{,})+f'_{,,}C_{,}\cos(l''t+\lambda'')\\
P'' &= \phantom{C_{,}} -b''C_{,}\sin(l'_{,,}t+\lambda'_{,,})+f''C'_{,,}\sin(l_{,}t+\lambda_{,})\\
Q'' &= C''+b''C_{,}\cos(l'_{,,}t+\lambda'_{,,})+f''C'_{,,}\cos(l_{,}t+\lambda_{,})
\end{aligned}\right.$$

aequationibus tamen (43) infra utemur. Quum in his ipsarum $P_{,}$, $Q_{,}$, $P'_{,,}$, etc. valoribus termini per tempus ipsum multiplicati desint, manifestum est, theorema art. 14. ad minimum usque ad secundam approximationem revera locum habere.

Eadem ratione qua in art. praec. inductus hoc loco pono

$$\begin{aligned}
\alpha_{,} &= \varepsilon_{,}-\eta'_{,,}\\
\alpha'_{,,} &= \varepsilon'_{,,}+\eta_{,}\\
\alpha'' &= \varepsilon''
\end{aligned}$$

quarum ope ex aequationibus (44) elicitur

$$(46)\ldots\left\{\begin{aligned}
o &= \varepsilon_{,}-\eta'_{,,}-\pi_{,}-\pi' \quad +\pi'_{,,}f'_{,,}+\pi_{,,}b''\\
o &= (\varepsilon_{,}+\eta_{,}-\pi'_{,,}-\pi''_{,})f'_{,,}+\pi''b''+\pi'\\
o &= (\varepsilon_{,} \quad -\pi''-\pi_{,,})b''+\pi_{,} \quad +\pi''_{,}f'_{,,}\\
o &= \varepsilon'_{,,}+\eta_{,}-\pi'_{,,}-\pi''_{,} \quad +\pi''f''+\pi'b_{,}\\
o &= (\varepsilon'_{,,} \quad -\pi''-\pi_{,,})f''+\pi_{,}b_{,}+\pi''_{,}\\
o &= (\varepsilon'_{,,}-\eta'_{,,}-\pi_{,}-\pi')b_{,}+\pi'_{,,} \quad +\pi_{,,}f''\\
o &= \varepsilon'' \quad -\pi''-\pi_{,,} \quad +\pi_{,}f_{,}+\pi''_{,}b'_{,,}\\
o &= (\varepsilon''-\eta'_{,,}-\pi_{,}-\pi')f_{,}+\pi'_{,,}b'_{,,}+\pi_{,,}\\
o &= (\varepsilon''+\eta_{,}-\pi'_{,,}-\pi''_{,})b'_{,,}+\pi'' \quad +\pi'f_{,}
\end{aligned}\right.$$

e quibus perspicuum est, $\varepsilon_{,}$, $\varepsilon'_{,,}$, ε'' esse diversas **unius** aequationis tertii gradus radices. Substitutis valoribus ipsarum $\alpha_{,}$, etc. per $\varepsilon_{,}$, etc. expressis in (42), emergunt

$$l_{,}=\varepsilon'_{,,}-\varepsilon'',\quad l'_{,,}=\varepsilon''-\varepsilon_{,},\quad l''=\varepsilon_{,}-\varepsilon'_{,,}.$$

quibus valores ipsarum $P_{,}$, $Q_{,}$, $P'_{,,}$ etc. functiones ipsarum $\varepsilon_{,}$, $\varepsilon'_{,,}$, ε'' reddi possunt. Positis in tribus ultimis aequationum (46) et $\varepsilon'' = o$ et $\eta_{,} = \eta'_{,,} = \eta'' = o$, habetur

$$\begin{aligned}
o &= -(\pi'' + \pi_{,,}) + \pi_{,} f_{,} + \pi''_{,} b'_{,,} \\
o &= -(\pi_{,} + \pi') f_{,} + \pi'_{,,} b'_{,,} + \pi_{,,} \\
o &= -(\pi'_{,,} + \pi''_{,}) b'_{,,} + \pi'' + \pi' f_{,}
\end{aligned}$$

quibus additis, nanciscimur

$$o = -(\pi'' + \pi_{,,}) + \pi_{,,} + \pi'' + [\pi_{,} + \pi' - (\pi_{,} + \pi')] f_{,} + [\pi''_{,} + \pi'_{,,} - (\pi'_{,,} + \pi''_{,})] b'_{,,}$$

quae est aequatio identica. Hinc sequitur unam radicum $\varepsilon_{,}$, $\varepsilon'_{,,}$, ε'' cifrae aequalem fore, si $\eta_{,}$, $\eta'_{,,}$, η'' essent cifrae aequales. Itaque quum hae quantitates cifrae non sint aequales, una radicum $\varepsilon_{,}$, $\varepsilon'_{,,}$, ε'' eiusdem ordinis esse debet ac quantitates $\eta_{,}$, $\eta'_{,,}$, η''.

Aequationes memoratu dignas praeterea appono has

$$\left.\begin{aligned}
o &= 1 - \frac{\eta'_{,,}}{\varepsilon_{,}} + \left(1 + \frac{\eta_{,}}{\varepsilon_{,}}\right) f'_{,,} + b'' \\
o &= 1 + \frac{\eta_{,}}{\varepsilon'_{,,}} + f'' + \left(1 - \frac{\eta'_{,,}}{\varepsilon'_{,,}}\right) b_{,} \\
o &= 1 + \left(1 - \frac{\eta'_{,,}}{\varepsilon''}\right) f_{,} + \left(1 + \frac{\eta_{,}}{\varepsilon''}\right) b'_{,,}
\end{aligned}\right\} \ldots\ldots(47)$$

quae facili opera ex (46) derivantur. Quum quantitates $\eta_{,}$, $\eta'_{,,}$, η'', uti mox patebit, sint ordinis secundi respectu ipsarum $C_{,}$, $C'_{,,}$, C'', et duae radicum $\varepsilon_{,}$, $\varepsilon'_{,,}$, ε'' ordinis o^{ti} respectu harum quantitatum, tertia vero radix eiusdem ordinis quam illae, aequationes praecedentes monstrant duas quantitatum $1 + b'' + f'_{,,}$, $1 + b_{,} + f''$, $1 + b'_{,,} + f_{,}$ secundi ordinis, reliquam vero finitam esse, et si ε'' sit radix eiusdem ordinis quam $\eta_{,}$, $\eta'_{,,}$, η'', $1 + b'_{,,} + f_{,}$ esse hanc quantitatem finitam.

Superest ut expressiones ipsarum $\eta_{,}$, $\eta'_{,,}$, η'' investigentur. Quem in finem, substitutis in aequatione pro ipsa $\frac{dK_{,}}{dt}$ art. 7. valoribus quotientium differentialium ipsius Ω respectu $P_{,}$, $Q_{,}$, $P'_{,,}$, etc. in art. 12. inventis, nanciscimur

$$\begin{aligned}
\frac{dK_{,}}{dt} = \eta_{,} &- \tfrac{1}{4}(\pi_{,} - \pi')(P_{,}^2 + Q_{,}^2) + \tfrac{1}{4}\pi_{,,}(P_{,,}P_{,} + Q_{,,}Q_{,})\cos(l'_{,,}t + \lambda'_{,,}) - \tfrac{1}{4}\pi_{,,}(P_{,,}Q_{,} - Q_{,,}P_{,})\sin(l'_{,,}t + \lambda'_{,,}) \\
&- \tfrac{1}{4}\pi'_{,,}(P'_{,,}P_{,} + Q'_{,,}Q_{,})\cos(l''t + \lambda'') - \tfrac{1}{4}\pi'_{,,}(P'_{,,}Q_{,} - Q'_{,,}P_{,})\sin(l''t + \lambda'')
\end{aligned}$$

cui similes sunt aequationes pro $\frac{dK'_{\prime\prime}}{dt}$ atque $\frac{dK''}{dt}$. Substitutis in his aequationibus valoribus ipsarum $P_{\prime}$, $Q_{\prime}$, $P'_{\prime\prime}$, etc. ex (45) desumendis, retentisque solummodo terminis constantibus, adipiscimur

$$\eta_{\prime} = \tfrac{1}{4}C_{\prime}^2 \{\pi_{\prime} - \pi' - \pi_{\prime\prime} b'' + \pi_{\prime\prime}' f_{\prime\prime}'\} + \tfrac{1}{4}C_{\prime\prime}'^2 \{(\pi_{\prime} - \pi') b_{\prime} - \pi_{\prime\prime} f'' + \pi'_{\prime\prime}\} b_{\prime} + \tfrac{1}{4}C''^2 \{(\pi_{\prime} - \pi') f_{\prime} - \pi_{\prime\prime} + \pi'_{\prime\prime} b'_{\prime\prime}\} f_{\prime}$$

$$\eta'_{\prime\prime} = \tfrac{1}{4}C_{\prime\prime}'^2 \{\pi'_{\prime\prime} - \pi_{\prime}'' - \pi' b_{\prime} + \pi'' f''\} + \tfrac{1}{4}C''^2 \{(\pi'_{\prime\prime} - \pi_{\prime}'') b'_{\prime\prime} - \pi' f_{\prime} + \pi''\} b'_{\prime\prime} + \tfrac{1}{4}C_{\prime}^2 \{(\pi'_{\prime\prime} - \pi_{\prime}'') f'_{\prime\prime} - \pi' + \pi'' b''\} f'_{\prime\prime}$$

$$\eta'' = \tfrac{1}{4}C''^2 \{\pi'' - \pi_{\prime\prime} - \pi_{\prime}'' b'_{\prime\prime} + \pi_{\prime} f_{\prime}\} + \tfrac{1}{4}C_{\prime}^2 \{(\pi'' - \pi_{\prime\prime}) b'' - \pi_{\prime}'' f'_{\prime\prime} + \pi_{\prime}\} b'' + \tfrac{1}{4}C_{\prime\prime}'^2 \{(\pi'' - \pi_{\prime\prime}) f'' - \pi_{\prime}'' + \pi_{\prime} b_{\prime}\} f''$$

Iam quantitates x, x', x'' atque $\varepsilon_{\prime}$, $\varepsilon'_{\prime\prime}$, ε'' hoc modo eliciendae sunt. Primum ex aequationibus (40) et (46), dum $\eta_{\prime}$, $\eta'_{\prime\prime}$, η'' negligantur, x, x', x'' atque $\varepsilon_{\prime}$, $\varepsilon'_{\prime\prime}$, ε'' nec non coefficientes $b_{\prime}$, $f_{\prime}$, $b'_{\prime\prime}$, etc., computentur, et valores earum hoc modo eruti una cum observationibus astronomicis ad approximatos constantium $C_{\prime}$, $C'_{\prime\prime}$, C'' valores eliciendos adhibeantur, e quibus adiumento praecedentium aequationum iam ipsarum $\eta_{\prime}$, $\eta'_{\prime\prime}$, η'' magis approximati valores computentur. Deinde his valoribus substitutis, aequationes (40) et (46) accuratiores ipsarum x, x', x'' atque $\varepsilon_{\prime}$, $\varepsilon'_{\prime\prime}$, ε'' valores suppeditant, et sic porro.

17.

Aequationes (31) et (32), quae relationes inter quantitates $P_{\prime}$, $Q_{\prime}$, $K_{\prime}$, $P'_{\prime\prime}$, etc. suppeditabant, relationibus inter constantes arbitrarias $C_{\prime}$, $v_{\prime}$, $k_{\prime}$, $C'_{\prime\prime}$ etc. inveniendis inserviunt. Iam ex aequatione (31) emersit simplicissima inter constantes $k_{\prime}$, $k'_{\prime\prime}$ atque k'' relatio, aequationes igitur (32) reliquas duas aequationes conditionales, quae inter constantes sex $C_{\prime}$, $C'_{\prime\prime}$, C'', $v_{\prime}$, $v'_{\prime\prime}$ atque v'' locum habere debent, suppeditabunt. Ex aequationibus (32) mutatis mutandis emergunt

$$(48)\ldots\ldots\begin{cases} o = P_{\prime\prime}' + \cos\tfrac{1}{2}I'' P_{\prime} \cos L'' + \cos\tfrac{1}{2}I'' Q_{\prime} \sin L'' + \cos\tfrac{1}{2}I_{\prime} P'' \cos L_{\prime} - \cos\tfrac{1}{2}I_{\prime} Q'' \sin L_{\prime} \\ o = Q_{\prime\prime}' + \cos\tfrac{1}{2}I'' Q_{\prime} \cos L'' - \cos\tfrac{1}{2}I'' P_{\prime} \sin L'' + \cos\tfrac{1}{2}I_{\prime} Q'' \cos L'' + \cos\tfrac{1}{2}I_{\prime} P'' \sin L_{\prime} \end{cases}$$

quae una cum (31) et (32) existunt, et revera in his continentur. Ex aequatione vero $k_{\prime} + k'_{\prime\prime} + k'' = o$, prodit

$$L_{\prime} + L'_{\prime\prime} + L'' = o$$

si non nisi ad constantes arbitrarias in ipsis $L_{\prime}$, $L'_{\prime\prime}$ atque L'' contentas respicitur. Iam si aequationum (32) priorem per $\cos L''$, et posteriorem per $\sin L''$ multiplicaverimus, emergit additis productis propter aequationem praecedentem

$$o = \cos\tfrac{1}{2} I'' P'_{\prime\prime} + P_{\prime} \cos L'' + Q_{\prime} \sin L'' + \cos\tfrac{1}{2} I'_{\prime\prime} P'' \cos L_{\prime} - \cos\tfrac{1}{2} I'_{\prime\prime} Q'' \sin L_{\prime}$$

si porro priorem per $-\sin L''$, et posteriorem per $\cos L''$ multiplicaverimus, fit additis productis

$$o = \cos\tfrac{1}{2} I'' Q'_{\prime\prime} + Q_{\prime} \cos L'' - P_{\prime} \sin L'' + \cos\tfrac{1}{2} I_{\prime} Q'' \cos L_{\prime} + \cos\tfrac{1}{2} I_{\prime} P'' \sin L_{\prime}$$

quae aequationes ad constantes arbitrarias in ipsis $P_{\prime}$, $Q_{\prime}$, $P'_{\prime\prime}$, etc. contentas restringendae sunt. Hae autem aequationes cum aequationibus (48) una existere nequeunt, nisi cosinus inclinationum mutuarum deleantur. Hinc concluditur, positis loco $L_{\prime}$, $L'_{\prime\prime}$, L'' solummodo constantibus arbitrariis, quae in earum valoribus continentur, aequationes has

$$\begin{aligned} o &= P_{\prime} + P'' \cos\lambda'_{\prime\prime} + Q'' \sin\lambda'_{\prime\prime} + P'_{\prime\prime} \cos\lambda'' - Q'_{\prime\prime} \sin\lambda'' \\ o &= Q_{\prime} + Q'' \cos\lambda'_{\prime\prime} - P'' \sin\lambda'_{\prime\prime} + Q'_{\prime\prime} \cos\lambda'' + P'_{\prime\prime} \sin\lambda'' \end{aligned}$$

esse reliquas duas aequationes conditionales inter constantes nostras arbitrarias, si in iis loco $P_{\prime}$, $Q_{\prime}$, $P'_{\prime\prime}$, etc. solummodo constantes arbitrariae in valoribus earum contentae substituantur. Substitutis valoribus ipsarum $P_{\prime}$, $Q_{\prime}$, $P'_{\prime\prime}$, etc. ex aequationibus (45), postquam tempus cifrae aequatum est, petendis, nanciscimur

$$\left.\begin{aligned} o &= C'_{\prime\prime}(1 + b_{\prime} + f'')\sin\lambda'' - C''(1 + b'_{\prime\prime} + f_{\prime})\sin\lambda'_{\prime\prime} \\ o &= C'_{\prime\prime}(1 + b_{\prime} + f'')\cos\lambda'' + C''(1 + b'_{\prime\prime} + f_{\prime})\cos\lambda'_{\prime\prime} + C_{\prime}(1 + b'' + f'_{\prime\prime}) \end{aligned}\right\} \ldots\ldots(49)$$

quae una cum

$$o = k_{\prime} + k'_{\prime\prime} + k''$$

aequationes conditionales inter constantes arbitrarias novem $C_{\prime}$, $C'_{\prime\prime}$, C'', $v_{\prime}$, $v'_{\prime\prime}$, v'', $k_{\prime}$, $k'_{\prime\prime}$, k'' sunt. Evidens est has aequationes tribus inter sex trianguli cuiuscunque plani partes locum habentibus aequationibus plane similes esse, et si aequationum differentialium (34) et (41) significationem in problemate quatuor corporum non respexeris, duo nova aequationum differentialium linearium cum coefficientibus variabilibus systemata inventa esse, quae rigorose integrari possint.

Quum, statuta ε' radice secundi ordinis, aequationes (47) monstrent, $1+b'+f'_{,,}$ atque $1+b_{,}+f''$ esse quantitates secundi ordinis, $1+b'_{,,}+f_{,}$ vero quantitatem finitam: aequationes (49) constantem C'' quantitatem tertii ordinis esse debere probant.

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§. III. *Perfecta aequationum differentialium paragraphi primae integratio.*

18.

Sint dextra aequationum (12) membra resp. per C, D atque E denotata, unde habetur

$$\frac{d\Upsilon}{dt}=C\,,\quad \frac{d\Psi}{dt}=D\,,\quad \frac{d\Xi}{dt}=E$$

Secundum praecedentia C atque D quantitates sunt primi, et E quantitas secundi ordinis, termini vero primi ordinis omnes, qui in ipsis C atque D continentur, in paragrapho praecedenti elicuimus et in aequationibus vel (24) vel (34) exposuimus. Hinc sequitur, positis

$$\begin{aligned}F=C-(y-\mu_{,}-\mu_{,,})\,\Psi\quad &-\sigma_{,}(\Upsilon'+2e')\sin(h_{,}t+2k_{,})+\sigma_{,}\Psi'\cos(h_{,}t+2k_{,})\\ &+\sigma_{,,}(\Upsilon''+2e'')\sin(h''t+2k'')+\sigma_{,,}\Psi''\cos(h''t+2k'')\\ G=D+(y-\mu_{,}-\mu_{,,})(\Upsilon+2e)&-\sigma_{,}(\Upsilon'+2e')\cos(h_{,}t+2k_{,})-\sigma_{,}\Psi'\sin(h_{,}t+2k_{,})\\ &-\sigma_{,,}(\Upsilon''+2e'')\cos(h''t+2k'')+\sigma_{,,}\Psi''\sin(h''t+2k'')\end{aligned}$$

esse

$$\begin{aligned}\frac{d\Upsilon}{dt}=\ (y-\mu_{,}-\mu_{,,})\,\Psi\quad &+\sigma_{,}(\Upsilon'+2e')\sin(h_{,}t+2k_{,})-\sigma_{,}\Psi'\cos(h_{,}t+2k_{,})\\ &-\sigma_{,,}(\Upsilon''+2e'')\sin(h''t+2k'')-\sigma_{,,}\Psi''\cos(h''t+2k'')+F\\ \frac{d\Psi}{dt}=-(y-\mu_{,}-\mu_{,,})(\Upsilon+2e)&+\sigma_{,}(\Upsilon'+2e')\cos(h_{,}t+2k_{,})+\sigma_{,}\Psi'\sin(h_{,}t+2k_{,})\\ &+\sigma_{,,}(\Upsilon''+2e'')\cos(h''t+2k'')-\sigma_{,,}\Psi''\sin(h''t+2k'')+G\end{aligned}$$

aequationes rigorosas, in quibus F et G quantitates secundi ordinis sint, et similes aequationes pro $\frac{d\Upsilon'}{dt}$, etc. obtineri. Quibus positis, ex ipsarum Υ, Ψ, Υ', etc. in praecedenti paragrapho erutis valoribus, qui terminos

earum primi ordinis continent, itaque usque ad quantitates secundi ordinis accurati sunt, quantitates $(n)\delta z$, w, $(n')\delta z'$, etc., quarum functiones ipsae F, G, F', etc. sunt, usque ad quantitates secundi ordinis accuratae computari possunt, quibus valoribus nec non ipsarum $P_{,}$, $Q_{,}$, etc. valoribus, qui et ipsi in praecedenti paragrapho usque ad quantitates secundi ordinis accurati exhibiti sunt, substitutis, ipsae F, G, etc. usque ad quantitates tertii ordinis accuratae solius variabilis t functiones factae sunt. Aequationes igitur praecedentes lineares factae sunt aequationes differentiales, quibus termini accedunt, qui functiones solius variabilis independentis t sunt. Quum, omissis his terminis, harum aequationum integralia iam invenerimus, per notam variationis constantium arbitrariarum methodum ad aequationes lineares applicatam aequationum praecedentium integralia inveniri possunt. Quo facto, valores variabilium Υ, Ψ, Υ', etc. usque ad quantitates tertii ordinis accuratos obtinebis, quo calculo repetito, usque ad quemlibet accurationis gradum pervenies.

19.

Methodus aequationum linearium, quae functiones solius variabilis independentis continent, integrandarum, quum in libris compluribus de calculo integrali agentibus iam copiose exposita sit, nihil attinet eam hoc loco explicare, quare nota illa praecepta ad aequationes praecedentes et earum similes statim applicabo. Aequationes (36) exhibent variabiles nostras functiones constantium arbitrariarum, quum vero ad finem nunc assequendum constantes arbitrarias functiones variabilium exhiberi oporteat, eliminatio instituenda est. Positis in aequationibus (36)

$$e\cos D = e_c,\ e\sin D = e_s,\ e'\cos D' = e'_c,\ e'\sin D' = e'_s,\ e''\cos D'' = e''_c,\ e''\sin D'' = e''_s$$

emergunt

$$\begin{aligned}
\Upsilon + 2e &= 2e_c\xi + 2e'_c s\cos(h_{,}t + 2k_{,}) + 2e'_s s\sin(h_{,}t + 2k_{,}) \\
&\quad + 2e''_c f\cos(h''t + 2k'') - 2e''_s f\sin(h''t + 2k''), \\
\Psi &= 2e_s\xi - 2e'_c s\sin(h_{,}t + 2k_{,}) + 2e'_s s\cos(h_{,}t + 2k_{,}) \\
&\quad + 2e''_c f\sin(h''t + 2k'') + 2e''_s f\cos(h''t + 2k'')
\end{aligned}$$

$$
\begin{aligned}
\Upsilon' + 2e' &= 2e'_c\xi + 2e''_c s' \cos(h'_{\prime\prime}t + 2k'_{\prime\prime}) + 2e''_s s' \sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) \\
&\quad + 2e_c f' \cos(h_{\prime}t + 2k_{\prime}) - 2e_s f' \sin(h_{\prime}t + 2k_{\prime}) \\
\Psi' &= 2e'_s\xi' - 2e''_c s' \sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) + 2e''_s s' \cos(h'_{\prime\prime}t + 2k'_{\prime\prime}) \\
&\quad + 2e_c f' \sin(h_{\prime}t + 2k_{\prime}) + 2e_s f' \cos(h_{\prime}t + 2k_{\prime}) \\
\Upsilon'' + 2e'' &= 2e''_c\xi'' + 2e_c s'' \cos(h''t + 2k'') + 2e_s s'' \sin(h''t + 2k'') \\
&\quad + 2e'_c f'' \cos(h'_{\prime\prime}t + 2k'_{\prime\prime}) - 2e'_s f'' \sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) \\
\Psi'' &= 2e''_s\xi'' - 2e_c s'' \sin(h''t + 2k'') + 2e_s s'' \cos(h''t + 2k'') \\
&\quad + 2e'_s f'' \sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) + 2e'_s f'' \cos(h'_{\prime\prime}t + 2k'_{\prime\prime})
\end{aligned}
$$

e quibus, eliminatione haud difficili instituta, elicitur

$$
\begin{aligned}
2e_c &= \frac{s'f''-\xi\xi''}{\Gamma}(\Upsilon + 2e) - \frac{ff''-s\xi''}{\Gamma}(\Upsilon' + 2e')\cos(h_{\prime}t + 2k_{\prime}) - \frac{ff''-s\xi''}{\Gamma}\Psi' \sin(h_{\prime}t + 2k_{\prime}) \\
&\quad - \frac{ss'-f\xi'}{\Gamma}(\Upsilon'' + 2e'')\cos(h''t + 2k'') + \frac{ss'-f\xi}{\Gamma}\Psi'' \sin(h''t + 2k'') \\
2e_s &= \frac{s'f''-\xi\xi''}{\Gamma}\,\Psi \quad + \frac{ff''-s\xi}{\Gamma}(\Upsilon' + 2e')\sin(h_{\prime}t + 2k_{\prime}) - \frac{ff''-s\xi''}{\Gamma}\Psi' \cos(h_{\prime}t + 2k_{\prime}) \\
&\quad - \frac{ss'-f\xi'}{\Gamma}(\Upsilon'' + 2e'')\sin(h''t + 2k'') - \frac{ss'-f\xi'}{\Gamma}\Psi'' \cos(h''t + 2k'') \\
2e'_c &= \frac{s''f-\xi''\xi}{\Gamma}(\Upsilon' + 2e') - \frac{f'f-s'\xi}{\Gamma}(\Upsilon'' + 2e'')\cos(h'_{\prime\prime}t + 2k'_{\prime\prime}) - \frac{f'f-s'\xi}{\Gamma}\Psi'' \sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) \\
&\quad - \frac{s's''-f'\xi''}{\Gamma}(\Upsilon + 2e)\cos(h_{\prime}t + 2k_{\prime}) + \frac{s's''-f'\xi''}{\Gamma}\Psi \sin(h_{\prime}t + 2k_{\prime}) \\
2e'_s &= \frac{s''f-\xi''\xi}{\Gamma}\,\Psi' \quad + \frac{f'f-s'\xi}{\Gamma}(\Upsilon'' + 2e'')\sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) - \frac{f'f-s'\xi}{\Gamma}\Psi'' \cos(h'_{\prime\prime}t + 2k'_{\prime\prime}) \\
&\quad - \frac{s's''-f'\xi''}{\Gamma}(\Upsilon + 2e)\sin(h_{\prime}t + 2k_{\prime}) - \frac{s's''-f'\xi''}{\Gamma}\Psi \cos(h_{\prime}t + 2k_{\prime}) \\
2e''_c &= \frac{sf'-\xi\xi'}{\Gamma}(\Upsilon'' + 2e'') - \frac{f''f'-s''\xi'}{\Gamma}(\Upsilon + 2e)\cos(h''t + 2k'') - \frac{f''f'-s''\xi}{\Gamma}\Psi \sin(h''t + 2k'') \\
&\quad - \frac{s''s-f''\xi}{\Gamma}(\Upsilon' + 2e')\cos(h'_{\prime\prime}t + 2k'_{\prime\prime}) + \frac{s''s-f''\xi}{\Gamma}\Psi'_{\prime} \sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) \\
2e''_s &= \frac{sf'-\xi\xi'}{\Gamma}\,\Psi'' \quad + \frac{f''f'-s''\xi'}{\Gamma}(\Upsilon + 2e)\sin(h''t + 2k'') - \frac{f''f'-s''\xi}{\Gamma}\Psi \cos(h''t + 2k'') \\
&\quad - \frac{s''s-f''\xi}{\Gamma}(\Upsilon' + 2e')\sin(h'_{\prime\prime}t + 2k'_{\prime\prime}) - \frac{s''s-f''\xi}{\Gamma}\Psi' \cos(h'_{\prime\prime}t + 2k'_{\prime\prime})
\end{aligned}
$$

ubi brevitatis caussa feci

$$
\Gamma = (sf' - \xi\xi')\xi'' - (ss'' - \xi f'')s' - (f'f'' - \xi's'')f
$$

Si hae aequationes post $\frac{de_c}{dt}$, $\frac{de_s}{dt}$, etc. loco e_c, e_s, etc. et F, G, F', etc. loco $(\Upsilon + 2e)$, Ψ, $(\Upsilon' + 2e')$, etc. substitutas integrantur, et integrata in praecedentibus substituuntur, obtinentur termini, qui ad expressiones (38) ipsarum Υ, Ψ, Υ', etc. additi perfecta aequationum (12) integralia subministrant. Aequationes hoc modo elicitae sub hac facile rediguntur forma

$$\begin{aligned}
\Upsilon = {} & 2e(\xi - 1) + 2e's\cos(h_{,}t + 2k_{,}) + 2e''f\cos(h''t + 2k'') + \int F\,dt \\
& - h_{,}\frac{s's'' - f'\xi''}{\Gamma}s\int\left\{\sin[h_{,}t - h_{,}(t)]\int F\,dt + \cos[h_{,}t - h_{,}(t)]\int G\,dt\right\}dt \\
& - h''\frac{f'f'' - \xi's''}{\Gamma}f\int\left\{\sin[h''t - h''(t)]\int F\,dt - \cos[h''t - h''(t)]\int G\,dt\right\}dt \\
& - h_{,}\frac{ff'' - s\xi''}{\Gamma}\xi\int\left\{\sin[h_{,}t + 2k_{,}]\int F'\,dt - \cos[h_{,}t + 2k_{,}]\int G'\,dt\right\}dt \\
& + h'_{,,}\frac{ss'' - \xi f''}{\Gamma}f\int\left\{\sin[-h'_{,,}t + h'_{,,}(t) + h_{,}(t) + 2k_{,}]\int F'\,dt - \cos[-h'_{,,}t + h'_{,,}(t) + h_{,}(t) + 2k_{,}]\int G'\,dt\right\}dt \\
& - h''\frac{ss' - f\xi'}{\Gamma}\xi\int\left\{\sin(h''t + 2k'')\int F''\,dt + \cos(h''t + 2k'')\int G''\,dt\right\}dt \\
& + h'_{,,}\frac{ff' - \xi s'}{\Gamma}s\int\left\{\sin[-h'_{,,}t + h'_{,,}(t) + h''(t) + 2k'']\int F''\,dt + \cos[-h'_{,,}t + h'_{,,}(t) + h''(t) + 2k'']\int G''\,dt\right\}dt
\end{aligned}$$

$$\begin{aligned}
\Psi = {} & -2e's\sin(h_{,}t + 2k_{,}) + 2e''f\sin(h''t + 2k'') + \int G\,dt \\
& + h_{,}\frac{s's'' - f'\xi''}{\Gamma}s\int\left\{\cos[h_{,}t - h_{,}(t)]\int F\,dt - \sin[h_{,}t - h_{,}(t)]\int G\,dt\right\}dt \\
& - h''\frac{f'f'' - \xi's''}{\Gamma}f\int\left\{\cos[h''t - h''(t)]\int F\,dt + \sin[h''t - h''(t)]\int G\,dt\right\}dt \\
& - h_{,}\frac{ff'' - s\xi''}{\Gamma}\xi\int\left\{\cos[h_{,}t + 2k_{,}]\int F'\,dt + \sin[h_{,}t + 2k_{,}]\int G'\,dt\right\}dt \\
& + h'_{,,}\frac{ss'' - \xi f''}{\Gamma}f\int\left\{\cos[-h'_{,,}t + h'_{,,}(t) + h_{,}(t) + 2k_{,}]\int F'\,dt + \sin[-h'_{,,}t + h'_{,,}(t) + h_{,}(t) + 2k_{,}]\int G'\,dt\right\}dt \\
& + h''\frac{ss' - f\xi'}{\Gamma}\xi\int\left\{\cos[h''t + 2k'']\int F''\,dt - \sin[h''t + 2k'']\int G''\,dt\right\}dt \\
& - h'_{,,}\frac{ff' - \xi s'}{\Gamma}s\int\left\{\cos[-h'_{,,}t + h'_{,,}(t) + h''(t) + 2k'']\int F''\,dt - \sin[-h'_{,,}t + h'_{,,}(t) + h''(t) + 2k'']\int G''\,dt\right\}dt
\end{aligned}$$

ad quas accedit

$$\Xi = -b + \int E\,dt$$

denotante $-b$ constantem arbitrariam huic integrali additam. De praecedentibus aequationibus annotandum est, ipsam (t) in integrationibus constantem tractandam, et post integrationes peractas in t mutandam esse.

Mutatis mutandis nanciscimur aequationes similes pro Υ', Ψ', Ξ', Υ'', Ψ'', Ξ''. Quum E sit quantitas secundi ordinis, termini primi ordinis ipsarum $n\delta z$, w, P, etc. in praecedentibus inventi, postquam in ipsa E substituti erunt, efficiunt, ut haec functio usque ad quantitates tertii ordinis accurata solius variabilis t evadat, quam in seriem infinitam evolutam statim integrare nobis liceat.

20.

Quum ad nz atque w computandas non ipsis Υ, Ψ atque Ξ, sed loco earum solummodo W sit opus, praestat hanc ex illis componere. Si expressionem

$$W = \Upsilon\left(\frac{\varrho}{a}\cos\varphi+\tfrac{3}{2}e\right) + \Psi\frac{\varrho}{a}\sin\varphi + \Xi$$

in seriem evolvimus, habemus primum

$$W = -\frac{dR^{(1)}_{,}}{de}\Upsilon\cos\gamma - \frac{R^{(1)}_{,}}{e}\sqrt{1-e^2}\,\Psi\sin\gamma + \Xi$$

et reliqui huius expressionis termini ope theorematis art. 9. Sect. IV. computari possunt, postquam loco Υ et Ψ earum evolutiones in series substitutae erunt. Quum vero maximus terminus et in $-\frac{dR^{(1)}_{,}}{de}$ et in $-\frac{R^{(1)}_{,}}{e}\sqrt{1-e^2}$ sit unitas ipsa, loco expressionis praecedentis ponere licet

(50)...... $$W = \Upsilon\cos\gamma + \Psi\sin\gamma + \Xi$$

si terminos omnes in ipsis F atque C per $-\frac{dR^{(1)}_{,}}{de}$, et terminos omnes in ipsis G atque D contentos per $-\frac{R^{(1)}_{,}}{e}\sqrt{1-e^2}$ multiplicatos esse animo concipimus. Quo enim facto, reliqui expressionis quoque (50) termini adiumento theorematis excitati computari possunt, postquam loco Υ et Ψ evolutiones earum in series substitutae erunt.

Substitutis aequationibus art. praec. in expressione (50), nanciscimur

$$\begin{aligned}
W = & -b + 2e(\xi-1)\cos\gamma + 2e's\cos(\gamma+h_{,}t+2k_{,}) + 2e''f\cos(\gamma-h''t-2k'') + \int\{F\cos\gamma + G\sin\gamma + E\}\,dt \\
& -h_{,}\frac{s's''-f'\xi'}{\Gamma}s\int\Big\{\sin[h_{,}t-h_{,}(t)]\int[F\cos\gamma+G\sin\gamma]dt - \cos[h_{,}t-h_{,}(t)]\int[F\sin\gamma-G\cos\gamma]dt\Big\}dt \\
& -h''\frac{f'f''-\xi's''}{\Gamma}f\int\Big\{\sin[h''t-h''(t)]\int[F\cos\gamma+G\sin\gamma]dt + \cos[h''t-h''(t)]\int[F\sin\gamma-G\cos\gamma]dt\Big\}dt \\
& -h_{,}\frac{ff''-s\xi''}{\Gamma}\xi\int\Big\{\sin[h_{,}t+2k_{,}]\int[F'\cos\gamma+G'\sin\gamma]dt + \cos[h_{,}t+2k_{,}]\int[F'\sin\gamma-G'\cos\gamma]dt\Big\}dt \\
& +h'_{,,}\frac{ss''-\xi f''}{\Gamma}f\int\left\{\begin{array}{l}\sin[-h'_{,,}t+h'_{,,}(t)+h_{,}(t)+2k_{,}]\int[F'\cos\gamma+G'\sin\gamma]dt \\ +\cos[-h'_{,,}t+h'_{,,}(t)+h_{,}(t)+2k_{,}]\int[F'\sin\gamma-G'\cos\gamma]dt\end{array}\right\}dt \\
& -h''\frac{ss'-f\xi'}{\Gamma}\xi\int\Big\{\sin[h''t+2k'']\int[F''\cos\gamma+G''\sin\gamma]dt - \cos[h''t+2k'']\int[F''\sin\gamma-G''\cos\gamma]dt\Big\}dt \\
& +h'_{,,}\frac{ff'-\xi s'}{\Gamma}s\int\left\{\begin{array}{l}\sin[-h'_{,,}t+h'_{,,}(t)+h''(t)+2k'']\int[F''\cos\gamma+G''\sin\gamma]dt \\ -\cos[-h'_{,,}t+h'_{,,}(t)+h''(t)+2k'']\int[F''\sin\gamma-G''\cos\gamma]dt\end{array}\right\}dt
\end{aligned}$$

Sit dextrum aequationis (10) membrum L nominatum, unde $\frac{dW}{dt} = L$, et sit praeterea

$$C\cos\gamma + D\sin\gamma + E = L$$

tum formulae art. 18. suppeditant

$$\begin{aligned}
F\cos\gamma + G\sin\gamma + E = \; & L + (y-\mu_{,}-\mu_{,,})[(\Upsilon+2e)\sin\gamma - \Psi\cos\gamma] \\
& -\sigma_{,}[(\Upsilon'+2e')\cos\gamma + \Psi'\sin\gamma]\sin(h_{,}t+2k_{,}) \\
& -\sigma_{,}[(\Upsilon'+2e')\sin\gamma - \Psi'\cos\gamma]\cos(h_{,}t+2k_{,}) \\
& +\sigma_{,,}[(\Upsilon''+2e'')\cos\gamma + \Psi''\sin\gamma]\sin(h''t+2k'') \\
& -\sigma_{,,}[(\Upsilon''+2e'')\sin\gamma - \Psi''\cos\gamma]\cos(h''t+2k'')
\end{aligned}$$

sed quoties termini, in quibus $\varkappa$ maior est quam ± 1, omittuntur, quantitas $\Upsilon\sin\gamma - \Psi\cos\gamma$ aequalis est ipsi $-\frac{dW}{d\gamma}$, $\Upsilon\cos\gamma + \Psi\sin\gamma$ aequalis ipsi $-\frac{d^2W}{d\gamma^2}$, $\Upsilon'\sin\gamma - \Psi'\cos\gamma$ aequalis ipsi $-\frac{dW'}{d\gamma'}$ in qua γ' in γ mutata est, id quod ita $\frac{dW_{\gamma}'}{d\gamma}$ denotabimus, $\Upsilon'\cos\gamma + \Psi'\sin\gamma$ aequalis ipsi $-\frac{d^2W_{\gamma}'}{d\gamma^2}$, et sic porro. Si igitur denotamus $F\cos\gamma + G\sin\gamma + E$ per M, habemus

$$M = L - (y - \mu_{,} - \mu_{,,})\frac{dW}{d\gamma} + \sigma_{,}\frac{d^2 W\gamma'}{d\gamma^2}\sin(h_{,}t + 2k_{,}) + \sigma_{,}\frac{dW\gamma'}{d\gamma}\cos(h_{,}t + 2k_{,})$$
$$- \sigma_{,,}\frac{d^2 W\gamma''}{d\gamma^2}\sin(h''t + 2k'') + \sigma_{,,}\frac{dW\gamma''}{d\gamma}\cos(h''t + 2k'')$$
$$+ 2e(y - \mu_{,} - \mu_{,,})\sin\gamma - 2e'\sigma_{,}\sin(\gamma + h_{,}t + 2k_{,}) - 2e''\sigma_{,,}\sin(\gamma - h''t - 2k'')$$

et praeterea

$$F\sin\gamma - G\cos\gamma = -\frac{dM}{d\gamma},\quad F\cos\gamma + G\sin\gamma = -\frac{d^2 M}{d\gamma^2}$$

formula igitur praecedens abit in

$$(51)\left\{\begin{aligned}
W = & -b + 2e(\xi - 1)\cos\gamma + 2e's\cos(\gamma + h_{,}t + 2k_{,}) + 2e''f\cos(\gamma - h''t - 2k'') + \int M\,dt \\
& + h_{,}\frac{s's'' - f'\xi''}{\Gamma}s\int\Big\{\sin[h_{,}t - h_{,}(t)]\int\frac{d^2 M}{d\gamma^2}dt - \cos[h_{,}t - h_{,}(t)]\int\frac{dM}{d\gamma}dt\Big\}dt \\
& + h''\frac{f'f'' - \xi's''}{\Gamma}f\int\Big\{\sin[h''t - h''(t)]\int\frac{d^2 M}{d\gamma^2}dt + \cos[h''t - h''(t)]\int\frac{dM}{d\gamma}dt\Big\}dt \\
& + h_{,}\frac{ff'' - s\xi''}{\Gamma}\xi\int\Big\{\sin[h_{,}t + 2k_{,}]\int\frac{d^2 M\gamma'}{d\gamma^2}dt + \cos[h_{,}t + 2k_{,}]\int\frac{dM\gamma'}{d\gamma}dt\Big\}dt \\
& - h'_{,,}\frac{ss'' - \xi f''}{\Gamma}f\int\Big\{\sin[-h'_{,,}t + h'_{,,}(t) + h_{,}(t) + 2k_{,}]\int\frac{d^2 M\gamma'}{d\gamma^2}dt + \cos[-h'_{,,}t + h'_{,,}(t) + h_{,}(t) + 2k_{,}]\int\frac{dM\gamma'}{d\gamma}dt\Big\}d \\
& + h''\frac{ss' - f\xi'}{\Gamma}\xi\int\Big\{\sin[h''t + 2k'']\int\frac{d^2 M\gamma''}{d\gamma^2}dt - \cos[h''t + 2k'']\int\frac{dM\gamma''}{d\gamma}dt\Big\}dt \\
& - h'_{,,}\frac{ff' - \xi s'}{\Gamma}s\int\Big\{\sin[-h'_{,,}t + h'_{,,}(t) + h''(t) + 2k'']\int\frac{d^2 M\gamma''}{d\gamma^2}dt - \cos[-h'_{,,}t + h'_{,,}(t) + h''(t) + 2k'']\int\frac{dM\gamma''}{d\gamma}dt\Big\}d
\end{aligned}\right.$$

et similes expressiones obtinentur pro W' atque W'', quae computationi harum quantitatum inserviunt. Quem in finem necesse est L habeatur, sed quum L sit dextrum aequationis (10) membrum, et in art. 7. Sect. III. expressio ipsius W in seriem evoluta data sit, habetur L expressioni ipsius Z l. c. datae aequalis, si in ea signa $\int$ atque dt deleta, et termini e corpore m'' pendentes, qui terminis e corpore m' pendentibus analogi sunt, additi erunt. Ex analysi in praecedentibus exposita sequitur, M esse quantitatem, quae terminos primi ordinis non continet, nec non $\mu_{,} + \mu_{,,}$ per $2e$ divisum coefficientem termini in expressione evoluta ipsius $(\overset{\circ}{T})$ per $\sin\gamma$ multiplicati, $\sigma_{,}$ per $2e'$ divisum coefficientem termini eiusdem expressionis per $\sin(\gamma + h_{,}t + 2k_{,})$ multiplicati, et $\sigma_{,,}$ per $2e''$ divisum coefficientem termini eiusdem expressionis per $\sin(\gamma - h''t - 2k'')$ multiplicati. Constans b ita determinanda est, ut in expressione ipsius $\overline{W}$ terminus constans, et ξ ita, ut in $\overline{W}$ terminus per $\sin g$ multiplicatus non adsit.

Termini $-b+2e(\xi-1)\cos\gamma$ praecedentis ipsius W expressionis constantem integrali, quod in art. 2. $\frac{d\xi}{d\tau}$ suppeditavit, addendam explet, quia illic loco integrae constantis numerum 1 addidimus. Termini quidem per $(\xi-1)^2$, etc. in expressione praecedenti explicite non adsunt, sed in approximationibus subsequentibus sua sponte nascuntur.

Complectamur breviter praecepta ad ipsius nz, w, $n'z'$, etc. computandas in praecedentibus exposita. Terminis tantum primis aequationis praecedentis receptis, habetur

$$W=-b+2e(\xi-1)\cos\gamma+2e's\cos(\gamma+h_{,}t+2k_{,})+2e''f\cos(\gamma-h''t-2k'')$$

unde

$$\overline{W}=-b+2e(\xi-1)\cos g+2e's\cos(g+h_{,}t+2k_{,})+2e''f\cos(g-h''t-2k'')$$

hinc, reiectis terminis ordinis superioris, nanciscimur per aequationes (7)

$$nz=g-nbt+2e(\xi-1)\sin g+\frac{2e's}{1+\frac{h_{,}}{n}}\sin(g+h_{,}t+2k_{,})+\frac{2e''f}{1-\frac{h''}{n}}\sin(g-h''t-2k'')$$

$$w=\qquad -e(\xi-1)\cos g-\frac{e's}{1+\frac{h_{,}}{n}}\cos(g+h_{,}t+2k_{,})-\frac{e''f}{1-\frac{h''}{n}}\cos(g-h''t-2k'')$$

quantitates vero b atque ξ, quum ita determinandae sint, ut ex expressione ipsius nz evanescant, statim reiici possunt, et quum praeterea $e(\xi-1)$ semper sit parvulus coefficiens ordini altiori adnumerandus quam reliqui coefficientes, idem ex hac quoque ipsius w expressione deleri potest. Habemus igitur

$$n\delta z=\frac{2e's}{1+\frac{h_{,}}{n}}\sin(g+h_{,}t+2k_{,})+\frac{2e''f}{1-\frac{h''}{n}}\sin(g-h''t-2k'')$$

$$w=-\frac{e's}{1+\frac{h_{,}}{n}}\cos(g+h_{,}t+2k_{,})-\frac{e''f}{1-\frac{h''}{n}}\cos(g-h''t-2k'')$$

et iisdem rationibus

$$W=2e's\cos(\gamma+h_{,}t+2k_{,})+2e''f\cos(\gamma-h''t-2k'')$$

quibus similes expressiones pro $n\delta z'$, w', W', etc. sunt. Substitutis his valoribus nec non valoribus analogis ipsarum $P_{,}$, $Q_{,}$, etc. in praecedentibus datis in expressione ipsius L sive $\frac{dZ}{dt}$ in art. 7. Sect. III. data, et tum

in expressione praecedenti ipsius M, dum semper termini in quibus $\varkappa$ maior est quam ± 1 omittuntur, ipsa M erit functio solius variabilis t, quae terminis primi ordinis, quorum argumenta ipsas g, g' et g'' non continent, caret, et usque ad quantitates tertii ordinis accurata est. Substituto hoc valore ipsius M in expressione (51), post integrationes peractas W usque ad quantitates tertii ordinis accuratam habebis, et, theoremate art. 9. Sect. IV. in usum vocato, adiumento formularum (7) $n\delta z$ atque w usque ad quantitates tertii ordinis accuratas computabis *). His accuratioribus valoribus et adiumento valoris ipsius $S+\varepsilon$ per aequationem (8) computandi idem calculus repetitur, quo ad valores ipsarum nz, w, etc. pervenies usque ad quantitates quarti ordinis accuratos, et sic porro.

Loco computationis modo descriptae etiam methodus coefficientium indeterminatorum, quae ad analyticas perturbationum coefficientium expressiones perducet, in usum vocari potest. Quem in finem necesse est, ut in calculi initio perturbationum omnium, quae vim habent, argumenta nota sint. Sit horum argumentorum unum quodque per $A \cos(\varkappa\gamma + \alpha t + \beta)$ denotatum, unde pones

$$W = \Sigma A \cos(\varkappa\gamma + \alpha t + \beta)$$

ubi argumenta omnia, quae vim habebunt, explicite apponenda sunt. Quibus praemissis, per aequationes (7) ipsarum $(n)z$, w, $(n)'z'$, w', etc. expressiones analyticae tanquam functiones coefficientium A, et, his valoribus substitutis, illa ipsius M in art. 7. Sect. III. data expressio, nec non analogae ipsarum M' atque M'' expressiones tanquam functiones eorundem coefficientium et quantitatum cognitarum exhiberi possunt. Quibus valoribus, nec non praecedente ipsius W valore in aequatione (51) substitutis, aequationem identicam nancisceris, qua singulorum coefficientium A expressiones analyticas invenire poteris. Illa tamen methodus plerumque huic praeferenda videtur.

---

*) Si praecisionem maximam brevissimo calculo assequi vis, in hac secunda approximatione termini tertii ordinis, quorum argumenta ipsas g, g' atque g'' non continent, et qui in hac approximatione eruti sunt, omitti debent, si perparvuli non sunt.

21.

Calculus in art. praec. expositus ad expressiones ipsarum $n\delta z$, w, $n'\delta z'$, etc. perducet terminorum per tempus ipsum multiplicatorum expertes, si ipsae M, M' atque M'' in evolutione earum terminos non continerent, huius formae

$$\left.\begin{aligned} M &= e\beta \sin\gamma + e'\delta \sin(\gamma + h_{,}t + 2k_{,}) + e''\nu \sin(\gamma - h''t - 2k'') \\ M_{\gamma}' &= e'\beta' \sin\gamma + e''\delta' \sin(\gamma + h_{,,}'t + 2k_{,,}') + e\,\nu' \sin(\gamma - h_{,}t - 2k_{,}) \\ M_{\gamma}'' &= e''\beta'' \sin\gamma + e\,\delta'' \sin(\gamma + h''t + 2k'') + e'\nu'' \sin(\gamma - h_{,,}'t - 2k_{,,}') \end{aligned}\right\} \ldots\ldots(52)$$

ubi coefficientes β, δ, ν, β', etc. constantes sunt, et ubi e, e', e'' apposui, quia termini hi, si adsunt, necessario per has excentricitates multiplicati esse debent. Substitutis vero his terminis in expressione (51), perfacile reperies, terminos per tempus ipsum multiplicatos in ipsa W orituros esse. Termini praecedentes revera existunt, nam in paragrapho secunda maximas tantum horum terminorum partes excerpsimus, et praeterea combinatio argumentorum perturbationum in quaque approximatione terminos tales gignit. Termini vero praecedentes illis multo minores sunt, et maximi eorum tertii ordinis, qua conditione methodus eorum tollendorum fundata est, quam confestim explicabo, et quae illi in art. 30. Sect. II. expositae plane similis est. Substitutione expressionis

$$W = 2e(\xi - 1)\cos\gamma + 2e's\cos(\gamma + h_{,}t + 2k_{,}) + 2e''f\cos(\gamma - h''t - 2k'')$$

et similium facile confirmatur identicas esse

$$M = e\beta \sin\gamma + e'\delta \sin(\gamma + h_{,}t + 2k_{,}) + e''\nu \sin(\gamma - h''t - 2k'')$$

atque

$$\begin{aligned} M = &- \varkappa \frac{dW}{d\gamma} - v \frac{d^2W_{\gamma}'}{d\gamma^2} \sin(h_{,}t + 2k_{,}) - v \frac{dW_{\gamma}'}{d\gamma} \cos(h_{,}t + 2k_{,}) \\ &+ \omega \frac{d^2W_{\gamma}''}{d\gamma^2} \sin(h''t + 2k'') - \omega \frac{dW_{\gamma}''}{d\gamma} \cos(h''t + 2k'') \\ &- 2e\varkappa \sin\gamma + 2e'v \sin(\gamma + h_{,}t + 2k_{,}) + 2e''\omega \sin(\gamma - h''t - 2k'') \end{aligned}$$

si quantitates $\varkappa$, v atque ω per aequationes determinantur has

$$\left.\begin{aligned} 2\xi\varkappa + 2f'v + 2s''\omega &= \beta \\ 2s\varkappa + 2\xi'v + 2f''\omega &= \delta, \\ 2f\varkappa + 2s'v + 2\xi''\omega &= \nu \end{aligned}\right\} \ldots\ldots(53)$$

Substituto vero praecedenti ipsius M valore in rigorosa aequatione differentiali hac

$$\begin{aligned}\frac{dW}{dt} = (y-\mu_{,}-\mu_{,,})\frac{dW}{d\gamma} - \sigma_{,}\frac{d^2W\gamma'}{d\gamma^2}\sin(h_{,}t+2k_{,}) - \sigma_{,}\frac{dW\gamma'}{d\gamma}\cos(h_{,}t+2k_{,}) \\ +\sigma_{,,}\frac{d^2W\gamma''}{d\gamma^2}\sin(h''t+2k'') - \sigma_{,,}\frac{dW\gamma''}{d\gamma}\cos(h''t+2k'') \\ -2e(y-\mu_{,}-\mu_{,,})\sin\gamma + 2e'\sigma_{,}\sin(\gamma+h_{,}t+2k_{,}) + 2e''\sigma_{,,}\sin(\gamma-h''t-2k'') + M\end{aligned}$$

emergit

$$\begin{aligned}\frac{dW}{dt} = (y-\varkappa-\mu_{,}-\mu_{,,})\frac{dW}{d\gamma} - (v+\sigma_{,})\frac{d^2W\gamma'}{d\gamma^2}\sin(h_{,}t+2k_{,}) - (v+\sigma_{,})\frac{dW\gamma'}{d\gamma}\cos(h_{,}t+2k_{,}) \\ +(\omega+\sigma_{,,})\frac{d^2W\gamma''}{d\gamma^2}\sin(h''t+2k'') - (\omega+\sigma_{,,})\frac{dW\gamma''}{d\gamma}\cos(h''t+2k'') \\ -2e(y-\varkappa-\mu_{,}-\mu_{,,})\sin\gamma + 2e'(v+\sigma_{,})\sin(\gamma+h_{,}t+2k_{,}) + 2e''(\omega+\sigma_{,,})\sin(\gamma-h''t-2k'')\end{aligned}$$

et similes aequationes nanciscimur pro $\frac{dW'}{dt}$ atque $\frac{dW''}{dt}$. Quae aequationes, quum eandem habeant formam atque aequationes differentiales pro $\frac{dW}{dt}$, $\frac{dW'}{dt}$ atque $\frac{dW''}{dt}$ ex aequationibus (34) emergentes, eadem habere debent integralia

$$W = 2e(\xi-1)\cos\gamma + 2e's\cos(\gamma+h_{,}t+2k_{,}) + 2e''f\cos(\gamma-h''t-2k'')$$

et similia pro W' atque W'', dummodo ξ, s, f, x, ξ', etc. non per aequationes (40), sed per has

$$(54)\ldots\left\{\begin{aligned} o &= [x-2\eta''-\varkappa-\mu_{,}-\mu_{,,}]\xi-(\omega+\sigma_{,,})s''-(v+\sigma_{,})f'' \\ o &= [x+2\eta'_{,,}-\varkappa'-\mu'-\mu'_{,,}]f''-(\omega'+\sigma')\xi-(v'+\sigma'_{,,})s'' \\ o &= [x \quad -\varkappa''-\mu''-\mu''_{,}]s''-(\omega''+\sigma''_{,})f''-(v''+\sigma'')\xi \\ o &= [x'+2\eta'_{,,}-\varkappa'-\mu'_{,,}-\mu']\xi'-(\omega'+\sigma')s-(v'+\sigma'_{,,})f'' \\ o &= [x' \quad -\varkappa''-\mu''-\mu''_{,}]f''-(\omega''+\sigma''_{,})\xi'-(v''+\sigma'')s \\ o &= [x'-2\eta''-\varkappa-\mu_{,}-\mu_{,,}]s-(\omega+\sigma_{,,})f''-(v+\sigma_{,})\xi' \\ o &= [x'' \quad -\varkappa''-\mu''-\mu''_{,}]\xi''-(\omega''+\sigma''_{,})s'-(v''+\sigma'')f \\ o &= [x''-2\eta''-\varkappa-\mu_{,}-\mu_{,,}]f-(\omega+\sigma_{,,})\xi''-(v+\sigma_{,})s' \\ o &= [x''+2\eta'_{,,}-\varkappa'-\mu'_{,,}-\mu']s'-(\omega'+\sigma')f-(v'+\sigma'_{,,})\xi'' \end{aligned}\right.$$

computentur, quae non minus quam illae ad unam aequationem tertii gradus, cuius radices sunt x, x' atque x'', perducunt. Considerati vero reliquis ipsius M terminis, idem integrale (51) obtinemus.

Ergo, postquam in approximatione secunda per evolutionem ipsarum M, M' atque M'' coefficientes β, δ, ν, β', etc. noti fuerint, computentur primum $\varkappa$, υ, ω, $\varkappa'$, etc. ex aequationibus (53) et similibus, sive aequationibus his

$$\varkappa = \frac{s'f''-\xi'\xi''}{2\Gamma}\beta + \frac{f'\xi''-s's''}{2\Gamma}\delta + \frac{\xi's''-f'f''}{2\Gamma}\nu$$
$$\upsilon = \frac{s\xi''-ff''}{2\Gamma}\beta + \frac{fs''-\xi\xi''}{2\Gamma}\delta + \frac{\xi f''-ss''}{2\Gamma}\nu$$
$$\omega = \frac{\xi'f-s's}{2\Gamma}\beta + \frac{\xi s'-ff'}{2\Gamma}\delta + \frac{sf'-\xi\xi'}{2\Gamma}\nu$$

et similibus, quae aequationes (53) resolutae sunt, in quo calculo valores ipsarum ξ, f, s, ξ', etc. ex approximatione prima noti adhibendi sunt; deinde ope aequationum (54) computentur valores accuratiores ipsarum x, s, f, x', etc. et denique, deletis terminis (52) ex expressionibus ipsarum M, M', M'', expressio (51) et eius similes ipsarum W, W', W'' suppeditant valores, quibus termini per tempus ipsum multiplicati non insunt. Non modo in approximatione secunda sed etiam in approximationibus omnibus subsequentibus idem calculus ad finem propositum perducit.

Termini

$$\theta \sin(-\gamma + h_{,}t + 2k_{,}) + \lambda \sin(\gamma + h''t + 2k'')$$

in M, et termini similes in ipsis M' atque M'' existentes, quorum maximae partes tertii ordinis sunt, primo aspectu etiam terminos per tempus ipsum multiplicatos in W, W' atque W'' procreare videntur, sed substitutione in aequatione (51) facta, facile reperies, terminos pure periodicos solummodo ex iis nasci posse.

22.

Quum perfectam aequationum pro Υ, Ψ, etc. integrationem in praecedentibus copiose tractaverimus, integrationem aequationum pro $P_{,}$, $Q_{,}$, etc., quae illis plane similes sunt, brevius exponere licet. Quum aequationes (36) et (38), positis resp. $Q_{,}$, $P_{,}$, 1, $C_{,}$, $D_{,}$, $b_{,}$, $C_{,,}$, l'', λ'', $D'_{,,}$, $f_{,}$, $C''_{,}$, $l'_{,,}$, $\lambda'_{,,}$, D'', etc. loco $\Upsilon+2e$, Ψ, ξ, $2e$, D, s, $2e'$, $h_{,}$, $2k_{,}$, D', f, $2e''$, h'', $2k''$, D'', etc. in (43) et (45) abeant, his mutationibus factis, integralia perfecta statim adscribi possunt haec

$$
\begin{aligned}
P_{,} = & -b_{,}C'_{,,}\sin(l''t+\lambda'')+f_{,}C''\sin(l'_{,,}t+\lambda'_{,,})+\int R_{,}dt \\
& + l''\frac{b_{,,}'b''-f_{,,}'}{\Gamma_{,}}b_{,}\int\left\{\cos[l''t-l''(t)]\int U_{,}dt-\sin[l''t-l''(t)]\int R_{,}dt\right\}dt \\
& - l'_{,,}\frac{f_{,,}'f''-b''}{\Gamma_{,}}f_{,}\int\left\{\cos[l'_{,,}t-l'_{,,}(t)]\int U_{,}dt+\sin[l'_{,,}t-l'_{,,}(t)]\int R_{,}dt\right\}dt \\
& - l''\frac{f_{,}f''-b_{,}}{\Gamma_{,}}\int\left\{\cos[l''t+\lambda'']\int U'_{,,}dt+\sin[l''t+\lambda'']\int R'_{,,}dt\right\}dt \\
& + l_{,}\frac{b_{,}b''-f''}{\Gamma_{,}}f_{,}\int\left\{\cos[-l_{,}t+l_{,}(t)+l''(t)+\lambda'']\int U'_{,,}dt+\sin[-l_{,}t+l_{,}(t)+l''(t)+\lambda'']\int R'_{,,}dt\right\}dt \\
& + l'_{,,}\frac{b_{,}b_{,,}'-f_{,}}{\Gamma_{,}}\int\left\{\cos[l'_{,,}t+\lambda'_{,,}]\int U''dt-\sin[l'_{,,}t+\lambda'_{,,}]\int R''dt\right\}dt \\
& - l_{,}\frac{f_{,}f_{,,}'-b_{,,}'}{\Gamma_{,}}b_{,}\int\left\{\cos[-l_{,}t+l_{,}(t)+l'_{,,}(t)+\lambda'_{,,}]\int U''dt-\sin[-l_{,}t+l_{,}(t)+l'_{,,}(t)+\lambda'_{,,}]\int R''dt\right\}dt
\end{aligned}
$$

$$
\begin{aligned}
Q_{,} = & \; C_{,}+b_{,}C'_{,,}\cos(l''t+\lambda'')+f_{,}C''\cos(l'_{,,}t+\lambda'_{,,})+\int U_{,}dt \\
& - l''\frac{b_{,,}'b''-f_{,,}'}{\Gamma_{,}}b_{,}\int\left\{\sin[l''t-l''(t)]\int U_{,}dt+\cos[l''t-l''(t)]\int R_{,}dt\right\}dt \\
& - l'_{,,}\frac{f_{,,}'f''-b''}{\Gamma_{,}}f_{,}\int\left\{\sin[l'_{,,}t-l'_{,,}(t)]\int U_{,}dt-\cos[l'_{,,}t-l'_{,,}(t)]\int R_{,}dt\right\}dt \\
& - l''\frac{f_{,}f''-b_{,}}{\Gamma_{,}}\int\left\{\sin[l''t+\lambda'']\int U'_{,,}dt-\cos[l''t+\lambda'']\int R'_{,,}dt\right\}dt \\
& + l_{,}\frac{b_{,}b''-f''}{\Gamma_{,}}f_{,}\int\left\{\sin[-l_{,}t+l_{,}(t)+l''(t)+\lambda'']\int U'_{,,}dt-\cos[-l_{,}t+l_{,}(t)+l''(t)+\lambda'']\int R'_{,,}dt\right\}dt \\
& - l'_{,,}\frac{b_{,}b_{,,}'-f_{,}}{\Gamma_{,}}\int\left\{\sin[l'_{,,}t+\lambda'_{,,}]\int U''dt+\cos[l'_{,,}t+\lambda'_{,,}]\int R''dt\right\}dt \\
& + l_{,}\frac{f_{,}f_{,,}'-b_{,,}'}{\Gamma_{,}}b_{,}\int\left\{\sin[-l_{,}t+l_{,}(t)+l'_{,,}(t)+\lambda'_{,,}]\int U''dt+\cos[-l_{,}t+l_{,}(t)+l'_{,,}(t)+\lambda'_{,,}]\int R''dt\right\}dt
\end{aligned}
$$

ubi

$$\Gamma_{,} = (b_{,}f'_{,,}-1)-(b_{,}b''-f'')b'_{,,}-(f'_{,,}f''-b'')f_{,}$$

et

$$R_{,}=H_{,}+(\alpha_{,}-\pi_{,}-\pi')Q_{,}+\pi'_{,,}Q'_{,,}\cos(l''t+\lambda'')+\pi'_{,,}P'_{,,}\sin(l''t+\lambda'')+\pi_{,,}Q''\cos(l'_{,,}t+\lambda'_{,,})-\pi_{,,}P''\sin(l'_{,,}t+\lambda'_{,,})$$
$$U_{,}=N_{,}-(\alpha_{,}-\pi_{,}-\pi')P_{,}+\pi'_{,,}Q'_{,,}\sin(l''t+\lambda'')-\pi'_{,,}P'_{,,}\cos(l''t+\lambda'')-\pi_{,,}Q''\sin(l'_{,,}t+\lambda'_{,,})-\pi_{,,}P''\cos(l'_{,,}t+\lambda'_{,,})$$

denotantibus $H_{,}$ atque $N_{,}$ resp. dextra aequationum art. 7. pro $\frac{dP_{,}}{dt}$ atque $\frac{dQ_{,}}{dt}$ membra, ita ut habeatur rigorose

$$\frac{dP_{,}}{dt}=H_{,},\quad \frac{dQ_{,}}{dt}=N_{,}$$

et similes aequationes pro reliquis P, Q denotatis quantitatibus. Ex sa-

tionibus in praecedentibus traditis manifestum est, $R_{,}$, $U_{,}$, $R'_{,,}$, etc. esse post evolutiones earum quantitates secundi ordinis, unde computatio aequationum in hoc articulo expositarum, si ea, quae ad ipsam γ spectant, excipis, eodem modo perficitur, quo computatio aequationis pro W. Praeterea quum termini maximi in $K_{,}$ contenti sint quantitates secundi ordinis, aequatio art. 7. pro $\frac{dK_{,}}{dt}$ post eius evolutionem immediate integrari potest. Denotante igitur $V_{,}$ dextrum huius aequationis evolutum membrum, habetur

$$K_{,} = k_{,} + \int V_{,}\, dt$$

ubi $V_{,}$ est functio solius variabilis t, et similes aequationes nanciscimur pro $K'_{,,}$ atque K'', in quibus resp. $\eta_{,}$, $\eta'_{,,}$, η'' ita determinandae sunt, ut termini per tempus ipsum multiplicati evanescant.

23.

Ex rationibus iisdem in praecedentibus respectu ipsius M explicatis ipsae $R_{,}$, $U_{,}$, $R'_{,,}$, etc. terminos continent hos

$$\begin{aligned}
R_{,} &= a_{,} C'_{,,} \cos(l''t+\lambda'') + c_{,} C'' \cos(l'_{,,}t+\lambda'_{,,}) + g_{,} C_{,} \\
U_{,} &= h_{,} C'_{,,} \sin(l''t+\lambda'') + u_{,} C'' \sin(l'_{,,}t+\lambda'_{,,}) \\
R'_{,,} &= a'_{,,} C'' \cos(l_{,}t+\lambda_{,}) + c'_{,,} C_{,} \cos(l''t+\lambda'') + g'_{,,} C'_{,,} \\
U'_{,,} &= h'_{,,} C'' \sin(l_{,}t+\lambda_{,}) + u'_{,,} C_{,} \sin(l''t+\lambda'') \\
R'' &= a'' C_{,} \cos(l'_{,,}t+\lambda'_{,,}) + c'' C'_{,,} \cos(l_{,}t+\lambda_{,}) + g'' C'' \\
U'' &= h'' C_{,} \sin(l'_{,,}t+\lambda'_{,,}) + u'' C'_{,,} \sin(l_{,}t+\lambda_{,})
\end{aligned}$$

quarum maximae partes tertii ordinis sunt, et quae in expressionibus ipsarum $P_{,}$ atque $Q_{,}$ in art. praec. datis terminos per tempus ipsum multiplicatos procreant, quos methodo eadem in praecedentibus adhibita tollere licet. Redactis aequationibus praecedentibus sub forma hac

$$\begin{aligned}
R_{,} = {} & \tfrac{1}{2}(a_{,}+h_{,}) C'_{,,} \cos(l''t+\lambda'') + \tfrac{1}{2}(c_{,}-u_{,}) C'' \cos(l'_{,,}t+\lambda'_{,,}) + g_{,} C_{,} \\
& + \tfrac{1}{2}(a_{,}-h_{,}) C'_{,,} \cos(l''t+\lambda'') + \tfrac{1}{2}(c_{,}+u_{,}) C'' \cos(l'_{,,}t+\lambda'_{,,}) \\
U_{,} = {} & \tfrac{1}{2}(a_{,}+h_{,}) C'_{,,} \sin(l''t+\lambda'') - \tfrac{1}{2}(c_{,}-u_{,}) C'' \sin(l'_{,,}t+\lambda'_{,,}) \\
& - \tfrac{1}{2}(a_{,}-h_{,}) C'_{,,} \sin(l''t+\lambda'') + \tfrac{1}{2}(c_{,}+u_{,}) C'' \sin(l'_{,,}t+\lambda'_{,,}) \\
\text{etc.} = {} & \text{etc.}
\end{aligned}$$

substitutisque ultimis per $\frac{1}{2}(a_{,}-h_{,})$ atque $\frac{1}{2}(c_{,}+u_{,})$ multiplicatis terminis,

nec non analogis quantitatum $R'_{,,}$, $U'_{,,}$, R'', U'' terminis in integralibus art. praec., facile comperitur hos terminos ipsarum $R_{,}$, $U_{,}$, etc. terminos per tempus ipsum multiplicatos in ipsis $P_{,}$, $Q_{,}$, etc. producere non posse. Supersunt igitur termini ipsarum $R_{,}$, $U_{,}$, etc. hi

$$(55)\ldots\ldots\left\{\begin{aligned} R_{,} &= \tfrac{1}{2}(a_{,}+h_{,})\,C'_{,,}\cos(l''t+\lambda'')+\tfrac{1}{2}(c_{,}-u_{,})\,C''\cos(l'_{,,}t+\lambda'_{,,})+g_{,}C_{,} \\ U_{,} &= \tfrac{1}{2}(a_{,}+h_{,})\,C'_{,,}\sin(l''t+\lambda'')-\tfrac{1}{2}(c_{,}-u_{,})\,C''\sin(l'_{,,}t+\lambda'_{,,}) \\ \text{etc.} &= \text{etc.}\end{aligned}\right.$$

e quibus solummodo termini per tempus ipsum multiplicati in valoribus ipsarum $P_{,}$, $Q_{,}$, etc. nasci possunt. Ad hos terminos tollendos animadverto, substitutis valoribus ipsarum $P_{,}$, $Q_{,}$, etc. ex (45) desumendis, resp. identicas esse aequationes has

$$\begin{aligned} R_{,} &= \beta_{,}Q_{,}+\gamma_{,}Q'_{,,}\cos(l''t+\lambda'')+\gamma_{,}P'_{,,}\sin(l''t+\lambda'')+\theta_{,}Q''\cos(l'_{,,}t+\lambda'_{,,})-\theta_{,}P''\sin(l'_{,,}t+\lambda'_{,,}) \\ U_{,} &= -\beta_{,}P_{,}+\gamma_{,}Q'_{,,}\sin(l''t+\lambda'')-\gamma_{,}P'_{,,}\cos(l''t+\lambda'')-\theta_{,}Q''\sin(l'_{,,}t+\lambda'_{,,})-\theta_{,}P''\cos(l'_{,,}t+\lambda'_{,,}) \\ \text{etc.} &= \text{etc.}\end{aligned}$$

et (55), dummodo coefficientes $\beta_{,}$, $\gamma_{,}$, $\theta_{,}$, etc. ex aequationibus his

$$(56)\ldots\ldots\left\{\begin{aligned} b_{,}\beta_{,} + \gamma_{,} + f''\theta_{,} &= \tfrac{1}{2}(a_{,}+h_{,}) \\ f_{,}\beta_{,} + h'_{,,}\gamma_{,} + \theta_{,} &= \tfrac{1}{2}(c_{,}-u_{,}) \\ \beta_{,} + f'_{,,}\gamma_{,} + b''\theta_{,} &= g_{,}\end{aligned}\right.$$

et earum similibus computentur. Substitutis his ipsarum $R_{,}$, $U_{,}$, etc. valoribus in aequationibus differentialibus rigorosis his

$$\begin{aligned} \frac{dP_{,}}{dt} &= -(\alpha_{,}-\pi_{,}-\pi')Q_{,}-\pi'_{,,}Q'_{,,}\cos(l''t+\lambda'')-\pi'_{,,}P'_{,,}\sin(l''t+\lambda'')-\pi_{,,}Q''\cos(l'_{,,}t+\lambda'_{,,})+\pi_{,,}P''\sin(l'_{,,}t+\lambda'_{,,})+R_{,} \\ \frac{dQ_{,}}{dt} &= (\alpha_{,}-\pi_{,}-\pi'_{,})P_{,}-\pi'_{,,}Q'_{,,}\sin(l''t+\lambda'')+\pi'_{,,}P'_{,,}\cos(l''t+\lambda'')+\pi_{,,}Q''\sin(l'_{,,}t+\lambda'_{,,})+\pi_{,,}P''\cos(l'_{,,}t+\lambda'_{,,})+U_{,} \\ &\text{etc.} \qquad\qquad \text{etc.}\end{aligned}$$

eadem integralia (45) obtinebimus, siquidem $\varepsilon_{,}$, $b_{,}$, $f_{,}$, $\varepsilon'_{,,}$, etc. non ex aequationibus (46), sed ex his

$$(57)\ldots\ldots\left\{\begin{aligned} o &= \varepsilon_{,}-\eta'_{,,}-\beta_{,}-\pi_{,}-\pi' \quad +(\pi'_{,,}-\gamma_{,})f'_{,,}+(\pi_{,,}-\theta_{,})b'' \\ o &= [\varepsilon_{,}+\eta_{,}-\beta'_{,,}-\pi'_{,,}-\pi''_{,}]f'_{,,}+(\pi''-\gamma'_{,,})b''+\pi'-\theta'_{,,} \\ o &= [\varepsilon_{,} \quad -\beta''-\pi''-\pi_{,,}]b''+\pi_{,}-\gamma'' \quad +(\pi''_{,}-\theta'')f'_{,,} \\ o &= \varepsilon'_{,,}+\eta_{,}-\beta'_{,,}-\pi'_{,,}-\pi''_{,} \quad +(\pi''-\gamma'_{,,})f''+(\pi'-\theta'_{,,})b_{,} \\ o &= [\varepsilon'_{,,} \quad -\beta''-\pi''-\pi_{,,}]f''+(\pi_{,}-\gamma'')b_{,}+\pi''_{,}-\theta''\end{aligned}\right.$$

$$\left.\begin{aligned}
o &= [\varepsilon'_{,,}-\eta'_{,,}-\beta_{,}-\pi_{,}-\pi']\,b_{,}+\pi'_{,,}-\gamma_{,} \quad +(\pi_{,,}-\theta_{,})f''\\
o &= \varepsilon'' \quad -\beta''-\pi''-\pi_{,,} \quad +(\pi_{,}-\gamma'')f_{,}+(\pi''_{,}-\theta'')\,b'_{,,}\\
o &= [\varepsilon''-\eta'_{,,}-\beta_{,}-\pi_{,}-\pi']f_{,}+(\pi'_{,,}-\gamma_{,})\,b'_{,,}+\pi_{,,}-\theta_{,}\\
o &= [\varepsilon''+\eta_{,}-\beta'_{,,}-\pi'_{,,}-\pi''_{,}]\,b'_{,,}+\pi''-\gamma'_{,,} \quad +(\pi'-\theta'_{,,})f_{,}
\end{aligned}\right\} \text{......(57)}$$

computentur. Habita igitur reliquorum ipsarum $R_{,}$, $U_{,}$, $R'_{,,}$, etc. terminorum ratione, integralia perfecta art. praec. semper locum habent, et in iis termini per tempus ipsum multiplicati sublati erunt, si valores accuratiores ipsarum $\varepsilon_{,}$, $b_{,}$, $f_{,}$, $\varepsilon'_{,,}$, etc. ex (57) computati, et termini (55) ipsarum $R_{,}$, $U_{,}$, $R'_{,,}$, etc. deleti fuerint. Aequationes (56) resolutae suppeditant ad ipsas $\beta_{,}$, $\gamma_{,}$, $\theta_{,}$ computandas aequationes has

$$\begin{aligned}
\beta_{,} &= \frac{f'_{,,}-b''b'_{,,}}{2\Gamma_{,}}(a_{,}+h_{,})+\frac{b''-f'_{,,}f''}{2\Gamma_{,}}(c_{,}-u_{,})+\frac{b'_{,,}f''-1}{\Gamma_{,}}\,g_{,}\\
\gamma_{,} &= \frac{b''f_{,}-1}{2\Gamma_{,}}(a_{,}+h_{,})+\frac{f''-b''b_{,}}{2\Gamma_{,}}(c_{,}-u_{,})+\frac{b_{,}-f_{,}f''}{\Gamma_{,}}\,g_{,}\\
\theta_{,} &= \frac{b'_{,,}-f'_{,,}f_{,}}{2\Gamma_{,}}(a_{,}+h_{,})+\frac{f'_{,,}b_{,}-1}{2\Gamma_{,}}(c_{,}-u_{,})+\frac{f_{,}-b'_{,,}b_{,}}{\Gamma_{,}}\,g_{,}
\end{aligned}$$

et similes pro ipsis $\beta'_{,,}$, $\gamma'_{,,}$, $\theta'_{,,}$, β'', etc. Evolutiones ipsarum Ω, Ω', Ω'' respectu ipsarum $P_{,}$, $Q_{,}$, $K_{,}$, $P'_{,,}$, etc. eodem modo perficiuntur, quo in art. 12. Sect. III. pro theoria Lunae absolutae sunt. Substitutis enim ab initio in ipsis Ω, Ω', Ω'', resp. $C_{,}$, $C'_{,,}$, C'' loco $2\sin\frac{1}{2}I_{,}$, $2\sin\frac{1}{2}I'_{,,}$, $2\sin\frac{1}{2}I''$, et $v_{,}$, $v'_{,,}$, v'' loco $N_{,}$, $N'_{,,}$, N'', habetur

$$\begin{aligned}
\delta P &= -b_{,}C'_{,,}\sin(l''t+\lambda'')+f_{,}C''\sin(l'_{,,}t+\lambda'_{,,})+\text{etc.}\\
\delta Q &= b_{,}C'_{,,}\cos(l''t+\lambda'')+f_{,}C''\cos(l'_{,,}t+\lambda'_{,,})+\text{etc.}\\
\text{etc.} &= \text{etc.}
\end{aligned}$$

atque

$$\left(\frac{d\Omega}{dP_{,}}\right)=\left(\frac{d\Omega}{dv_{,}}\right)\frac{1}{C_{,}},\quad \left(\frac{d\Omega}{dQ_{,}}\right)=\left(\frac{d\Omega}{dC_{,}}\right),\quad \left(\frac{d\Omega}{dK_{,}}\right)=\left(\frac{d\Omega}{dk_{,}}\right)$$

etc. etc.

$$\left(\frac{d^2\Omega}{dP_{,}^2}\right)=\left(\frac{d^2\Omega}{dv_{,}^2}\right)\frac{1}{C_{,}^2}+\left(\frac{d\Omega}{dC_{,}}\right)\frac{1}{C_{,}};\quad \left(\frac{d^2\Omega}{dP_{,}\,.\,dQ_{,}}\right)=\left(\frac{d^2\Omega}{dv_{,}\,.\,dC_{,}}\right)\frac{1}{C_{,}}-\left(\frac{d\Omega}{dv_{,}}\right)\frac{1}{C_{,}^2}$$

$$\left(\frac{d^2\Omega}{dQ_{,}^2}\right)=\left(\frac{d^2\Omega}{dC_{,}^2}\right),$$

etc. etc.

24.

Si ipsae nz, $n'z'$, $n''z''$, w, w', w'', $P_{,}$, $Q_{,}$, $K_{,}$, $P'_{,,}$, etc. atque $S+\varepsilon$, $S'+\varepsilon'$, $S''+\varepsilon''$ per hanc methodum computatae et in quantitatibus art. 8. $f_{,}t$, $\varphi_{,}t$, $f_{,,}t$, etc. nominatis substitutae erunt, hae quantitates functiones solius variabilis t factae sunt, et aequationes differentiales eiusdem articuli pro p_0, q_0, u, p_{00}, etc. adiumento aequationum (64) Sect. II., quae in artt. 2. seqq. Sect. VI. evolutae sunt, integrari possunt, quaecunque est anguli Γ per has integrationes introducti magnitudo. Quantitatibus p_0, q_0, p'_0, etc. sive loco earum p_{00}, q_{00}, p'_{00}, etc. sive loco earum p_{000}, q_{000}, p'_{000}; etc. hoc modo computatis, latitudo corporum versus planum fundamentale et reductio longitudinis ad idem planum eodem modo, qui in Sectt. II. et VI. explicatus est, computantur, quo facto, problema perfecte solutum est.

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§. IV. *De significatione determinationeque constantium arbitrariarum in integralibus praecedentibus introductarum.*

25.

Integrationes ad p_0, q_0 et u obtinendas instituendae duas introducunt constantes arbitrarias, quas, ut analogiam perfectam persequar, Γ_0 atque Θ_0 nominabo, item integrationes ad p'_0, q'_0, u' obtinendas introducunt constantes Γ'_0 atque Θ'_0, integrationes ad p_{00}, q_{00} atque u obtinendas constantes Γ_{00} atque Θ_{00} et sic porro, integrationes quoque eaedem introducunt angulos determinatos resp. $x_0 t$, $x'_0 t$, $x''_0 t$, $x_{00} t$, etc. denotandos. Quibus positis, si inclinationes mutuas evanescere ponimus, necessario esse debent $u = u' = u''$, et tertia aequatio (64) Sect. II., si ad omnes ipsarum du, du' et du'', in art. 8. huius Sectionis explicatas aequationes relata fuerit, suppeditabit

$$u = u' = u'' = \cos \Gamma_0 = \cos \Gamma'_0 = \cos \Gamma''_0 = \cos \Gamma_{00} = \cos \Gamma'_{00} = \text{etc.}$$

praeterea prima aequatio (64) Sect. II. subministrat

$$\begin{array}{lll}
p_0 = \sin\Gamma_0 \sin(\alpha_{,} t - \Theta_0), & p'_0 = \sin\Gamma'_0 \sin(\alpha'_{,,} t - \Theta'_0), & p''_0 = \sin\Gamma''_0 \sin(\alpha'' t - \Theta''_0) \\
p_{00} = \sin\Gamma_{00} \sin(\alpha'' t - \Theta_{00}), & p'_{00} = \sin\Gamma'_{00} \sin(\alpha_{,} t - \Theta'_{00}), & p''_{00} = \sin\Gamma''_{00} \sin(\alpha'_{,,} t - \Theta''_{00}) \\
p_{000} = \sin\Gamma_{000} \sin(\alpha'_{,,} t - \Theta_{000}), & p'_{000} = \sin\Gamma'_{000} \sin(\alpha'' t - \Theta'_{000}); & p''_{000} = \sin\Gamma''_{000} \sin(\alpha_{,} t - \Theta''_{000})
\end{array}$$

e quibus aequationes pro q_0, q'_0, etc. nancisceris, si sinus arcuum $\alpha_{,} t - \Theta_0$, $\alpha'_{,,} t - \Theta'_0$, etc. in eorum cosinus mutaveris. Porro, statutis semper adhuc inclinationibus mutuis cifrae aequalibus, expressiones finitae ipsarum p_0, q_0, p'_0, etc. art. 8. suppeditant

$$\begin{array}{ll}
p_0 = \sin i \sin[\chi - \omega - \pi + v_{,} + k_{,} + \alpha_{,} t], & p'_0 = \sin i \sin[\chi - \omega - \pi' + v'_{,,} + k'_{,,} + \alpha'_{,,} t], \\
 & p''_0 = \sin i \sin[\chi - \omega - \pi'' + v'' + k'' + \alpha'' t] \\
p_{00} = \sin i \sin[\chi - \omega - \pi + v'' - k'' + \alpha'' t], & p'_{00} = \sin i \sin[\chi - \omega - \pi' + v_{,} - k_{,} + \alpha_{,} t], \\
 & p''_{00} = \sin i \sin[\chi - \omega - \pi'' + v'_{,,} - k'_{,,} + \alpha'_{,,} t] \\
p_{000} = \sin i \sin[\chi - \omega - \pi + v'_{,,} - k'' + k_{,} + \alpha'_{,,} t], & p'_{000} = \sin i \sin[\chi - \omega - \pi' + v'' - k_{,} + k'_{,,} + \alpha'' t], \\
 & p''_{000} = \sin i \sin[\chi - \omega - \pi'' + v_{,} - k'_{,,} + k'' + \alpha_{,} t]
\end{array}$$

e quibus valores analogos ipsarum q_0, q'_0, etc. nancisceris, si sinus arcuum $\chi - \omega - \pi + v_{,} + k_{,} + \alpha_{,} t$, etc. in eorum cosinus mutaveris. Denique rationibus iisdem uti in art. 31. Sect. II. ex aequationibus (16) in casu, quem nunc tractamus, eliciuntur hae

$$\begin{array}{lll}
v_{,} + k_{,} = \pi - E, & v_{,} - k_{,} = \pi' - E, & v_{,} - k'_{,,} + k'' = \pi'' - E \\
v'_{,,} + k'_{,,} = \pi' - E', & v'_{,,} - k'_{,,} = \pi'' - E', & v'_{,,} - k'' + k_{,} = \pi - E' \\
v'' + k'' = \pi'' - E'', & v'' - k'' = \pi - E'', & v'' - k_{,} + k'_{,,} = \pi' - E''
\end{array}$$

designantibus E, E' atque E'' tres arcus constantes et indeterminatos. Comparatis his aequationibus omnibus, facili opera emergunt

$$\Gamma_0 = \Gamma'_0 = \Gamma''_0 = \Gamma_{00} = \Gamma'_{00} = \Gamma''_{00} = \Gamma_{000} = \Gamma'_{000} = \Gamma''_{000}$$
$$\Theta_0 = \Theta'_{00} = \Theta''_{000}, \quad \Theta'_0 = \Theta''_{00} = \Theta_{000}, \quad \Theta''_0 = \Theta_{00} = \Theta'_{000}$$

Constantes igitur duodeviginti Γ atque Θ denotatae per has aequationes ad quatuor independentes reductae sunt.

26.

Si terminus ipsarum $f_{,}t$, $\varphi_{,}t$, $f_{,,}t$ atque $\varphi_{,,}t$, qui terminos primi ordinis respectu ipsarum $C_{,}$, $C'_{,,}$, C'' et 0^{ti} ordinis respectu massarum proferre potest, solummodo consideratur, habetur

$$f_{,}t = \pi_{,}Q_{,}\,;\quad \varphi_{,}t = \pi_{,}P_{,}\,,\quad f_{,,}t = \pi_{,,}Q''\,,\quad \varphi_{,,}t = \pi_{,,}P''$$

in quibus valores (45) ipsarum $P_{,}$, $Q_{,}$, etc. substituendi sunt. Quibus factis, adiumento aequationum (62) Sect. II. eliciuntur

$$\begin{aligned}
c = (\pi_{,}-\pi_{,,}b'')C_{,}\,,\quad H &= \{(\pi_{,}b_{,}-\pi_{,,}f'')\,C'_{,,}\cos(l''t+\lambda'')+(\pi_{,}f_{,}-\pi_{,,})\,C''\cos(l'_{,,}t+\lambda'_{,,})\}\,u \\
L &= -\{(\pi_{,}b_{,}-\pi_{,,}f'')\,C'_{,,}\sin(l''t+\lambda'')-(\pi_{,}f_{,}-\pi_{,,})\,C''\sin(l'_{,,}t+\lambda'_{,,})\}\,u \\
M &= \{(\pi_{,}b_{,}-\pi_{,,}f'')\,C'_{,,}\cos(l''t+\lambda'')+(\pi_{,}f_{,}-\pi_{,,})\,C''\cos(l'_{,,}t+\lambda'_{,,})\}\,p_0 \\
&\quad -\{(\pi_{,}b_{,}-\pi_{,,}f'')\,C'_{,,}\sin(l''t+\lambda'')-(\pi_{,}f_{,}-\pi_{,,})\,C''\sin(l'_{,,}t+\lambda'_{,,})\}\,q_0
\end{aligned}$$

Iam applicatis evolutionibus integralium (64) Sect. II. in art. 2. Sect. VI. datis ad hos ipsarum c, H, L atque M valores, neglectisque coefficientium terminis, qui secundi et altioris ordinis respectu ipsarum $C_{,}$, $C'_{,,}$ atque C'' sunt, emergunt

$$\begin{aligned}
p_0 &= \sin\Gamma\sin(x_0t-\Theta_0) + \cos\Gamma\,\frac{\pi_{,}b_{,}-\pi_{,,}f''}{\alpha'_{,,}}\,C'_{,,}\sin(l''t+\lambda'') \\
&\quad - \cos\Gamma\,\frac{\pi_{,}f_{,}-\pi_{,,}}{\alpha''}\,C''\sin(l'_{,,}t+\lambda'_{,,}) \\
q_0 &= \sin\Gamma\cos(x_0t-\Theta_0)+\cos\Gamma\,\frac{\pi_{,}-\pi_{,,}b''}{\alpha_{,}}\,C_{,}+\cos\Gamma\,\frac{\pi_{,}b_{,}-\pi_{,,}f''}{\alpha'_{,,}}\,C'_{,,}\cos(l''t+\lambda'') \\
&\quad + \cos\Gamma\,\frac{\pi_{,}f_{,}-\pi_{,,}}{\alpha''}\,C''\cos(l'_{,,}t+\lambda'_{,,}) \\
u &= \cos\Gamma-\sin\Gamma\,\frac{\pi_{,}-\pi_{,,}b''}{\alpha_{,}}\,C_{,}\cos(x_0t-\Theta_0)-\sin\Gamma\,\frac{\pi_{,}b_{,}-\pi_{,,}f''}{\alpha'_{,,}}\,C'_{,,}\cos[(l''-x_0)t+\lambda''+\Theta_0] \\
&\quad -\sin\Gamma\,\frac{\pi_{,}f_{,}-\pi_{,,}}{\alpha''}\,C''\cos[(l'_{,,}+x_0)t+\lambda'_{,,}-\Theta_0]
\end{aligned}$$

e quibus mutatis mutandis expressiones analogae ipsarum p'_0, q'_0, u', p''_0, q''_0 atque u'' emergunt. Idem calculus, adhibitis aequationibus conditionalibus inter constantes Γ_0 etc. atque Θ_0 etc. in art. praec. inventis, suppeditat

$$p_{00} = \sin\Gamma \sin(x_{00}t - \Theta''_0) \qquad + \cos\Gamma \frac{\pi_, - \pi_{,,} b''}{\alpha_,} C_, \sin(l'_{,,}t + \lambda'_{,,})$$

$$- \cos\Gamma \frac{\pi_, b_, - \pi_{,,} f''}{\alpha'_{,,}} C'_{,,} \sin(l_, t + \lambda_,)$$

$$q_{00} = \sin\Gamma \cos(x_{00}t - \Theta''_0) + \cos\Gamma \frac{\pi_, f_, - \pi_{,,}}{\alpha''} C'' + \cos\Gamma \frac{\pi - \pi_{,,} b''}{\alpha_,} C_, \cos(l'_{,,}t + \lambda'_{,,})$$

$$+ \cos\Gamma \frac{\pi_, b_, - \pi_{,,} f''}{\alpha'_{,,}} C'_{,,} \cos(l_, t + \lambda_,)$$

$$u = \cos\Gamma - \sin\Gamma \frac{\pi_, f_, - \pi_{,,}}{\alpha''} C'' \cos(x_{00}t - \Theta''_0) - \sin\Gamma \frac{\pi_, - \pi_{,,} b''}{\alpha_,} C_, \cos[(l'_{,,} - x_{00})t + \lambda'_{,,} + \Theta''_0]$$

$$- \sin\Gamma \frac{\pi_, b_, - \pi_{,,} f''}{\alpha'_{,,}} C'_{,,} \cos[(l_, + x_{00})t + \lambda_, - \Theta''_0]$$

$$p_{000} = \sin\Gamma \sin(x_{000}t - \Theta'_0) \qquad + \cos\Gamma \frac{\pi_, f_, - \pi_{,,}}{\alpha''} C'' \sin(l_, t + \lambda_,)$$

$$- \cos\Gamma \frac{\pi_, - \pi_{,,} b''}{\alpha_,} C_, \sin(l'' t + \lambda'')$$

$$q_{000} = \sin\Gamma \cos(x_{000}t - \Theta'_0) + \cos\Gamma \frac{\pi_, b_, - \pi_{,,} f''}{\alpha'_{,,}} C'_{,,} + \cos\Gamma \frac{\pi_, f_, - \pi_{,,}}{\alpha''} C'' \cos(l_, t + \lambda_,)$$

$$+ \cos\Gamma \frac{\pi_, - \pi_{,,} b''}{\alpha_,} C_, \cos(l'' t + \lambda'')$$

$$u = \cos\Gamma - \sin\Gamma \frac{\pi_, b_, - \pi_{,,} f''}{\alpha'_{,,}} C'_{,,} \cos(x_{000}t - \Theta'_0) - \sin\Gamma \frac{\pi_, f_, - \pi_{,,}}{\alpha''} C'' \cos[(l_, - x_{000})t + \lambda_, + \Theta'_0]$$

$$- \sin\Gamma \frac{\pi_, - \pi_{,,} b''}{\alpha_,} C_, \cos[(l'' + x_{000})t + \lambda'' - \Theta'_0]$$

e quibus mutatis mutandis expressiones analogae ipsarum p'_{00}, q'_{00}, u', p''_{00}, q''_{00}, u'', p'_{000}, q'_{000}, u', p''_{000}, q''_{000}, u'' eliciuntur. Si supponitur, ε'' esse radicem aequationum (57) eam, quae secundi ordinis est, secundum art. 17. constans C'' tertii ordinis est, coefficientes vero terminorum expressionum praecedentium, qui per C'' multiplicati sunt, propter divisorem α'' primi ordinis esse videntur. Quum aequationes tamen (46) facili calculo monstrent, in hoc casu numeratorem $\pi_, f_, - \pi_{,,}$ necessario secundi ordinis esse debere, termini ipsarum p_0, q_0, etc. per C'' multiplicati tertii ordinis sunt, et in hoc calculo, ubi terminos primi tantum ordinis respicimus, omitti potuissent. Ratio vero, cur non omissi sint, est, quod mutatis mutandis in expressionibus ipsarum p'_0, q'_0 etc. terminos per $C_,$ atque $C'_{,,}$ multiplicatos proferunt, qui primi ordinis sunt.

Aequationes finitae ipsarum p_0, q_0, p_{00}, etc. art. 8. suppeditant

$$p_0 = p_{00}\cos(l'_{,,}t+\lambda'_{,,}) - q_{00}\sin(l'_{,,}t+\lambda'_{,,}) = p_{000}\cos(l''t+\lambda'') + q_{000}\sin(l''t+\lambda'')$$
$$q_0 = q_{00}\cos(l'_{,,}t+\lambda'_{,,}) + p_{00}\sin(l'_{,,}t+\lambda'_{,,}) = q_{000}\cos(l''t+\lambda'') - p_{000}\sin(l''t+\lambda'')$$

Substitutis valoribus praecedentibus ipsarum p_0, q_0, p_{00}, etc. in his aequationibus, emergunt aequationes conditionales hae

$$x_0 = x_{00} - l'_{,,} = x_{000} + l'', \quad \Theta_0 = \Theta''_0 + \lambda'_{,,} = \Theta'_0 - \lambda''$$

quae mutatis mutandis subministrant

$$x'_0 = x'_{00} - l'' = x'_{000} + l_{,}\,, \quad \Theta'_0 = \Theta_0 + \lambda'' = \Theta''_0 - \lambda_{,}\,,$$
$$x''_0 = x''_{00} - l_{,} = x''_{000} + l'_{,,}, \quad \Theta''_0 = \Theta'_0 + \lambda_{,} = \Theta_0 - \lambda'_{,,}$$

quarum ope expressiones tres praecedentes ipsius u, quae diversae indolis esse videntur, congruunt. Eaedem aequationes praeterea suppeditant

$$\Theta'_0 = \Theta_0 + \lambda'', \quad \Theta''_0 = \Theta_0 - \lambda'_{,,}$$

unde constantes arbitrariae Γ atque Θ, quarum numerus ab initio duodeviginti erat, ad duas constantes Γ_0 atque Θ_0 independentes reductae sunt.

Eodem denique modo, quo in art. 33. Sect. II. usi sumus, demonstratur integrationes, quas reductio longitudinis corporum m, m' atque m'' postulat, unam tantum constantem ibidem Π denotatam introducere, quae longitudinem nodi ascendentis plani, ad quod Γ spectat, cum plano fundamentali denotat.

27.

Ad significationem ipsarum Γ et Θ_0 indagandam eligamus expressiones ipsarum p_0, p'_{00}, p''_{000}, q_0, etc. ex articulo praecedente. Quae, omissis terminis tertii ordinis per C'' multiplicatis, factoque tempore cifrae aequali, sunt

$$p_0 = -\sin\Gamma\sin\Theta_0 + \cos\Gamma\frac{\varkappa_{,}b_{,} - \varkappa_{,,}f''}{\alpha'_{,,}}C'_{,,}\sin\lambda''$$
$$p'_{00} = -\sin\Gamma\sin\Theta_0 + \cos\Gamma\frac{\varkappa'_{,,} - \varkappa' b_{,}}{\alpha'_{,,}}C'_{,,}\sin\lambda''$$
$$p''_{000} = -\sin\Gamma\sin\Theta_0 + \cos\Gamma\frac{\varkappa''f'' - \varkappa''_{,}}{\alpha'_{,,}}C'_{,,}\sin\lambda''$$

$$q_0 \;= \sin\Gamma\cos\Theta_0 + \cos\Gamma\,\frac{\pi_{,}-\pi_{,,}b''}{\alpha_{,}}\,C_{,} + \cos\Gamma\,\frac{\pi_{,}b_{,}-\pi_{,,}f''}{\alpha'_{,,}}\,C'_{,,}\cos\lambda''$$

$$q'_{00} = \sin\Gamma\cos\Theta_0 + \cos\Gamma\,\frac{\pi'_{,,}f'_{,,}-\pi'}{\alpha_{,}}\,C_{,} + \cos\Gamma\,\frac{\pi'_{,,}-\pi' b_{,}}{\alpha'_{,,}}\,C'_{,,}\cos\lambda''$$

$$q''_{000} = \sin\Gamma\cos\Theta_0 + \cos\Gamma\,\frac{\pi''b''-\pi_{,}''f'_{,,}}{\alpha_{,}}\,C_{,} + \cos\Gamma\,\frac{\pi''f''-\pi_{,}''}{\alpha'_{,,}}\,C'_{,,}\cos\lambda''$$

$$u \;= \cos\Gamma - \sin\Gamma\,\frac{\pi_{,}-\pi_{,,}b''}{\alpha_{,}}\,C_{,}\cos\Theta_0 - \sin\Gamma\,\frac{\pi_{,}b_{,}-\pi_{,,}f''}{\alpha'_{,,}}\,C'_{,,}\cos(\lambda''+\Theta_0)$$

$$u' = \cos\Gamma - \sin\Gamma\,\frac{\pi'_{,,}f'_{,,}-\pi'}{\alpha_{,}}\,C_{,}\cos\Theta_0 - \sin\Gamma\,\frac{\pi'_{,,}-\pi' b_{,}}{\alpha'_{,,}}\,C'_{,,}\cos(\lambda''+\Theta_0)$$

$$u'' = \cos\Gamma - \sin\Gamma\,\frac{\pi''b''-\pi_{,}''f'_{,,}}{\alpha_{,}}\,C_{,}\cos\Theta_0 - \sin\Gamma\,\frac{\pi''f''-\pi_{,}''}{\alpha'_{,,}}\,C'_{,,}\cos(\lambda''+\Theta_0)$$

Eliminatis $C_{,}$ atque $C'_{,,}$ ex his expressionibus, nanciscimur

$$\left.\begin{aligned} -\sin\Gamma\sin\Theta_0 &= Ap_0 + Bp'_{00} + Cp''_{000} \\ \sin\Gamma\cos\Theta_0 &= Aq_0 + Bq'_{00} + Cq''_{000} \\ \cos\Gamma &= Au + Bu' + Cu'' \end{aligned}\right\} \ldots\ldots(58)$$

ubi A, B, C coefficientes noti sunt, qui per eliminationem duarum quantitatum ex tribus aequationibus primi gradus prodeunt. Si vero aequationes has $o = 1 + b'' + f'_{,,}$, $o = 1 + b_{,} + f''$, quae omissis quantitatibus secundi ordinis respectu quantitatum finitarum b'', etc., ex (47) derivatae sunt, et aequationem facile confirmandam hanc $\pi_{,}\pi'_{,,}\pi'' = \pi'\pi''_{,}\pi_{,,}$ in usum vocaveris, invenies, eliminatione revera instituta,

$$A = \frac{\pi'\pi''}{\pi'\pi''+\pi_{,}\pi''+\pi_{,,}\pi'},\quad B = \frac{\pi_{,}''\pi_{,}}{\pi_{,}''\pi_{,}+\pi_{,}\pi'_{,,}+\pi_{,}''\pi'},\quad C = \frac{\pi_{,,}\pi'_{,,}}{\pi_{,,}\pi'_{,,}+\pi'_{,,}\pi''+\pi_{,,}\pi_{,}''}$$

In aequationibus vero art. 12. his $\pi_{,} = -\frac{an}{2\sqrt{1-e^2}}l_4$, $\pi' = -\frac{a'n'}{2\sqrt{1-e'^2}}h'_4$, etc. est $-\frac{l_4}{2}$ huius formae $\frac{m'}{M+m}D_{,}$, $-\frac{h'_4}{2}$ huius formae $\frac{m}{M+m'}D_{,}$ etc. ubi $D_{,}$ constans est ex semiaxibus maioribus a atque a' pendens; hinc sequitur, denotantibus $E_{,}$, $E'_{,,}$, E'' functiones semiaxium maiorum, haberi

$$\pi_{,} = \frac{m'}{a^2 n\sqrt{1-e^2}}E_{,},\quad \pi''_{,,} = \frac{m''}{a'^2 n'\sqrt{1-e'^2}}E'_{,,},\quad \pi'' = \frac{m}{a''^2 n''\sqrt{1-e''^2}}E''$$

$$\pi' = \frac{m}{a'^2 n'\sqrt{1-e'^2}}E_{,},\quad \pi''_{,} = \frac{m'}{a''^2 n''\sqrt{1-e''^2}}E'_{,,},\quad \pi_{,,} = \frac{m''}{a^2 n\sqrt{1-e^2}}E''$$

quibus aequationes (58) transeunt in has

$$-\operatorname{tg}\Gamma\sin\Theta_0=\frac{ma^2n\sqrt{1-e^2}.\sin i\sin(\theta+F)+m'a'^2n'\sqrt{1-e'^2}.\sin i'\sin(\theta'+F)+m''a''^2n''\sqrt{1-e''^2}.\sin i''\sin(\theta''+F)}{ma^2n\sqrt{1-e^2}.\cos i+m'a'^2n'\sqrt{1-e'^2}.\cos i'+m''a''^2n''\sqrt{1-e''^2}.\cos i''}$$

$$\operatorname{tg}\Gamma\cos\Theta_0=\frac{ma^2n\sqrt{1-e^2}.\sin i\cos(\theta+F)+m'a'^2n'\sqrt{1-e'^2}.\sin i'\cos(\theta'+F)+m''a''^2n''\sqrt{1-e''^2}.\sin i''\cos(\theta''+F)}{ma^2n\sqrt{1-e^2}.\cos i+m'a'^2n'\sqrt{1-e'^2}.\cos i'+m''a''^2n''\sqrt{1-e''^2}.\cos i''}$$

ubi θ, θ', θ'' longitudines nodorum ascendentium orbitarum m, m', m'' cum plano proiectionis seu fundamentali, tempori $t=o$ respondentes denotant, et, secundum art. 8.,

$$F=-\pi+v_,+k_,=-\pi'+v_,-k_,=-\pi''+v_,-k'_{,,}+k''\quad,$$

quae quantitates ad minimum usque ad quantitates secundi ordinis in hoc calculo ceteroquin omissas, sibi aequales sunt.

Aequationes praecedentes monstrant, Γ esse inclinationem plani invariabilis versus planum fundamentale, et Θ_0 arcum plani invariabilis a nodo eius ascendente cum plano fundamentali usque ad punctum quoddam adhuc determinandum ductum.

28.

Ex expressione latitudinis significationes reliquarum constantium facile inveniuntur. Habemus ad instar art. 33. Sect. II.

$$s=q_0\sin[\overline{f}+(y+\alpha_,-\eta_,)t+v_,+k_,]-p_0\cos[\overline{f}+(y+\alpha_,-\eta_,)t+v_,+k_,]$$

ubi $\overline{f}$ denotat anomaliam veram, et ex hac expressione obtinemus mutatis mutandis expressiones sinuum latitudinum s', s'' corporum m' atque m''. Substitutis valoribus ipsarum q_0, p_0 etc. in praecedentibus datis, emergunt

$$s=\sin\Gamma\sin[\overline{f}+(y+\alpha_,-\eta_,-x_0)t+v_,+k_,+\Theta_0]+\cos\Gamma\frac{\pi_,-\pi_{,,}b''}{\alpha_,}C_,\sin[\overline{f}+(y+\alpha_,-\eta_,)t+v_,+k_,]$$
$$+\cos\Gamma\frac{\pi_,b_,-\pi_{,,}f''}{\alpha'_{,,}}C'_{,,}\sin[\overline{f}+(y+\alpha'_{,,}-\eta_,+\eta'')t+v'_{,,}+k_,-k'')+\cos\Gamma\frac{\pi_,b_,-\pi_{,,}}{\alpha''}C''\sin[\overline{f}+(y+\alpha''+\eta'')t+v''-k'']$$

$$s'=\sin\Gamma\sin[\overline{f}'+(y'+\alpha_,+\eta_,-x'_{00})t+v_,-k_,+\Theta_0]+\cos\Gamma\frac{\pi_{,,}'f_{,,}'-\pi'}{\alpha_,}C_,\sin[\overline{f}'+(y'+\alpha_,+\eta_,)t+v_,-k_,]$$
$$+\cos\Gamma\frac{\pi_{,,}'-\pi'b_{,,}'}{\alpha_{,,}'}C'_{,,}\sin[\overline{f}'+(y'+\alpha'_{,,}-\eta'_{,,})t+v'_{,,}+k'_{,,}]+\cos\Gamma\frac{\pi_{,,}'b_{,,}'-\pi'f_,}{\alpha''}C''.\sin[\overline{f}'+(y'+\alpha''-\eta'_{,,}+\eta_,)t+v''+k'_{,,}-k_,]$$

$$s''=\sin\Gamma\sin[\overline{f}''+(y''+\alpha_,+\eta'_{,,}-\eta''-x''_{000})t+v_,-k'_{,,}+k''+\Theta_0)+\cos\Gamma\frac{\pi''b''-\pi_,''f_{,,}'}{\alpha_,}C_,\sin[\overline{f}''+(y''+\alpha_,-\eta''+\eta'_{,,})t+v_,+k''-k'_{,,}]$$
$$+\cos\Gamma\frac{\pi''f''-\pi_,''}{\alpha'_{,,}}C'_{,,}\sin[\overline{f}''+(y''+\alpha'_{,,}+\eta'_{,,})t+v'_{,,}-k'_{,,})+\cos\Gamma\frac{\pi''-\pi_,''b_{,,}'}{\alpha''}C''\sin[\overline{f}''+(y''+\alpha''-\eta'')t+v''+k'']$$

quae expressiones hoc modo depinguntur.

Delineetur triangulum sphaericum rectangulum, cuius hypotenusa repraesentet plani invariabilis pars, et sit amplitudinis $(y+\alpha_{,}-\eta_{,}-x_{0})t+v_{,}+k_{,}+\Theta_{0}$, et cuius catheta altera faciat angulum finitum Γ cum hypotenusa, ideo repraesentet plani fundamentalis pars; si huius trianguli alterius cathetae sinus S nominatus fuerit, erit $S = \sin\Gamma \sin[(y+\alpha_{,}-\eta_{,}-x_{0})t+v_{,}+k_{,}+\Theta_{0})$. Produc S extra planum invariabile et duc circulum maximum (A) a producta S usque ad planum invariabile extensum, qui sit amplitudinis $(y+\alpha_{,}-\eta_{,})t+v_{,}+k_{,}$ et faciat angulum primi ordinis $=\frac{\pi_{,}-\pi_{,,}b''}{\alpha_{,}}C_{,}$ cum plano invariabili. Produc iterum S, et duc circulum maximum (B) ab iterum producta S usque ad circulum (A) extensum, qui sit amplitudinis $(y+\alpha'_{,,}-\eta_{,}+\eta'')t+v'_{,,}-k''+k_{,}$ et faciat angulum primi ordinis $=\frac{\pi_{,}b_{,}-\pi_{,,}f''}{\alpha'_{,,}}C'_{,,}$ cum circulo (A). Denique produc denuo S, et duc circulum maximum (C) a denuo producta S usque ad circulum (B) extensum, qui sit amplitudinis $(y+\alpha''+\eta'')t+v''-k''$ et faciat angulum tertii ordinis $=\frac{\pi_{,}f_{,}-\pi_{,,}}{\alpha''}C''$ cum circulo (B). Quibus factis, intersectio circuli (C) cum S erit locus perihelii corporis m pro tempore t, nec non $-x_{0}t+\Theta_{0}$ arcus plani invariabilis a nodo eius ascendenti cum plano fundamentali usque ad nodum eius cum circulo maximo sive plano (A) extensus, $(\alpha_{,}-\alpha'_{,,}-\eta'')t+v_{,}-v'_{,,}+k''$ sive $l''t+\lambda''$ arcus plani (A) a hoc nodo usque ad nodum plani (B) cum (A) extensus, $(\alpha'_{,,}-\alpha''-\eta_{,})t+v'_{,,}-v''+k_{,}$ sive $l_{,}t+\lambda_{,}$ arcus plani (B) a hoc nodo usque ad nodum plani (C) cum (B) extensus, et denique $(y+\alpha''+\eta'')t+v''-k''$ arcus plani (C) a hoc nodo usque ad perihelii corporis m locum, qui non minus quam hic planorum (A), (B), (C) situs tempori t correspondet, extensus.

Fac ut trianguli illius sphaerici rectanguli hypotenusa sit amplitudinis $(y'+\alpha_{,}+\eta_{,}-x'_{00})t+v_{,}-k_{,}+\Theta_{0}$ et lege eadem qua supra duc circulos maximos (A'), (B'), (C'), quorum quisque resp. inclinationem cum priore habeat $\frac{\pi'_{,,}f'_{,,}-\pi'}{\alpha_{,}}C_{,}$, $\frac{\pi'_{,,}-\pi'b_{,,}}{\alpha'_{,,}}C'_{,,}$, $\frac{\pi'_{,,}b'_{,}-\pi'f_{,}}{\alpha''}C''$, tum erit $(y'+\alpha''-\eta'_{,,}+\eta_{,})t+v''+k'_{,,}-k_{,}$ arcus plani (C') a nodo huius plani cum plano (B') usque ad locum perihelii corporis m' tempori t respondentem extensus, et arcus $-x'_{00}t+\Theta_{0}$, $l''t+\lambda''$, $l_{,}t+\lambda_{,}$ ad planum invariabile et plana (A'), (B'), (C') eandem rationem insistunt, ut supra ad planum invariabile et plana (A), (B), (C).

Porro fac ut eiusdem trianguli sphaerici rectanguli hypotenusa sit amplitudinis $(y''+\alpha_{,}+\eta'_{,,}-\eta''-x''_{000})t+v_{,}-k'_{,,}+k''+\Theta_{0}$, et duc circulos maximos

(A''), (B''), (C''), quorum quisque cum priore inclinationem resp. habeat $\frac{\pi'' b'' - \pi_{,}'' f'_{,,}}{\alpha_{,}} C_{,}$, $\frac{\pi'' f'' - \pi_{,}''}{\alpha'_{,,}} C'_{,,}$, $\frac{\pi'' - \pi_{,}'' b'_{,,}}{\alpha''} C''$, tum erit $(y'' + \alpha'' - \eta'')t + v'' + k''$ arcus plani (C'') a nodo eius cum plano (B'') usque ad locum perihelii corporis m'' tempori t respondentem extensus, et arcus $-x''_{000} + \Theta_0$, $l''t + \lambda''$, $l_{,}t + \lambda_{,}$ iterum ad planum invariabile et plana (A''), (B''), (C'') eandem rationem insistunt quam illi.

Denique animadvertendum est, nodos hos planorum (A), (A'), (A'') cum plano invariabili, planorum (B), (B'), (B'') resp. cum (A), (A'), (A'') et planorum (C), (C'), (C'') resp. cum (B), (B'), (B'') esse ascendentes, quoties inclinatio respectiva $\frac{\pi_{,} - \pi_{,,} b''}{\alpha_{,}} C_{,}$, $\frac{\pi_{,} b_{,} - \pi_{,,} f''}{\alpha'_{,,}} C'_{,,}$, etc. est positiva, descendentes vero, quoties inclinatio est negativa quantitas.

Si in his figuris statueris $t = o$, significationem constantium $C_{,}$, $v_{,}$, $k_{,}$, $C'_{,,}$, etc. habes, et delineatione analoga significatio constantium e, e', e'', quae quasi excentricitates veluti $C_{,}$, $C'_{,,}$, C'' quasi inclinationes mutuae sunt, explicari potest.

Congeramus constantes arbitrarias independentes omnes, quae in formulis huius Sectionis continentur.

Independentes sunt rationes massarum m, m', m'' ad massam M 3

Porro motus medii n, n', n'', e quibus tertia Keppleri lege semiaxes maiores a, a', a'' pendent 3

Porro quasi excentricitates e, e', e'' 3

Porro independentes (c), (c'), (c''), quae in anomaliis mediis continentur et anomalias medias tempori $t = o$ respondentes denotant . . 3

Quum inter novem constantes $C_{,}$, $C'_{,,}$, C'', $v_{,}$, $v'_{,,}$, v'', $k_{,}$, $k'_{,,}$, k'' tres in praecedentibus explicatae aequationes conditionales adsint, hae constantes sex independentibus aequiparant 6

Denique independentes sunt Γ, Θ_0 atque Π, quarum priores modo explicatae sunt, et quarum tertia longitudinem nodi ascendentis plani invariabilis cum plano fundamentali denotat 3

Quarum omnium est numerus 21

sive ter septem, quarum ter sex elementorum ellipticorum trium corporum m, m' atque m'' vice funguntur.

29.

Ad constantes arbitrarias observationibus astronomicis determinandas recipiantur in primo calculo solummodo termini primi ordinis tum in coefficientibus cum in perturbationum periodis. Quo facto, habemus secundum praecedentia

$$\begin{aligned}
nz &= g+2e's\sin[g+(y-y')t+2k_,] && +2e''f\sin[g-(y''-y)t+2k_,+2k'_{,,}]\\
n'z' &= g'+2e''s'\sin[g'+(y'-y'')t+2k'_{,,}] && +2ef'\sin[g'-(y-y')t-2k_,]\\
n''z'' &= g''+2es''\sin[g''+(y''-y)t-2k_,-2k'_{,,}] && +2e'f''\sin[g''-(y'-y'')t-2k'_{,,}]\\
w &= -e's\cos[g+(y-y')t+2k_,] && -e''f\cos[g-(y''-y)t+2k_,+2k'_{,,}]\\
w' &= -e''s'\cos[g'+(y'-y'')t+2k'_{,,}] && -ef'\cos[g'-(y-y')t-2k_,]\\
w'' &= -es''\cos[g''+(y''-y)t-2k_,-2k'_{,,}] && -e'f''\cos[g''-(y'-y'')t-2k'_{,,}]
\end{aligned}$$

$$\bar{f} = nz+A_{\rm I}\sin nz+A_2\sin 2nz+\text{etc.}, \qquad \bar{f}' = n'z'+A'_{\rm I}\sin n'z'+\text{etc.},$$
$$\bar{f}'' = n''z''+A''_{\rm I}\sin n''z''+\text{etc.}$$

$$lr = la+w+B_0+B_{\rm I}\cos nz+B_2\cos 2nz+\text{etc.}, \qquad lr' = la'+w'+B'_0+B'_{\rm I}\cos n'z'+\text{etc.},$$
$$lr'' = la''+w''+B''_0+B''_{\rm I}\cos n''z''+\text{etc.}$$

quae, denotantibus $A_{\rm I}$, A_2, etc. $A'_{\rm I}$ etc. $A''_{\rm I}$ etc. coefficientes notos aequationis centri et B_0, $B_{\rm I}$ etc. B'_0, etc. B''_0, etc. coefficientes notos evoluti logarithmi radii vectoris, resp. ope excentricitatum e, e' atque e'' computandos, anomalias veras $\bar{f}$, $\bar{f}'$, $\bar{f}''$ atque logarithmos radiorum vectorum r, r', r'' suppeditant. Porro

$$\begin{aligned}
s = \sin\Gamma\sin(\bar{f}+yt+v_,+k_,+\Theta_0) &+ \cos\Gamma\frac{\pi_,-\pi_{,,}b''}{\alpha_,}C_,\sin[\bar{f}+(y+\alpha_,)t+v_,+k_,]\\
&+ \cos\Gamma\frac{\pi_,b_,-\pi_{,,}f''}{\alpha'_{,,}}C'_{,,}\sin[\bar{f}+(y+\alpha'_{,,})t+v'_{,,}+2k_,+k'_{,,}]
\end{aligned}$$

$$\begin{aligned}
s' = \sin\Gamma\sin(\bar{f}'+y't+v_,-k_,+\Theta_0) &+ \cos\Gamma\frac{\pi_{,,}'f_{,,}'-\pi'}{\alpha_,}C_,\sin[\bar{f}'+(y'+\alpha_,)t+v_,-k_,]\\
&+ \cos\Gamma\frac{\pi_{,,}'-\pi'b_,}{\alpha_{,,}'}C'_{,,}\sin[\bar{f}'+(y'+\alpha'_{,,})t+v'_{,,}+k'_{,,}]
\end{aligned}$$

$$\begin{aligned}
s'' = \sin\Gamma\sin(\bar{f}''+y''t+v_,-k_,-2k'_{,,}+\Theta_0) &+ \cos\Gamma\frac{\pi''b''-\pi_,''f_{,,}'}{\alpha_,}C_,\sin[\bar{f}''+(y''+\alpha_,)t+v_,-k_,-2k'_{,,}]\\
&+ \cos\Gamma\frac{\pi''f''-\pi_,''}{\alpha'_{,,}}C'_{,,}\sin[\bar{f}''+(y''+\alpha'_{,,})t+v'_{,,}-k'_{,,}]
\end{aligned}$$

quae sinus latitudinum s, s', s'' versus planum fundamentale subministrant. Denique, denotantibus l, l', l'' longitudines ad planum fundamentale reductas, habemus per aequationem (10) Sect. VI.

$$l = \bar{f} + y\,t + \nu_{,} + k_{,} + \Theta_0 + \Pi + R - \delta s \frac{\operatorname{tg} \Gamma \cos(\bar{f} + yt + \nu_{,} + k_{,} + \Theta_0)}{\cos^2 B}$$

$$l' = \bar{f}' + y'\,t + \nu_{,} - k_{,} + \Theta_0 + \Pi + R' - \delta s' \frac{\operatorname{tg} \Gamma \cos(\bar{f}' + y't + \nu_{,} - k_{,} + \Theta_0)}{\cos^2 B'}$$

$$l'' = \bar{f}'' + y''t + \nu_{,} - 2k'_{,,} - k_{,} + \Theta_0 + \Pi + R'' - \delta s'' \frac{\operatorname{tg} \Gamma \cos(\bar{f}'' + y''t + \nu_{,} - 2k_{,,}' - k_{,} + \Theta_0)}{\cos^2 B''}$$

ubi primus ipsarum s, s', s'' terminus resp. per $\sin B$, $\sin B'$, $\sin B''$, reliqui vero earum termini per δs, $\delta s'$, $\delta s''$ repraesentati sunt, et praeterea habentur

$$\operatorname{tg} R = -\frac{\operatorname{tg}^2 \tfrac{1}{2} \Gamma \sin 2(\bar{f} + yt + \nu_{,} + k_{,} + \Theta_0)}{1 + \operatorname{tg}^2 \tfrac{1}{2} \Gamma \cos 2(\bar{f} + yt + \nu_{,} + k_{,} + \Theta_0)}$$

$$\operatorname{tg} R' = -\frac{\operatorname{tg}^2 \tfrac{1}{2} \Gamma \sin 2(\bar{f}' + y't + \nu_{,} - k_{,} + \Theta_0)}{1 + \operatorname{tg}^2 \tfrac{1}{2} \Gamma \cos 2(\bar{f}' + y't + \nu_{,} - k_{,} + \Theta_0)}$$

$$\operatorname{tg} R'' = -\frac{\operatorname{tg}^2 \tfrac{1}{2} \Gamma \sin 2(\bar{f}'' + y''t + \nu_{,} - 2k_{,,}' - k_{,} + \Theta_0)}{1 + \operatorname{tg}^2 \tfrac{1}{2} \Gamma \cos 2(\bar{f}'' + y''t + \nu_{,} - 2k_{,,}' - k_{,} + \Theta_0)}$$

Iam si massae m, m', m'' et semiaxes maiores orbitarum noti sunt, quantitates μ, μ', μ'', σ, σ', σ'', $\pi_{,}$, π', π'', etc. computari possunt, et tum adiumento aequationum (40) et (46), omissis $\eta_{,}$, $\eta'_{,,}$, η'', coefficientium s, s', s'', f, etc., $b_{,}$, $f_{,}$ etc. nec non periodorum y, y', y'', $\alpha_{,}$, $\alpha'_{,,}$ valores approximati eliciuntur. Quibus factis, aequationes praecedentes constantes arbitrarias independentes duodeviginti n, n', n'', (c), (c'), (c''), e, e', e'', $C_{,}$, $C'_{,,}$, $\nu_{,}$, $\nu'_{,,}$, $k_{,}$, $k'_{,,}$, Γ, Θ_0, Π continent, quae, electo plano fundamentali ad libitum, per methodos notas observationibus astronomicis determinandae sunt; et quo maior est observationum ad hanc determinationem adhibitarum numerus et quanto longius temporis intervallum complectitur, tanto accuratiores constantium valores eliciuntur, quia eo magis reliquarum minorum perturbationum periodicarum effectus destruitur. Constantibus his determinatis, aequatio $k'' = -k_{,} - k'_{,,}$ suppeditat ipsam k'' et aequationes hae

$$C'' \sin\lambda'_{''} = \frac{1+b_{,}+f''}{1+b_{,,}'+f_{,}} C_{''} \sin(v_{,}-v'_{''}+k'')$$

$$C'' \cos\lambda'_{''} = -\frac{1+b_{,}+f''}{1+b_{,,}'+f_{,}} C'_{''} \cos(v_{,}-v'_{''}+k'') - \frac{1+b''+f_{,,}'}{1+b_{,,}'+f_{,}} C_{,}$$

$$v'' = \lambda'_{''}+v_{,}-k'_{''}$$

ipsas C'' atque v''. Iam nunc ope formularum in hac Sectione expositarum non modo $\eta_{,}$, $\eta'_{''}$, η'', sed etiam reliqui aequationum (54) et (57) termini, nec non perturbationes reliquae magna cum praecisione computari possunt, quo facto valores constantium arbitrariarum observationibus nanciscimur accuratiores, quibus computatio perturbationum repetitur, et sic porro. Plerumque tamen constantium illarum ex primo calculo eruti valores ad perturbationes omnes, quae vim habent, satis accurate computandas sufficiunt, quin etiam in casibus quibusdam specialibus formulis in huius articuli initio expositis statim perturbationum reliquarum maiores terminos aggregare nobis licet.

In hac expositione semiaxium maiorum et massarum ab initio valores approximatos esse notos supposuimus, et quidem illi, qui ex motibus pendent mediis, quos ex observationibus longum temporis intervallum complectentibus simplicissimo calculo eruere possumus, semper noti haberi possunt. Verum enim vero si massae perturbantes ignotae sunt, periodorum y, y', y'', $\alpha_{,}$, $\alpha'_{''}$ valores observationibus eliciendi sunt, quibus adiumento aequationum (37) vel (44) approximati massarum valores computari possunt.

Haec sunt, quae generaliter de constantium arbitrariarum determinatione statuere licet, in casibus vero specialibus haec vel illa systematis quatuor corporum conditio determinationem hanc sublevabit atque simplificabit.

30.

Abscissis aequationum in praecedentibus evolutarum terminis, qui ad quartum corpus m'' pertinent, $P_{,}$, $Q_{,}$ atque $K_{,}$ eaedem evadunt, uti in praecedenti Lunae theoria. Fiunt enim

$$P_{,} = \int R_{,}dt\,;\ \ Q_{,} = C_{,} + \int U_{,}dt\,;\ \ K_{,} = k_{,} + \int V_{,}dt\,,\ \ \alpha_{,} = g_{,} + \pi_{,} + \pi'_{,}$$

quae, substitutis dextrorum membrorum valoribus huic casui respondentibus, cum illis plane congruunt. Porro nanciscimur ex formulis huius Sectionis formulas ad problema trium corporum spectantes has

$$\begin{aligned} W = & -b + 2e(\xi - 1)\cos\gamma + 2e's\cos(\gamma + h_{,}t + 2k_{,}) + \int M\,dt \\ & - h_{,}\frac{f's}{\Gamma}\int\Big\{\sin[h_{,}t - h_{,}(t)]\int\frac{d^2M}{d\gamma^2}\,dt - \cos[h_{,}t - h_{,}(t)]\int\frac{dM}{d\gamma}\,dt\Big\}\,dt \\ & - h_{,}\frac{s\xi}{\Gamma}\int\Big\{\sin[h_{,}t + 2k_{,}]\int\frac{d^2M_{\gamma}'}{d\gamma^2}\,dt + \cos[h_{,}t + 2k_{,}]\int\frac{dM_{\gamma}'}{d\chi}\,dt\Big\}\,dt \end{aligned} \tag{76}$$

ubi $\Gamma = sf' - \xi\xi'$ et

$$o = (y - \varkappa - \mu_{,})\xi - (v + \sigma_{,})f' \quad , \quad o = (y' - \varkappa' - \mu')\xi' - (\varpi' + \sigma')s$$

$$o = (y - 2\eta_{,} - \varkappa' - \mu')f' - (\varpi' + \sigma')\xi, \quad o = (y' + 2\eta_{,} - \varkappa - \mu_{,})s - (v + \sigma_{,})\xi'$$

quae cum formulis pro motu Lunae investigando supra datis haud plane congruunt. Cuius rei caussa in eo posita est, quod theoria Lunae non nisi casus specialis problematis trium corporum est. Significatio ipsarum $\mu_{,}$, $\sigma_{,}$, etc. in art. 20. explicata monstrat esse in theoria Lunae

$$2e\mu_{,} = n[\vartheta_{1}^{0,0,1} - \vartheta_{-1}^{0,0,1}], \quad 2e'\sigma_{,} = n\vartheta_{1}^{0,0,\delta}$$

denotantibus $\vartheta_{1}^{0,0,1}$, etc. quantitates in art. 11. Sect. IV. introductas; ex indole vero harum quantitatum in eadem Sectione allata patet, propter parvulam Lunae a Terra distantiam esse hunc ipsius $\sigma_{,}$ valorem perparvulum et quidem ad tertium ordinem referendum, si hic valor ipsius $\mu_{,}$ primi ordinis habetur. Porro quum $\varkappa' + \mu'$ atque $\varpi' + \sigma'$ ad motum Solis pertinentes per Lunae massam multiplicatae sint, hae illis adhuc multo minores sunt, et respectu illarum negligi possunt. Quamobrem aequationes praecedentes $y' = o$, $f' = o$ atque s aequalem esse quantitati secundi ordinis suppeditant, cuius rationem separatim habere non est opus. Ergo habemus pro theoria Lunae

$$W = -b + 2e(\xi - 1)\cos\gamma + \int M\,dt\,; \quad y = \varkappa + \mu$$

quae cum formulis Sectionum praecedentium plane congruunt.

Formulae in hac Sectione evolutae monstrant, dummodo radices aequationum (54) et (57) omnes reales et inaequales sint, in motu systematis quatuor corporum, quod consideravimus, terminos per tempus ipsum multiplicatos non adesse, etiamsi, plurimis deinceps approximationibus institutis, quemlibet praecisionis gradum attingere velis. Iam olim demon-

stratum est, aequationum earum, quae neglectis quantitatibus $\eta_{,}$, $\eta'_{,,}$, η'' ex (37) et (44) prodeunt, radices omnes reales et inaequales esse, si directio motuum corporum omnium eadem est, qua re et quum termini, quibus aequationes (54) et (57) ab aequationibus (37) et (44) differunt, parvuli sint, quoties massae perturbantes, excentricitates et inclinationes mutuae parvulae sunt, concludere licet, radices quoque aequationum (54) et (57) omnes reales et inaequales esse, quoties corporum motuum eadem directio est, et massae perturbantes, excentricitates et inclinationes mutuae adeo parvulae sunt, ut series infinitae, in quas formulae nostrae evolvendae sunt, convergant. Quae quum ita sint, systema quatuor corporum, quod consideravimus, nec non idem plurium corporum systema stabile esse, sequitur. Casus duarum aut plurium radicum aequalium per se quidem minime probabilis est, sed si forte acciderit, ut quantitas

$$in + i'n' + i''n'' + i'''\alpha + i^{IV}h_{,} + i^{V}h'_{,,} + i^{VI}h'' + i^{VII}l_{,} + i^{VIII}l'_{,,} + i^{IX}l''$$

pro quibusdam indicum i, i', i'', etc. valoribus valde parvula foret, permagnae et omnem limitem superantes perturbationes oriri posse videntur, quia haec quantitas in perturbationum coefficientium denominatoribus aderit. Theorema vero demonstrari potest, inclyto ab ill. Laplace de motuum commensurabilitate detecto theoremati simile, quo terminum permagnum ab illo parvulo divisore ortum limitem quendam superare non posse probatur, et quoties ille denominator in corporum motus initio tam parvulus fuerit, ut maiorem coefficientem efficiat, mutuam corporum attractionem sufficere ad eum rigorose cifrae aequalem reddendum, quo factum fuerit, ut motus commensurabiles evadant, et tales remaneant. Quae vero ad corporum, quorum motus commensurabiles sunt, perturbationes spectant, nec non plura alia, in aliud tempus differre debemus, ne hoc volumen nimis crescat.

CORRIGENDA.

Pag. 5. lin. 8. a v. loco $\frac{(n-1)}{2}$ lege $\frac{n(n-1)}{2}$

— 8. — 2. a c. — dm — dm'

— 10. — 10. a v. — $(y-y')$ — $(y-y')^2$

— 46. — 4. a c. — ψt — $\psi_, t$

— 56. — 4. a c. — $r^2 dv_,^2$ — $dr^2 + r^2 dv_,^2$

— 81. — 1. a v. — v — v'

— 85. — 12. a c. — triangulus — triangulum.

— 96. ultimae aequationi numerus (53) est apponendus.

— 103. lin. 1. — — 2. — — 3. — — 4. a v. loco $+(n)(\alpha-\eta-\varepsilon)t$ lege $-(n)(\alpha-\eta-\varepsilon)t$

— 110. — 13. a v. loco $-\int\left\{\frac{\alpha-\eta}{x}L+\ldots\right.$ lege $+\int\left\{\frac{\alpha-\eta}{x}L+\ldots\right.$

— 119. — 8. a c. — triangulus — triangulum.

— — — 7. a c. — sphaericus rectangulus — sphaericum rectangulum.

— — — 6. a c. — formatus — formatum.

— 131. — ult. ultimo termino ad dextram appone signum $\}$

— 126. — 10. a v. loco $\frac{d(\mathring{T})}{dg}(n)\,\partial\zeta$ lege $\frac{d(\mathring{T})}{dg}(n)\,\partial z$

— 144. — 3. a v. — $\frac{y_{,,}}{\sqrt{1-(e)^2}}$ — $\frac{(n)\,y_{,,}}{\sqrt{1-(e)^2}}$

— 203. — 8. a c. — φ' — $\bar{\varphi}'$

— 220. — 21. a v. — columma — columna.

— 229. — 4. a c. — $\frac{\overset{1}{y}_,}{\sqrt{1-e^2}}\;\frac{d^2\cdot\frac{(r)^2}{(a)^2}}{dg^2} - w\overline{\left(\frac{dW}{d\gamma}\right)}$ lege $\frac{\overset{1}{y}_,}{\sqrt{1-e^2}}\;\frac{d^2\cdot\frac{(r)^2}{(a)^2}}{dg^2}\Big] - w\overline{\left(\frac{dW}{d\gamma}\right)}$

— 234. — 4. a v. — $\frac{dB}{dK}$ lege $\frac{dA}{dK}$

— 239. — 9. a c. — (h_x-1) lege (h_x+1)

— 240. — 3. a c. — $\frac{d\zeta}{dt}$ — $\frac{d\zeta}{d\tau}$

— 255. — ult. — $v-\alpha+\eta=2v$ lege $v-\alpha+\eta=2\eta$

— 256. — 6. a c. — habita — habito

— 281. — 6. a c. — $\frac{d\{0,1,-1,0\}_,}{da}$ — $\frac{d\{0,1,-1,0\}_,}{da'}$

— 289. — 1. a v. — $(M''-M_,+L'_{,,})$ — $\sin(M''-M_,+L'_{,,})$

— — — 3. a v. — $(M_,-M'_{,,}+L'')$ — $\sin(M_,-M'_{,,}+L'')$

— — ubique — $dL'_{,,}$ et dL'' — $dK'_{,,}$ et resp. dK''.